POTENCIJALNA ENERGIJA DEFORMACIJE

ZADATAK 8.

Za prikazani sistem treba primjenom drugog Castiglianova teorema nacrtati dijagrame momenata savijanja, poprečnih i uzdužnih sila.

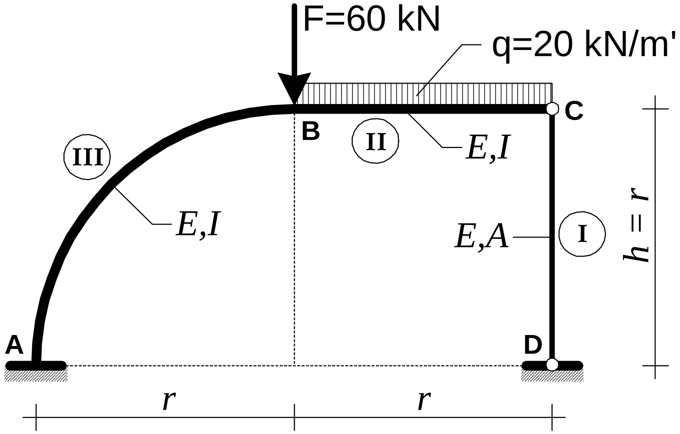
Zadano je:

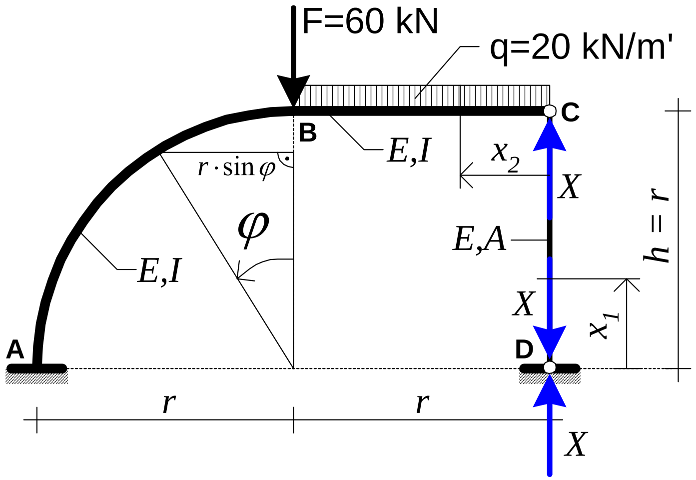
$E = 2{,}1 \cdot 10^5 \;\; MPa$

$I = 3{,}0 \cdot 10^8 \;\; mm^4$

$A = 600 \;\; mm^4$

$r = 2{,}0 \;\; m\,.$





Sistem je jednaput statički neodređen.

Kao prekobrojna veličina uzeta je tlačna sila X u štapu **CD**, koju ćemo odrediti iz uvjeta da je pomak na mjestu i u smjeru te sile jednak nuli (točka **D** je ležaj – nema pomaka). Prema drugom Castiglianovom teoremu parcijalna derivacija potencijalne energije deformacije po sili jednaka je pomaku na mjestu i u smjeru te sile:

$$\delta_{DV} = \frac{\partial U}{\partial X} = 0 \qquad \delta_{DV} = \frac{\partial U}{\partial X} = \int_0^r \frac{N_{x_1}}{E \cdot A} \cdot \frac{\partial N_{x_1}}{\partial X} \cdot dx_1 + \int_0^r \frac{M_{x_2}}{E \cdot I} \cdot \frac{\partial M_{x_2}}{\partial X} \cdot dx_2 + \int_0^{\frac{\pi}{2}} \frac{M_{\varphi}}{E \cdot I} \cdot \frac{\partial M_{\varphi}}{\partial X} \cdot r \cdot d\varphi = 0 \qquad r \cdot d\varphi = ds \qquad U = U_I + U_{II} + U_{III}$$

Na dijelu **I** (štap **CD**) postoji samo uzdužna sila, a na dijelovima **II** i **III** zanemarujemo utjecaje poprečne i uzdužne sile na ukupnu potencijalnu energiju deformacije, pa ostaju samo momenti savijanja.

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Dio **I**: $N_{x_1} = X$ $\qquad \dfrac{\partial N_{x_1}}{\partial X} = 1$

Dio **II**: $M_{x_2} = X \cdot x_2 - q \cdot \dfrac{x_2^2}{2}$ $\qquad \dfrac{\partial M_{x_2}}{\partial X} = x_2$

Dio **III**: $M_{\varphi} = X \cdot r \cdot (1+\sin\varphi) - q \cdot r^2 \cdot \left(\dfrac{1}{2} + \sin\varphi\right) - F \cdot r \cdot \sin\varphi$ $\qquad \dfrac{\partial M_{\varphi}}{\partial X} = r \cdot (1+\sin\varphi)$

$$\delta_{DV} = \frac{1}{E \cdot A} \cdot \int_0^r X \cdot 1 \cdot dx_1 + \frac{1}{E \cdot I} \int_0^r \left(X \cdot x_2 - q \cdot \frac{x_2^2}{2} \right) \cdot x_2 \cdot dx_2 + \frac{1}{E \cdot I} \int_0^{\frac{\pi}{2}} \left[X \cdot r \cdot (1+\sin\varphi) - q \cdot r^2 \cdot \left(\frac{1}{2} + \sin\varphi \right) - F \cdot r \cdot \sin\varphi \right] \cdot [r \cdot (1+\sin\varphi)] \cdot r \cdot d\varphi = 0$$

$$\delta_{DV} = \frac{X \cdot r}{E \cdot A} + \frac{1}{E \cdot I} \int_0^r \left(X \cdot x_2^2 - q \cdot \frac{x_2^3}{2} \right) \cdot dx_2 + \frac{1}{E \cdot I} \int_0^{\frac{\pi}{2}} \left[X \cdot r^3 \cdot (1 + 2 \cdot \sin\varphi + \sin^2\varphi) - q \cdot r^4 \cdot \left(\frac{1}{2} + \frac{1}{2} \cdot \sin\varphi + \sin\varphi + \sin^2\varphi \right) - F \cdot r^3 \cdot (\sin\varphi + \sin^2\varphi) \right] \cdot d\varphi = 0 \quad (*)$$

$$\frac{X \cdot r}{E \cdot A} + \frac{1}{E \cdot I} \cdot \left| X \cdot \frac{x_2^3}{3} - q \frac{x_2^4}{8} \right|_0^r + \frac{1}{E \cdot I} \cdot X \cdot r^3 \cdot \left(\frac{\pi}{2} + 2 \cdot 1 + \frac{\pi}{4} \right) - \frac{1}{E \cdot I} \cdot q \cdot r^4 \cdot \left(\frac{1}{2} \cdot \frac{\pi}{2} + \frac{1}{2} \cdot 1 + 1 + \frac{\pi}{4} \right) - \frac{1}{E \cdot I} \cdot F \cdot r^3 \cdot \left(1 + \frac{\pi}{4} \right) = 0$$

$$X \cdot \left[\frac{r}{E \cdot A} + \frac{1}{E \cdot I} \cdot \frac{r^3}{3} + \frac{1}{E \cdot I} \cdot r^3 \cdot \left(\frac{\pi}{2} + 2 + \frac{\pi}{4} \right) \right] = \frac{1}{E \cdot I} \cdot q \cdot \frac{r^4}{8} + \frac{1}{E \cdot I} \cdot q \cdot r^4 \cdot \left(\frac{\pi}{4} + \frac{3}{2} + \frac{\pi}{4} \right) + \frac{1}{E \cdot I} \cdot r^3 \cdot F \cdot \left(1 + \frac{\pi}{4} \right) \quad \Big| \cdot \frac{E \cdot I}{r^3}$$

$$X \cdot \left[\frac{I}{A \cdot r^2} + \frac{1}{3} + \left(\frac{\pi}{2} + 2 + \frac{\pi}{4} \right) \right] = q \cdot \frac{r}{8} + q \cdot r \left(\frac{\pi}{4} + \frac{3}{2} + \frac{\pi}{4} \right) + F \cdot \left(1 + \frac{\pi}{4} \right)$$

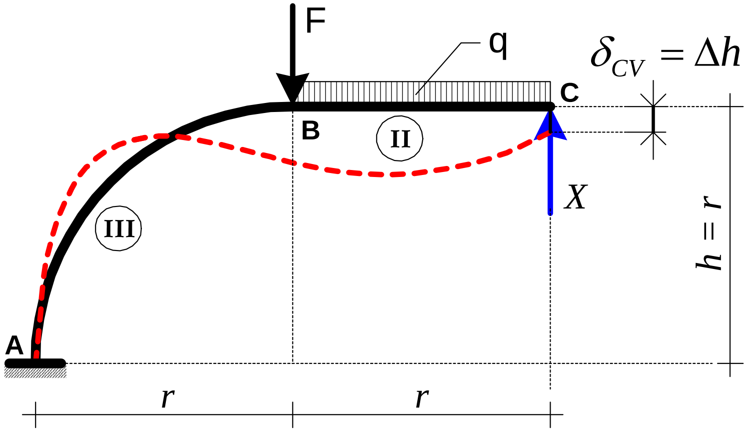
$$X \cdot \left[\frac{3 \cdot 10^8}{600 \cdot 2000^2} + \frac{1}{3} + 4{,}3562 \right] = 20 \cdot \frac{2000}{8} + 20 \cdot 2000 \cdot 3{,}0708 + 60000 \cdot 1{,}7854$$

$$X \cdot (0{,}125 + 0{,}3333 + 4{,}3562) = 5000 + 122832 + 107124$$

$$X \cdot 4{,}8145 = 234956 \qquad \Rightarrow \qquad X = 48801{,}74 \ N = \underline{48{,}80 \ kN}$$

Do rješenja se može doći i na ovaj način razmišljanja:

Kao prekobrojna veličina uzeta je tlačna sila X u štapu **CD**, koju ćemo odrediti iz uvjeta da je pomak točke **C** na mjestu i u smjeru te sile jednak skraćenju štapa **CD**.



$$\delta_{CV} = \frac{\partial U}{\partial X} = -\Delta h = -\frac{X \cdot r}{E \cdot A}$$

$$\delta_{CV} = \frac{\partial U}{\partial X} = \int_0^r \frac{M_{x_2}}{E \cdot I} \cdot \frac{\partial M_{x_2}}{\partial X} \cdot dx_2 + \int_0^{\frac{\pi}{2}} \frac{M_\varphi}{E \cdot I} \cdot \frac{\partial M_\varphi}{\partial X} \cdot r \cdot d\varphi = -\frac{X \cdot r}{E \cdot A}$$

$$r \cdot d\varphi = ds \qquad U = U_{II} + U_{III}$$

Predznak minus na desnoj strani je zato što je pomak točke **C** suprotan sili X.

Kad je određena prekobrojna veličina sile X, izračunat ćemo veličine momenata savijanja, poprečnih i uzdužnih sila na karakterističnim mjestima i nacrtati dijagrame.

<u>Momenti savijanja</u>

$$M_C = 0 \qquad M_{x_2 = \frac{r}{2}} = X \cdot \frac{r}{2} - q \cdot \frac{\left(\frac{r}{2}\right)^2}{2} = +38{,}80 \ kNm \qquad M_B = X \cdot r - q \cdot \frac{r^2}{2} = +57{,}60 \ kNm$$

$$M_{\varphi = \frac{\pi}{4}} = X \cdot r \cdot \left(1 + \sin\frac{\pi}{4}\right) - q \cdot r^2 \cdot \left(\frac{1}{2} + \sin\frac{\pi}{4}\right) - F \cdot r \cdot \sin\frac{\pi}{4} = 166{,}61 - 96{,}57 - 84{,}85 = -14{,}81 \ kNm$$

$$M_{\varphi = \frac{\pi}{2}} = M_A = X \cdot 2r - q \cdot r \cdot \left(\frac{r}{2} + r\right) - F \cdot r = -44{,}80 \ kNm$$

Nul točka dijagrama momenata savijanja:

$$M_{\varphi_0} = X \cdot r \cdot (1 + \sin\varphi) - q \cdot r^2 \cdot \left(\frac{1}{2} + \sin\varphi\right) - F \cdot r \cdot \sin\varphi = 0 \quad \Rightarrow \quad \sin\varphi_o = 0{,}5625 \quad \Rightarrow \quad \varphi_0 = 34{,}23^o$$

<u>Poprečne sile</u>

$T_C^{lijevo} = -X = -48{,}80\ kN$

$T_B^{desno} = -X + q \cdot r = -8{,}80\ kN$

$T_B^{lijevo} = -X + q \cdot r + F = +51{,}20\ kN$

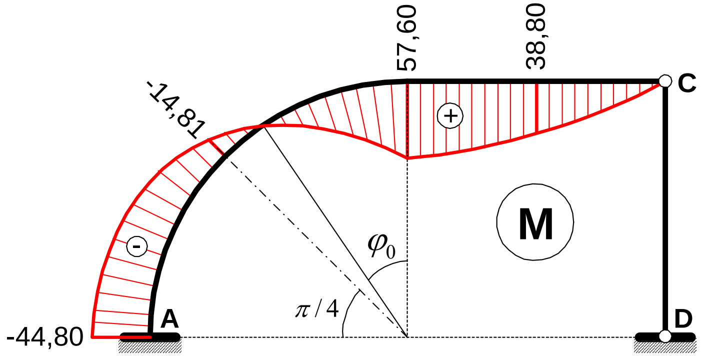
$T_A = 0$

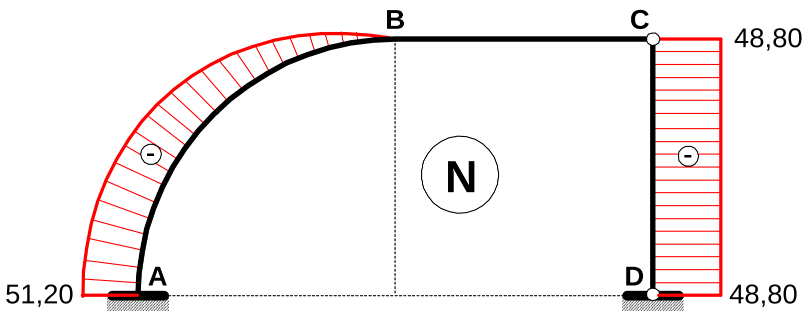
<u>Uzdužne sile</u>

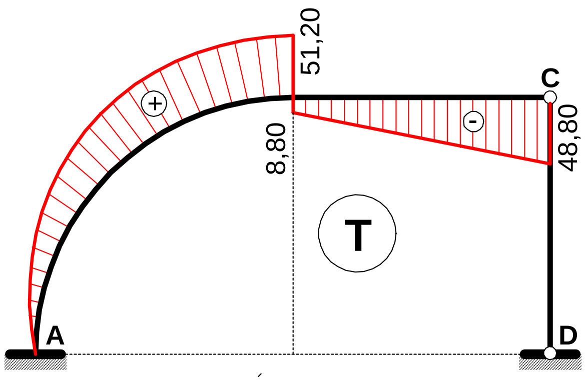
$N_{CD} = -X = -48{,}80\ kN$

$N_C = N_B = 0$

$N_A = R_{AV} = +X - q \cdot r - F = -51{,}20\ kN$







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Pri rješavanju sličnih zadataka, na zakrivljenim dijelovima sistema s radijusom zakrivljenosti r, mogu se pojaviti integrali koji sadrže funkcije $d\varphi$, $\sin\varphi$, $\cos\varphi$, $\sin\varphi\cdot\cos\varphi$, $\sin^2\varphi$ i $\cos^2\varphi$.

Rješenja tih integrala u granicama od 0 do $\frac{\pi}{2}$ i od 0 do π su:

$$\int_0^{\frac{\pi}{2}} d\varphi = +\frac{\pi}{2} \qquad \int_0^{\pi} d\varphi = +\pi$$

$$\int_0^{\frac{\pi}{2}} \sin\varphi\cdot d\varphi = +1 \qquad \int_0^{\pi} \sin\varphi\cdot d\varphi = +2$$

$$\int_0^{\frac{\pi}{2}} \cos\varphi\cdot d\varphi = 1+ \qquad \int_0^{\pi} \cos\varphi\cdot d\varphi = 0$$

$$\int_0^{\frac{\pi}{2}} \sin\varphi\cdot\cos\varphi\cdot d\varphi = +\frac{1}{2} \qquad \int_0^{\pi} \sin\varphi\cdot\cos\varphi\cdot d\varphi = 0$$

$$\int_0^{\frac{\pi}{2}} \sin^2\varphi\cdot d\varphi = +\frac{\pi}{4} \qquad \int_0^{\pi} \sin^2\varphi\cdot d\varphi = +\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^2\varphi\cdot d\varphi = +\frac{\pi}{4} \qquad \int_0^{\pi} \cos^2\varphi\cdot d\varphi = +\frac{\pi}{2}$$

Ova rješenja su korištena i pri rješavanju integrala u izrazu (*) na 2. stranici ovog zadatka.