

POTENCIJALNA ENERGIJA DEFORMACIJE

ZADATAK 9.

Za zatvoreni kružni prsten treba primjenom drugog Castiglianova teorema izračunati prekobrojnu veličinu, nacrtati dijagrame momenata savijanja, poprečnih i uzdužnih sila i odrediti pomak točke **A**.

Zadano je:

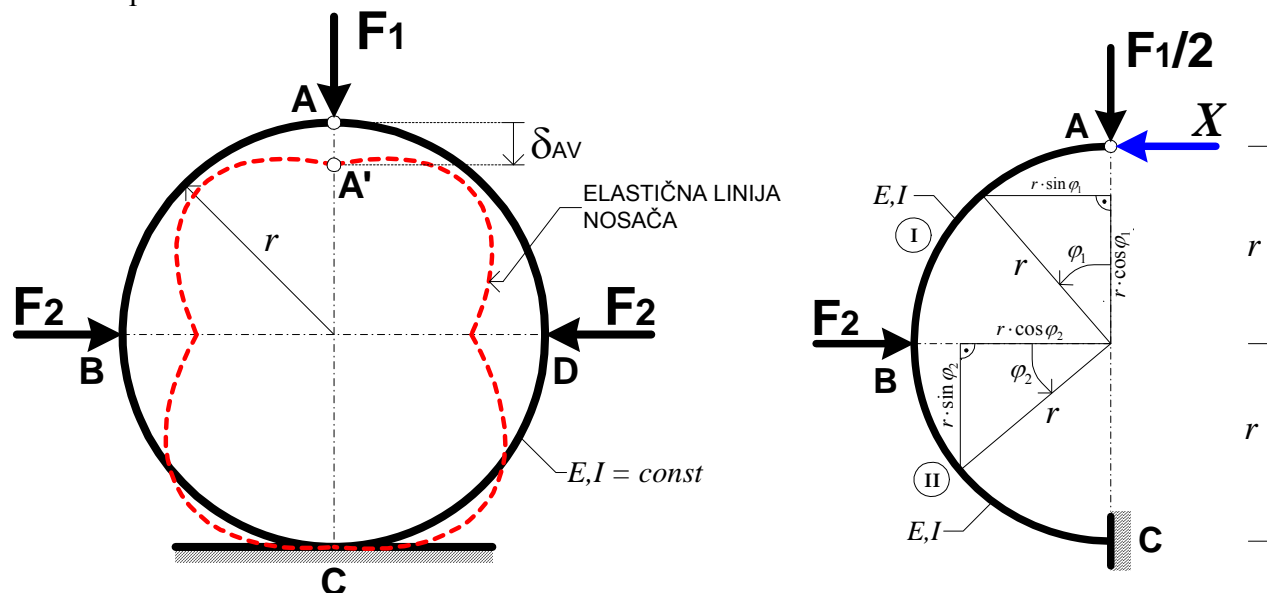
$$E = 2,1 \cdot 10^5 \text{ MPa}$$

$$I = 2,0 \cdot 10^6 \text{ mm}^4$$

$$r = 1,0 \text{ m}$$

$$F_1 = 20 \text{ kN}$$

$$F_2 = F_1$$



Sistem je jedanput statički neodređen ($s = 3 \cdot n - m = 3 \cdot n - 2 \cdot z_1 = 3 \cdot 1 - 2 \cdot 1 = 1$) i ima jednu (vertikalnu) os simetrije.

Za odabir osnovnog sistema iskoristavamo os simetrije, promatramo samo lijevu polovicu prstena, a na osi simetrije zadovoljavamo rubne uvjete.

Zbog simetrije presjek **C** se ne zaokreće tijekom deformiranja prstena (vidi oblik elastične linije), pa lijevu polovicu prstena možemo razmatrati kao da je **upeta u presjeku C**.

Na drugoj strani simetrije u presjeku **A**, poprečna sila jednaka je polovici sile F_1 , moment je jednak nuli zbog zgloba, pa kao prekobrojna veličina ostaje samo uzdužna sila X . Na toj strani zadovoljavamo rubni uvjet: **zbog osi simetrije horizontalni pomak točke A je jednak nuli**, pa prekobrojnu veličinu, uzdužnu silu X , određujemo iz uvjeta da je pomak na mjestu i u smjeru te sile jednak nuli.

Prema drugom Castiglianovom teoremu parcijalna derivacija potencijalne energije deformacije po sili jednaka je pomaku na mjestu i u smjeru te sile:

$$\delta_{AH} = \frac{\partial U}{\partial X} = 0 \quad \delta_{AH} = \frac{\partial U}{\partial X} = \int_0^s \frac{M}{E \cdot I} \cdot \frac{\partial M}{\partial X} \cdot ds = 0 \quad U = U_I + U_{II}$$

Utjecaje uzdužne i poprečne sile na ukupnu potencijalnu energiju deformacije smo zanemarili.

$$\text{Dio I:} \quad M_{\varphi_1} = -\frac{F_1}{2} \cdot r \cdot \sin \varphi_1 + X \cdot r \cdot (1 - \cos \varphi_1) \quad \frac{\partial M_{\varphi_1}}{\partial X} = +r \cdot (1 - \cos \varphi_1)$$

$$\text{Dio II:} \quad M_{\varphi_2} = -\frac{F_1}{2} \cdot r \cdot \cos \varphi_2 + X \cdot r \cdot (1 + \sin \varphi_2) - F_2 \cdot r \cdot \sin \varphi_2 \quad \frac{\partial M_{\varphi_2}}{\partial X} = +r \cdot (1 + \sin \varphi_2)$$

$$\delta_{AH} = \frac{\partial U}{\partial X} = \frac{1}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} M_{\varphi_1} \cdot \frac{\partial M_{\varphi_1}}{\partial X} \cdot r \cdot d\varphi_1 + \frac{1}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} M_{\varphi_2} \cdot \frac{\partial M_{\varphi_2}}{\partial X} \cdot r \cdot d\varphi_2 = 0 \quad | \cdot (E \cdot I) \quad r \cdot d\varphi = ds$$

$$\int_0^{\frac{\pi}{2}} M_{\varphi_1} \cdot \frac{\partial M_{\varphi_1}}{\partial X} \cdot r \cdot d\varphi_1 + \int_0^{\frac{\pi}{2}} M_{\varphi_2} \cdot \frac{\partial M_{\varphi_2}}{\partial X} \cdot r \cdot d\varphi_2 = 0$$

$$\int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot r \cdot \sin \varphi_1 + X \cdot r \cdot (1 - \cos \varphi_1) \right] \cdot [r \cdot (1 - \cos \varphi_1)] \cdot r \cdot d\varphi_1 + \int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot r \cdot \cos \varphi_2 + X \cdot r \cdot (1 + \sin \varphi_2) - F_2 \cdot r \cdot \sin \varphi_2 \right] \cdot [r \cdot (1 + \sin \varphi_2)] \cdot r \cdot d\varphi_2 = 0 \quad | : r^3$$

$$\int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot (\sin \varphi_1 - \sin \varphi_1 \cdot \cos \varphi_1) + X \cdot (1 - 2 \cdot \cos \varphi_1 + \cos^2 \varphi_1) \right] \cdot d\varphi_1 + \int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot (\cos \varphi_2 + \sin \varphi_2 \cdot \cos \varphi_2) + X \cdot (1 + 2 \cdot \sin \varphi_2 + \sin^2 \varphi_2) - F_2 \cdot (\sin \varphi_2 + \sin^2 \varphi_2) \right] \cdot d\varphi_2 = 0$$

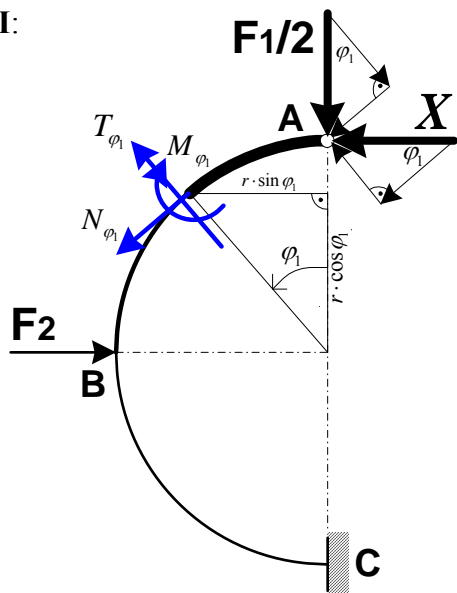
$$X \cdot \left(\frac{\pi}{2} - 2 \cdot 1 + \frac{\pi}{4} \right) + X \cdot \left(\frac{\pi}{2} + 2 \cdot 1 + \frac{\pi}{4} \right) = \frac{F_1}{2} \cdot \left(1 - \frac{1}{2} \right) + \frac{F_1}{2} \cdot \left(1 + \frac{1}{2} \right) + F_2 \cdot \left(1 + \frac{\pi}{4} \right) \quad F_2 = F_1 = F$$

$$X \cdot (0,356194 + 4,356194) = F \cdot (0,25 + 0,75 + 1,785398)$$

$$X \cdot 4,712388 = F \cdot 2,785398 \quad \Rightarrow \quad X = 0,59108 \cdot F = \underline{11,824 \text{ kN}}$$

Kad je određena prekobrojna veličina uzdužne sile $X = 11,824 \text{ kN}$, izračunat ćemo veličine momenata savijanja, poprečnih i uzdužnih sila na karakterističnim mjestima i nacrtati dijagrame.

Dio I:

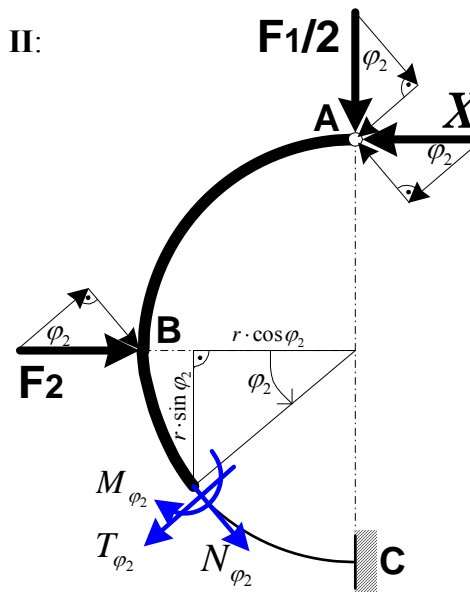


$$\sum M = 0 \Rightarrow M_{\varphi_1} = -\frac{F_1}{2} \cdot r \cdot \sin \varphi_1 + X \cdot r \cdot (1 - \cos \varphi_1)$$

$$\sum F_t = 0 \Rightarrow T_{\varphi_1} = +\frac{F_1}{2} \cdot \cos \varphi_1 - X \cdot \sin \varphi_1$$

$$\sum F_n = 0 \Rightarrow N_{\varphi_1} = -\frac{F_1}{2} \cdot \sin \varphi_1 - X \cdot \cos \varphi_1$$

Dio II:



$$M_{\varphi_2} = -\frac{F_1}{2} \cdot r \cdot \cos \varphi_2 + X \cdot r \cdot (1 + \sin \varphi_2) - F_2 \cdot r \cdot \sin \varphi_2$$

$$T_{\varphi_2} = -\frac{F_1}{2} \cdot \sin \varphi_2 - X \cdot \cos \varphi_2 + F_2 \cdot \cos \varphi_2$$

$$N_{\varphi_2} = -\frac{F_1}{2} \cdot \cos \varphi_2 + X \cdot \sin \varphi_2 - F_2 \cdot \sin \varphi_2$$

Momenti savijanja

$$\text{Dio I:} \quad M_A = 0 \quad M_{\varphi_1 = \frac{\pi}{4}} = -\frac{F_1}{2} \cdot r \cdot \sin \frac{\pi}{4} + X \cdot r \cdot \left(1 - \cos \frac{\pi}{4}\right) = -3,61 \text{ kNm}$$

$$M_{\varphi_1 = \frac{\pi}{2}} = M_B = -\frac{F_1}{2} \cdot r + X \cdot r = -10 + 11,824 = +1,824 \text{ kNm}$$

$$\text{Dio II:} \quad M_{\varphi_2 = \frac{\pi}{4}} = -\frac{F_1}{2} \cdot r \cdot \cos \frac{\pi}{4} + X \cdot r \cdot \left(1 + \sin \frac{\pi}{4}\right) - F_2 \cdot r \cdot \sin \frac{\pi}{4} = -1,028 \text{ kNm}$$

$$M_{\varphi_2 = \frac{\pi}{2}} = M_C = +X \cdot 2 \cdot r - F_2 \cdot r = +23,648 - 20 = +3,648 \text{ kNm}$$

Poprečne sile

$$\text{Dio I:} \quad T_A^{lijevo} = +\frac{F_1}{2} = +10 \text{ kN}$$

$$T_{\varphi_1 = \frac{\pi}{4}} = +\frac{F_1}{2} \cdot \cos \frac{\pi}{4} - X \cdot \sin \frac{\pi}{4} = -1,30 \text{ kN}$$

$$T_{\varphi_1 = \frac{\pi}{2}} = T_B^{gore} = -X = -11,824 \text{ kN}$$

$$\text{Dio II:} \quad T_B^{dolje} = -X + F = +8,176 \text{ kN}$$

$$T_{\varphi_2 = \frac{\pi}{4}} = -\frac{F_1}{2} \cdot \sin \frac{\pi}{4} - X \cdot \cos \frac{\pi}{4} + F_2 \cdot \cos \frac{\pi}{4} = -1,30 \text{ kN}$$

$$T_{\varphi_2 = \frac{\pi}{2}} = T_C^{lijevo} = -\frac{F_1}{2} = -10 \text{ kN}$$

Uzdužne sile

$$N_A = -X = -11,824 \text{ kN}$$

$$N_{\varphi_1 = \frac{\pi}{4}} = -\frac{F_1}{2} \cdot \sin \frac{\pi}{4} - X \cdot \cos \frac{\pi}{4} = -15,43 \text{ kN}$$

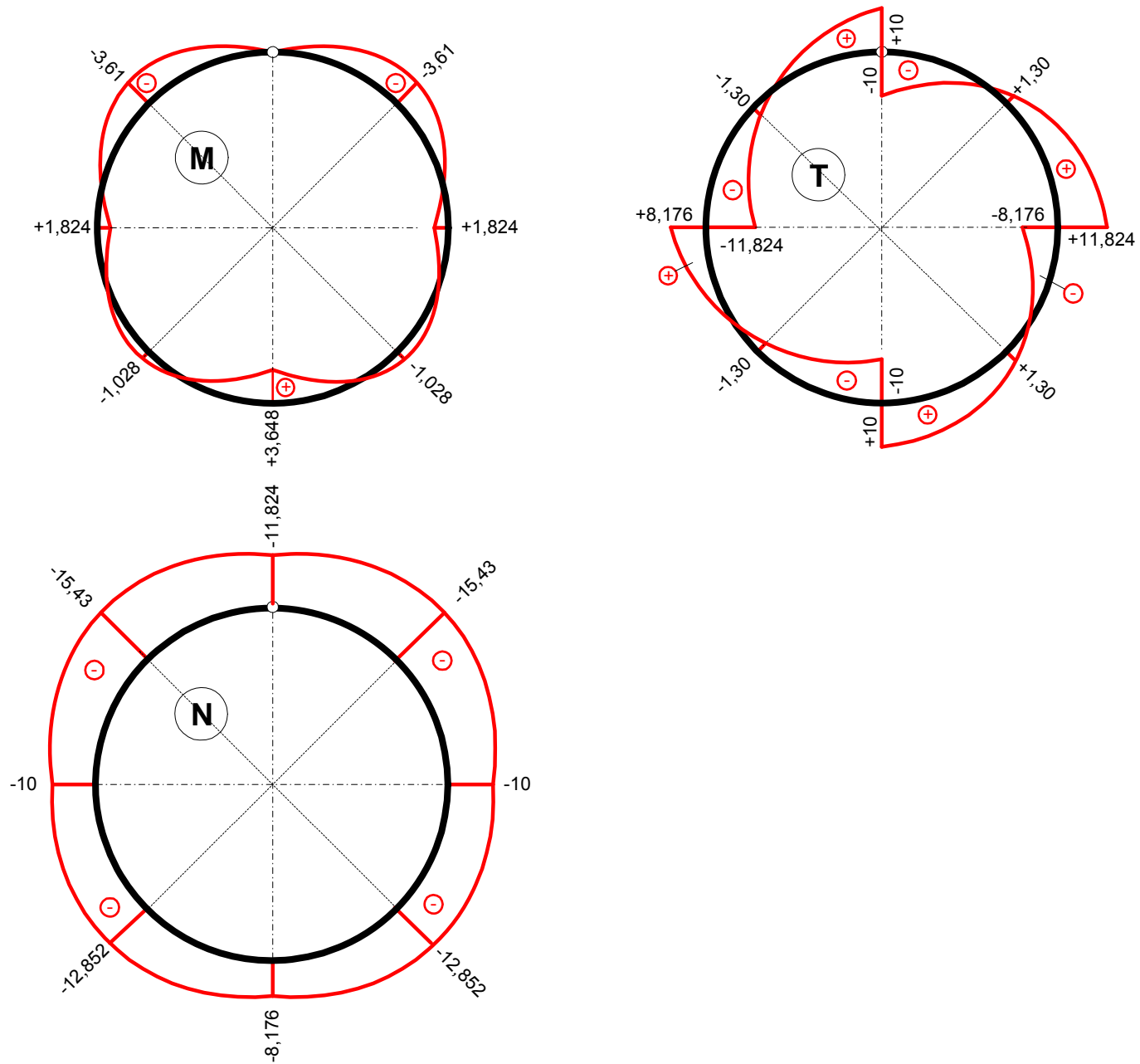
$$N_{\varphi_1 = \frac{\pi}{2}} = N_B^{gore} = -\frac{F_1}{2} = -10 \text{ kN}$$

$$N_B^{dolje} = -\frac{F_1}{2} = -10 \text{ kN}$$

$$N_{\varphi_2 = \frac{\pi}{4}} = -\frac{F_1}{2} \cdot \cos \frac{\pi}{4} + X \cdot \sin \frac{\pi}{4} - F_2 \cdot \sin \frac{\pi}{4} = -12,852 \text{ kN}$$

$$N_{\varphi_2 = \frac{\pi}{2}} = N_C^{lijevo} = +X - F_2 = -8,176 \text{ kN}$$

DIJAGRAMI UNUTARNJIH SILA



POMAK TOČKE **A**

Prema drugom Castiglianovom teoremu parcijalna derivacija potencijalne energije deformacije po sili jednaka je pomaku na mjestu i u smjeru te sile:

$$\delta_{AV} = \frac{\partial U}{\partial F_1} \quad \delta_{AV} = \frac{\partial U}{\partial F_1} = \int_0^s \frac{M}{E \cdot I} \cdot \frac{\partial M}{\partial F_1} \cdot ds \quad U = U_I + U_{II}$$

Utjecaje uzdužne i poprečne sile na ukupnu potencijalnu energiju deformacije smo zanemarili.

$$\text{Dio I:} \quad M_{\varphi_1} = -\frac{F_1}{2} \cdot r \cdot \sin \varphi_1 + X \cdot r \cdot (1 - \cos \varphi_1) \quad \frac{\partial M_{\varphi_1}}{\partial F_1} = -\frac{1}{2} \cdot r \cdot \sin \varphi_1$$

$$\text{Dio II:} \quad M_{\varphi_2} = -\frac{F_1}{2} \cdot r \cdot \cos \varphi_2 + X \cdot r \cdot (1 + \sin \varphi_2) - F_2 \cdot r \cdot \sin \varphi_2 \quad \frac{\partial M_{\varphi_2}}{\partial F_1} = -\frac{1}{2} \cdot r \cdot \cos \varphi_2$$

$$\delta_{AV} = \frac{\partial U}{\partial F_1} = \frac{1}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} M_{\varphi_1} \cdot \frac{\partial M_{\varphi_1}}{\partial F_1} \cdot r \cdot d\varphi_1 + \frac{1}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} M_{\varphi_2} \cdot \frac{\partial M_{\varphi_2}}{\partial F_1} \cdot r \cdot d\varphi_2 \quad r \cdot d\varphi = ds$$

$$\delta_{AV} = \frac{r^3}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot \sin \varphi_1 + X \cdot (1 - \cos \varphi_1) \right] \cdot \left(-\frac{1}{2} \cdot \sin \varphi_1 \right) + \frac{r^3}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot \cos \varphi_2 + X \cdot (1 + \sin \varphi_2) - F_2 \cdot \sin \varphi_2 \right] \cdot \left(-\frac{1}{2} \cdot \cos \varphi_2 \right)$$

$$\delta_{AV} = \frac{r^3}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} \left[\frac{F_1}{4} \sin^2 \varphi_1 - \frac{X}{2} \cdot (\sin \varphi_1 - \sin \varphi_1 \cdot \cos \varphi_1) \right] \cdot d\varphi_1 + \frac{r^3}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} \left[\frac{F_1}{4} \cos^2 \varphi_2 - \frac{X}{2} \cdot (\cos \varphi_2 + \sin \varphi_2 \cdot \cos \varphi_2) + \frac{F_2}{2} \cdot \sin \varphi_2 \cdot \cos \varphi_2 \right] \cdot d\varphi_2$$

$$\delta_{AV} = \frac{r^3}{E \cdot I} \cdot \left[\frac{F_1}{4} \cdot \frac{\pi}{4} - \frac{X}{2} \cdot \left(1 - \frac{1}{2} \right) \right] + \frac{r^3}{E \cdot I} \cdot \left[\frac{F_1}{4} \cdot \frac{\pi}{4} - \frac{X}{2} \cdot \left(1 + \frac{1}{2} \right) + \frac{F_2}{2} \cdot \frac{1}{2} \right]$$

$$\delta_{AV} = \frac{r^3}{E \cdot I} \cdot [3926,99 - 2956,0] + \frac{r^3}{E \cdot I} \cdot (3926,99 - 8868,0 + 5000,0) = \frac{r^3}{E \cdot I} \cdot 1029,98 \left(\frac{\text{mm}^3}{\frac{\text{N}}{\text{mm}^2} \cdot \text{mm}^4} \cdot \text{N} = \text{mm} \right)$$

$$\delta_{AV} = \frac{r^3}{E \cdot I} \cdot 1029,98 = \frac{1000^3}{2,1 \cdot 10^5 \cdot 2,0 \cdot 10^6} \cdot 1029,98 = \underline{+2,452 \text{ mm}}$$