Chapter 16

Linear Small Amplitude Wave Theories

16-1 Basic Equations and Formulation of a Surface Wave Problem

16-1.1 Notation and Continuity

The motion is defined with respect to the three axes in a Cartesian coordinates system. OX, OY, and OZ are the mutually perpendicular axes. The OZ axis is taken to be vertical and positive upwards. Any point is defined by the coordinates x, y, and z. The depth is defined by $z = -d$, and is assumed to be constant (see Fig. 16-1). Viscosity forces are neglected. The motion is assumed to be irrotational and the fluid is incompressible.

$$\textbf{curl V} = 0 \quad \text{or} \quad \zeta = \eta = \xi = 0$$

(Note that η in this chapter will be used for the free surface elevation and not for vorticity.) Also,

$$\text{div } \mathbf{V} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

These assumptions result in a number of simplifications.

curl V $= 0$ ensures the existence of a single-valued velocity potential function $\phi(x,y,z,t)$ from which the velocity field can be derived. Thus, the potential function can arbitrarily be defined as $\mathbf{V} = \textbf{grad}\ \phi$ or $\mathbf{V} = -\textbf{grad}\ \phi$. The latter definition is used in this chapter, i.e., $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$, $w = -\partial\phi/\partial z$. The velocity potential function

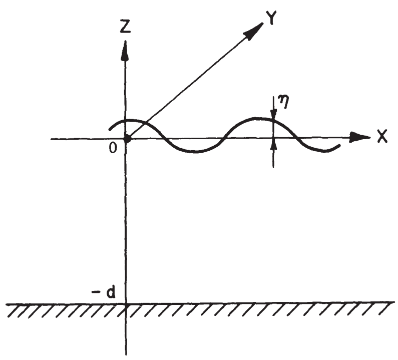

Figure 16-1

Notation.

has to be found from the continuity equation, the momentum equation and the boundary conditions.

The continuity equation div **V** = 0 is expressed in terms of ϕ by the equation $\nabla^2\phi = 0$. In Cartesian coordinates, it is written as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

16-1.2 The Momentum Equation

The momentum equation for an irrotational flow is given by the following form of the Bernoulli equation (see Section 10-1.2). The minus sign in Equation 16-1 is due to the new definition of ϕ.

$$\underbrace{-\frac{\partial\phi}{\partial t}}_{\text{Local inertia term}} + \underbrace{\frac{1}{2}V^2}_{\text{Convective inertia term}} + \underbrace{\frac{p}{\rho}}_{\text{Pressure term}} + \underbrace{gz}_{\text{Gravity term}} = f(t) \qquad (16\text{-}1)$$

In this equation, $f(t)$ may depend on t but not on the space variables. The fact that the flow is assumed to be irrotational means that the Bernoulli law is valid throughout the fluid and not only along streamlines.

This equation is nonlinear because of the convective inertia term. This term may be expressed as a function of the potential function ϕ so that

$$\frac{1}{2}V^2 = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]$$

The nonlinearity of the motion can be seen clearly. In the case of very slow motion, the convective term is neglected and the Bernoulli equation is written as

$$-\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz = f(t)$$

Periodic gravity wave theories often satisfy the condition for slow motion with a fairly good degree of accuracy. The corresponding solutions are mathematically exact when the motion tends to be infinitely small.

16-1.3 Boundary Conditions

16-1.3.1 At a fixed boundary, the fluid velocity is tangential to the boundary, that is, the normal component V_n is zero. In terms of velocity potential ϕ, this condition is written $\partial\phi/\partial n = 0$. In particular on a horizontal bottom

$$w\Big|_{z=-d} = -\frac{\partial\phi}{\partial z}\Big|_{z=-d} = 0$$

16-1.3.2 One of the difficulties encountered in determining the nature of wave motion is due to the fact that one of the boundaries—the free surface—is unknown, except in the case of infinitely small motion in which the free surface is, at the beginning, assumed to be a horizontal line. Hence, another unknown $z = \eta$ appears in wave problems. If one assumes that the free surface, in the most general case of a three dimensional motion, is given by the equation $z = \eta(x,y,t)$, then the variation of z with respect to time t is

$$\frac{dz}{dt} = \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}\frac{dx}{dt} + \frac{\partial\eta}{\partial y}\frac{dy}{dt}$$

Introducing the values

$$\frac{dx}{dt} = u = -\frac{\partial\phi}{\partial x} \qquad \frac{dy}{dt} = v = -\frac{\partial\phi}{\partial y}$$

$$\frac{dz}{dt} = w\Big|_{z=\eta} = -\frac{\partial\phi}{\partial z}\Big|_{z=\eta}$$

the free surface equation becomes

$$\frac{\partial\phi}{\partial z}\Big|_{z=\eta} = -\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\Big|_{z=\eta}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\Big|_{z=\eta}\frac{\partial\eta}{\partial y}$$

This equation is nonlinear and is the *kinematic condition* at the free surface.

Another equation—the dynamic equation—is given by the Bernoulli equation in which the pressure p is considered as constant (and equal to atmospheric pressure). Hence the free surface dynamic condition becomes

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta = f(t)$$

Thus, generally ϕ and η appear to be given by the solution of $\nabla^2\phi = 0$ with two simultaneous nonlinear boundary conditions at the free surface and a linear boundary condition at the bottom

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-d} = 0$$

16-1.4 The Free Surface Condition in the Case of Very Slow Motion

In the case of slow motion the Bernoulli equation

$$-\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz = f(t)$$

becomes

$$-\left.\frac{\partial \phi}{\partial t}\right|_{z=\eta} + g\eta = 0$$

at the free surface. The result is:

$$\eta = \frac{1}{g}\left.\frac{\partial \phi}{\partial t}\right|_{z=\eta}$$

provided the function $f(t)$ and any additive constant can be included in the value of $\partial\phi/\partial t$.

Since the motion is assumed to be infinitely small, η may be written

$$\frac{1}{g}\left.\frac{\partial \phi}{\partial t}\right|_{z=0}$$

This approximation leads to an error of the order of those already done in neglecting the convective inertia term.

Consider the kinematic condition: $\partial\eta/\partial x$ and $\partial\eta/\partial y$ are the components of the slope of the free surface and are small as in the case of slow motion (see Fig. 16-2).

The nonlinear terms $(\partial\phi/\partial x)(\partial\eta/\partial x)$ and $(\partial\phi/\partial y)(\partial\eta/\partial y)$ may be neglected. The normal component of the fluid velocity at the free surface is now equal to the normal velocity of the surface itself. This gives with sufficient approximation

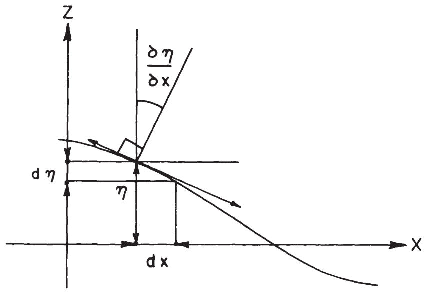

Figure 16-2 *Notation.*

$$\frac{\partial \eta}{\partial t} = -\left.\frac{\partial \phi}{\partial z}\right|_{z=0}$$

η may now be easily eliminated from the dynamic and kinematic conditions. The derivative of η with respect to t in the dynamic condition gives

$$\frac{\partial \eta}{\partial t} = \frac{1}{g}\cdot\left.\frac{\partial^2 \phi}{\partial t^2}\right|_{z=0}$$

$\partial\eta/\partial t$ can be eliminated by equating the two above equations. This yields:

$$\left[\frac{\partial \phi}{\partial z} + \frac{1}{g}\cdot\frac{\partial^2 \phi}{\partial t^2}\right]_{z=0} = 0$$

which is called the *Cauchy–Poisson condition* at the free surface.

16-1.5 Formulation of a Surface Wave Problem

16-1.5.1 Thus ϕ and η appear to be solutions of the following system:

1. Continuity

$$\nabla^2\phi = 0 \qquad \begin{cases} -d \le z \le \eta(x,y,t) \\ \left.\begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}\right\} \text{no boundary} \end{cases}$$

2. Fixed boundary $\partial\phi/\partial n = 0$. In particular, at the bottom

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-d} = 0$$

3. Free surface $z = \eta(x,y,t)$
   a. Kinematic condition:

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = -\frac{\partial \eta}{\partial t} + \left.\frac{\partial \phi}{\partial x}\right|_{z=\eta}\frac{\partial \eta}{\partial x} + \left.\frac{\partial \phi}{\partial y}\right|_{z=\eta}\frac{\partial \eta}{\partial y}$$

   b. Dynamic condition:

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta = 0$$

   where $f(t)$ is now included in $\partial\phi/\partial t$. However, even in this case, this last equation may be different from zero, and equal to a given function $f(x,y,t)$ in the case of a disturbance created at the free surface.

So formulated, the solution of the system of equations presented in this section is still difficult to determine. First, the equations are nonlinear, and second, the free surface is unknown and is time-dependent.

16-1.5.2 In the case of slow motion, η may be eliminated from the two free surface conditions resulting in the simple Cauchy–Poisson condition

$$\left[\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right]_{z=0} = 0$$

This leaves only one unknown, ϕ, to be determined from

$$\nabla^2\phi = 0 \qquad \begin{cases} -d \le z \le \eta = 0 \\ -\infty < x < \infty \\ -\infty < y < \infty \end{cases}$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-d} = 0 \quad \text{and} \quad \left[\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right]_{z=0} = 0$$

16-2 Method of Solutions

16-2.1 General Approach

16-2.1.1 When all the equations are homogeneous and linear, the principle of superposition states that any number of individual solutions may be superimposed to form new functions which constitute solutions themselves. In a linear equation ϕ and its derivatives occur only in the first degree in every term. For example, if ϕ_1 and ϕ_2 are two separate solutions, $a\phi_1 + b\phi_2$ is also a solution, a and b being two arbitrary constants. This basic principle is very important and will be used in the following sections.

16-2.1.2 Most of the solutions with which we are concerned in this chapter are harmonic. This stems from the fact that harmonic functions are quite natural solutions of the basic equations. The solutions characterizing periodic motions may be considered as superposition of harmonic components.

The solution of $\phi(x,y,z,t)$ is usually of the form

$$\phi = f(x,y,z)\cos(kt + \varepsilon),$$

where $k = 2\pi/T$ and T is the wave period. Another form of the solution is

$$\phi = \text{Re}\, f(x,y,z)e^{i(kt+\varepsilon)}$$

Recall that

$$e^{i(kt+\varepsilon)} = \cos(kt + \varepsilon) + i\sin(kt + \varepsilon)$$

"Re" means the real part of the function and ε is the phase of ϕ with respect to the origin of time, $t = 0$. In the following, "Re" will be omitted and it is to be understood that only the real parts of the mathematical expressions are considered.

Introducing this form of ϕ in the free surface condition

$$\frac{1}{g}\cdot\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial z} = 0$$

gives

$$\left(\frac{k^2}{g} - \frac{\partial}{\partial z}\right)\phi = 0$$

16-2.1.3 If it is assumed that ϕ is given by a product of functions of each variable alone, then the basic equation $\nabla^2\phi = 0$ may be solved by the separation of variables method. From physical considerations, it may be expected that the solution ϕ will be given by the product of the functions of the horizontal components $U(x,y)$, the vertical component $P(z)$, and the time $f(t)$. Hence

$$\phi = U(x,y) \cdot P(z) \cdot f(t).$$

This value of $\phi(x,y,z,t)$ can be used in the continuity equation, $\nabla^2\phi = 0$. Algebraic manipulation of the result will give:

$$-\frac{\partial^2 U/\partial x^2 + \partial^2 U/\partial y^2}{U(x,y)} = \frac{d^2P/dz^2}{P(z)}$$

This may be written as

$$-\frac{\nabla^2 U}{U} = \frac{P''}{P}$$

Notice that the functions of x and y are on one side of the equal sign, while the functions of z are on the other. The variables have been separated.

It must be said that it was not certain at the beginning that it would have been possible to separate the variables as has been done. However, it will be shown later that this process may be performed for solutions of $\nabla^2\phi = 0$. The right-hand side of the above equation is a function of z. The left-hand side is a function of x and y. Since x and y can vary independently of z and vice versa, the only way in which the function of x and y and the function of z can always be equal (as stated by the above equation) is if the left-hand side and the right-hand side are both equal to the same constant m^2 where m may be real or imaginary.

It will be easily seen that if m is imaginary there is no physical meaning to the solutions in the case of wave motion. Thus, m is chosen to be real and m^2 is always positive.

The equations

$$\frac{P''}{P} = -\frac{\nabla^2 U}{U} = m^2$$

are now reduced to

$$\frac{d^2P}{dz^2} - m^2 P(z) = 0$$

and

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + m^2 U(x,y) = 0$$

These equations will often be written in the shorter form:

$$\begin{cases} \left(\dfrac{d^2}{dz^2} - m^2\right)P = 0 \\ (\nabla^2 + m^2)U = 0 \end{cases}$$

The last equation is the well-known *Helmholtz equation* (also called the wave equation) of mathematical physics.

16-2.2 Wave Motion along a Vertical

The equation

$$\left(\frac{d^2}{dz^2} - m^2\right)P = 0$$

may easily be integrated, giving the general solution

$$P = Ae^{mz} + Be^{-mz}$$

where A and B are constants.

The boundary condition at the bottom,

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-d} = 0$$

gives for any fixed value of x, y, and t:

$$\left.\frac{dP}{dz}\right|_{z=-d} = 0$$

When this boundary condition is applied to the solution for P the result is

$$mAe^{-md} - mBe^{+md} = 0$$

Hence

$$Ae^{-md} = Be^{+md}$$

Consider the original solution,

$$P = Ae^{mz} + Be^{-mz}$$

Multiply each term on the right by $e^{-md}e^{+md}$. Let

$$Ae^{-md} = Be^{+md} = \tfrac{1}{2}D$$

and substitute this in the equation for P. The result is

$$P = \frac{D}{2}\left(e^{m(z+d)} + e^{-m(z+d)}\right)$$

That is $P = D \cosh m(z + d)$.

Now, substituting P in the expression of ϕ gives

$$\phi = D \cosh m(d + z)U(x,y)f(t)$$

16-2.3 *Introduction of the Free Surface Condition: General Solution*

The solution for $f(t)$ is given by the Cauchy–Poisson condition at the free surface:

$$\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right)_{z=0} = 0$$

Substitute the value of ϕ obtained in the previous section in this equation. Only the case when $z = 0$ needs to be considered here, since this is the free surface condition. When the resulting equation is divided by the value of ϕ, the following is obtained:

$$f''/f = -gm \tanh md$$

If we let $k^2 = gm \tanh md$, the solution for f is given by the equation $f'' + k^2 f = 0$. The characteristic equation $r^2 + k^2 = 0$ gives $r = \pm ik$. Hence

$$f = \alpha e^{ikt} + \beta e^{-ikt}$$

where α, β are constant coefficients which depend upon the boundary conditions, k is the frequency $2\pi/T$, and T is the wave period. When $\beta = 0$ and the coefficient α is included in the coefficient D, it is found

$$\phi = D \cosh m(d + z)U(x,y)e^{ikt}$$

Since there exist an infinite (but discrete) number of values for k_n and m_n which satisfy the equation

$$k_n^2 = m_n g \tanh m_n d$$

a general solution for ϕ can be written as

$$\phi = \sum_{n=0}^{\infty} D_n \cosh m_n(d + z)U_n(x,y) \exp(ik_n t + \varepsilon_n)$$

where ε_n is a phase constant.

Consider the case of a monochromatic wave. It is convenient to express D as a function of the wave height $2a$. From the free surface dynamic equation for slow motion,

$$\eta = \frac{1}{g}\frac{\partial\phi}{\partial t}\bigg|_{z=0}$$

one obtains

$$\eta = \frac{ikD}{g} \cosh md\,U(x,y)e^{ikt}$$

Considering only the real part, this can be written as (recall $i^2 = -1$):

$$\eta = -\frac{kD}{g} \cosh md\,U(x,y) \sin kt$$

The expressions for ϕ and η become more convenient if we write (aU) for the amplitude of η. Then

$$a = -\frac{kD}{g} \cosh md$$

Hence

$$D = -\frac{ag}{k}\frac{1}{\cosh md}$$

and

$$\phi = -\frac{ag}{k}\frac{\cosh m(d + z)}{\cosh md} U(x,y)e^{ikt}$$

Substituting the relationship $k^2 = mg \tanh md$ leads to

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} U(x,y) e^{ikt}$$

With the value of D that has been found, the expression for P becomes

$$P(z) = -\frac{ak}{m} \frac{\cosh m(d+z)}{\sinh md} = -\frac{ag}{k} \frac{\cosh m(d+z)}{\cosh md}$$

Under these conditions the wave height at any point is $2aU(x,y)$. $U(x,y)$ is the relative value of the wave height with respect to a plane or a point where it is simply $2a$.

16-3 Two-Dimensional Wave Motion

16-3.1 Integration of the Wave Equation

The differential equation to be solved is $(\nabla^2 + m^2)U = 0$. A general solution of this equation does not exist, but a number of solutions may be found, corresponding to particular boundary conditions. In the case of a two-dimensional wave such as motion encountered in a wave flume.

$$\frac{\partial \phi}{\partial y} = 0 \qquad \frac{\partial U}{\partial y} = 0$$

This reduces the wave equation to

$$\left(\frac{\partial^2}{\partial x^2} + m^2\right) U = 0$$

Solving this, one finds that the solutions for U are given by any linear combination of e^{-imx} and e^{imx} such as,

$$U = A' e^{imx} + B' e^{-imx}$$

In particular if $U = e^{-imx}$, then

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} e^{i(kt - mx)}$$

or

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} \cos (kt - mx)$$

This is the velocity potential function of a progressive wave traveling in the OX direction.

If $U = e^{imx}$, the velocity potential function of a wave traveling in the opposite direction is obtained.

If the solution for U is:

$$U = \frac{1}{2} (e^{imx} + e^{-imx}) = \cos mx$$

or

$$U = \frac{1}{2i} (e^{imx} - e^{-imx}) = \sin mx$$

then

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} mx \cos kt$$

This is the velocity potential function of a standing wave. If A' is different from B', a partial standing wave is obtained. In practice, the values for A' and B' are given by vertical boundary conditions (wave reflection, etc.).

In the most general case of a two-dimensional irregular wave, as may be observed at sea, the velocity potential function ϕ is:

$$\phi = \sum_{n=0}^{\infty} -a_n \frac{k_n}{m_n} \frac{\cosh m_n(d+z)}{\sinh m_n d} \exp\left[i(k_n t - m_n x + \varepsilon_n)\right]$$

where ε_n is a phase constant.

When there are two waves only, traveling in the same direction, the velocity potential function describing the "beating" phenomena may be obtained easily.

16-3.2 Physical Meaning: Wavelength

It is easy to see the physical meaning of the coefficient m. Since ϕ and consequently η is periodic with respect to space, $m = 2\pi/L$ and L is the wavelength.

The wavelength is given by

$$k^2 = mg \tanh md$$

and then

$$\left(\frac{2\pi}{T}\right)^2 = \frac{2\pi}{L} g \tanh \frac{2\pi}{L} d$$

that is,

$$L = \frac{gT^2}{2\pi} \tanh \frac{2\pi d}{L}$$

and the wave celerity:

$$C = \frac{L}{T} = \frac{gT}{2\pi} \tanh \frac{2\pi d}{L}$$

In particular when d/L is small (shallow water)

$$\tanh \frac{2\pi d}{L} \cong \frac{2\pi d}{L} \qquad L = T(gd)^{1/2} \qquad C = (gd)^{1/2}$$

and

$$L = T(gd)^{1/2} \qquad C = (gd)^{1/2}$$

When d/L is large (deep water), $\tanh 2\pi d/L = 1$, and $L = gT^2/2\pi$, $C = gT/2\pi$. The values of L and C are given as functions of the depth d and the wave period T on the following nomographs (Figs. 16-3, 16-4, and 16-5).

16-3.3 Flow Patterns

The velocity components are $u = -\partial\phi/\partial x$, $w = -\partial\phi/\partial z$ and the particle orbits are:

$$x = -\int_0^t \frac{\partial\phi}{\partial x} dt \qquad z = -\int_0^t \frac{\partial\phi}{\partial z} dt$$

In the case of a progressive wave:

$$u = ka \frac{\cosh m(d + z)}{\sinh md} \sin (kt - mx)$$

$$w = ka \frac{\sinh m(d + z)}{\sinh md} \cos (kt - mx)$$

The particle orbits are determined by assuming that the motion around a fixed point x_0, z_0 is small, so that one can consider x and z constant in the integration.

$$x = x_0 - a \frac{\cosh m(d + z_0)}{\sinh md} \cos (kt - mx_0)$$

and

$$z = z_0 + a \frac{\sinh m(d + z_0)}{\sinh md} \sin (kt - mx_0)$$

Squaring and adding these two last equations to eliminate t, the equation of an ellipse is obtained

$$\frac{(x - x_0)^2}{A^2} + \frac{(z - z_0)^2}{B^2} = 1$$

It is now seen that x_0 and z_0 are at the center of the ellipse, i.e., can be considered as the position of the particle at rest with the horizontal semimajor axis

$$A = a \frac{\cosh m(d + z_0)}{\sinh md}$$

and the vertical semiminor axis:

$$B = a \frac{\sinh m(d + z_0)}{\sinh md}$$

$B = a$ at the free surface, and $B = 0$ at the bottom (Fig. 16-6). The free surface equation is

$$\eta = \frac{1}{g} \frac{\partial\phi}{\partial t} = a \sin (kt - mx)$$

When $d \to \infty$, $(A/B) \to 1$ and the orbits are circles of radius $R = a \exp (4\pi^2 z/gT^2)$.

In the case of a standing wave, it would be easily found that the paths of particles are straight lines given by

$$\frac{z - z_0}{x - x_0} = -\tanh m(d + z_0) \cot mx_0$$

or

$$\frac{z - z_0}{x - x_0} = \tanh m(d + z_0) \tan mx_0$$

(see Fig. 16-7). They are parabolas at a second order of approximation.

16-3.4 Partial Standing Wave

A partial standing wave is caused by the superimposition of two waves of the same period but travelling in opposite directions and with different amplitudes (Fig. 16-8).

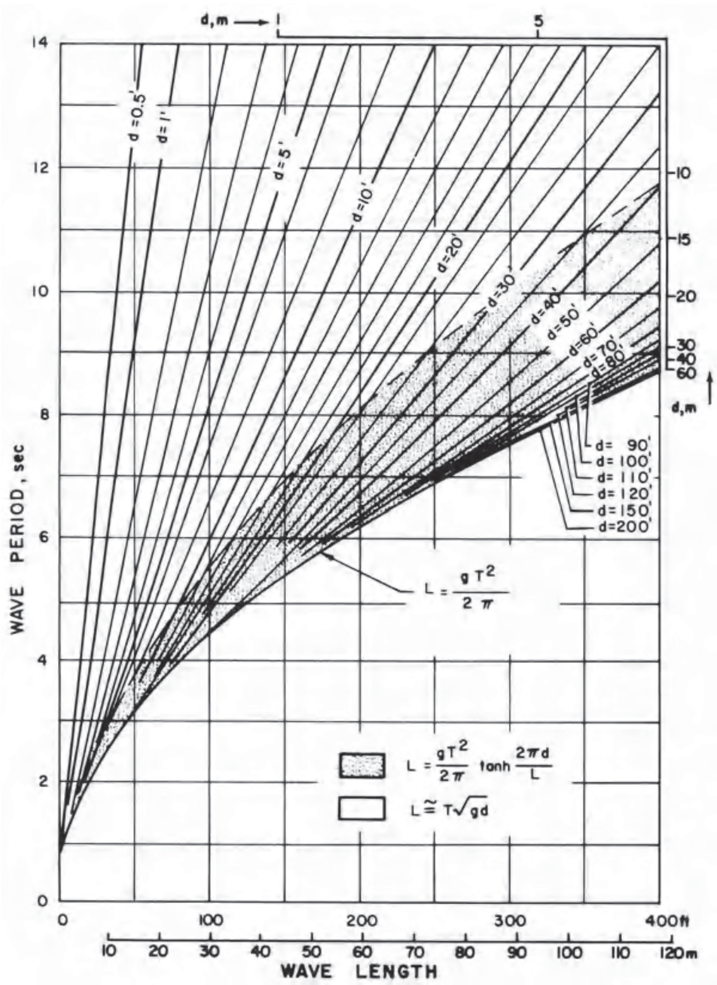

Figure 16-3
Wavelength vs depth and period.

The first-order potential function is

$$\phi = -\frac{k}{m}\frac{\cosh m(d+z)}{\sinh md}$$
$$\times\ [\alpha_1 \sin(kt + mx) + \alpha_2 \sin(kt - mx)]$$

The amplitude at the antinode is $(\alpha_1 + \alpha_2)$ and at the node is $(\alpha_1 - \alpha_2)$. It is possible to determine the individual wave height of the two progressive waves by measuring the amplitude at the antinode (maximum) A and at the node B (minimum). Then,

$$\alpha_1 = \tfrac{1}{2}(A + B)$$

$$\alpha_2 = \tfrac{1}{2}(A - B)$$