POTENCIJALNA ENERGIJA DEFORMACIJE

ZADATAK 9.

Za zatvoreni kružni prsten treba primjenom drugog Castiglianova teorema izračunati prekobrojnu veličinu, nacrtati dijagrame momenata savijanja, poprečnih i uzdužnih sila i odrediti pomak točke **A**.

Zadano je:

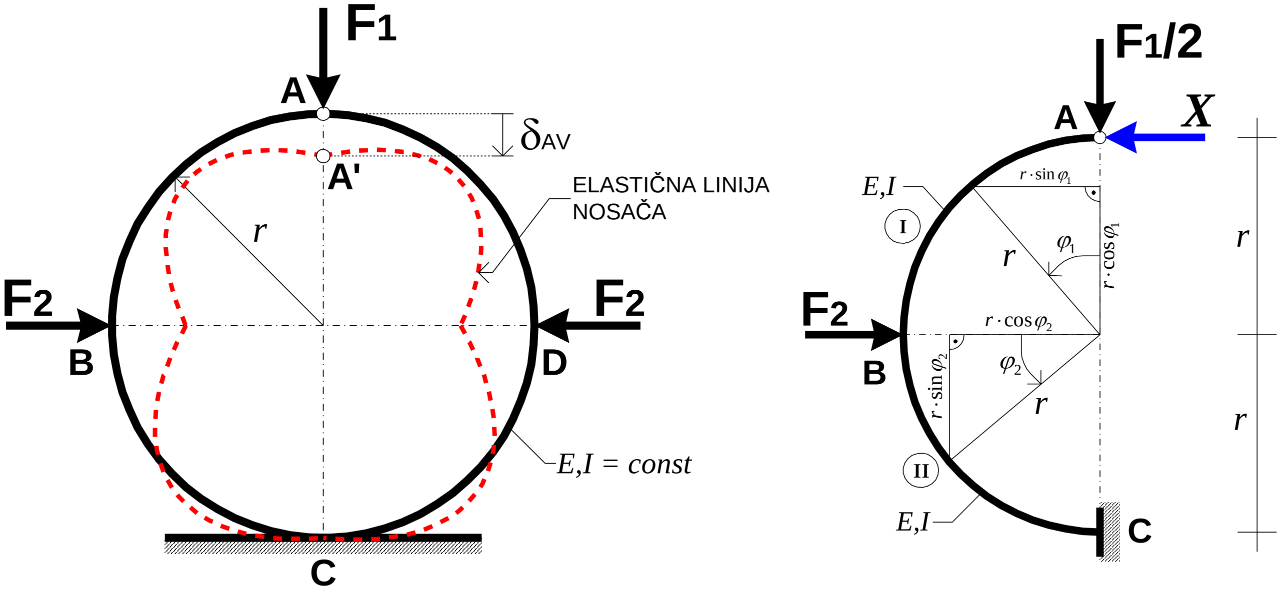
$E = 2{,}1 \cdot 10^5 \ MPa$

$I = 2{,}0 \cdot 10^6 \ mm^4$

$r = 1{,}0 \ m$

$F_1 = 20 \ kN$

$F_2 = F_1$



Sistem je jedanput statički neodređen $(s = 3 \cdot n - m = 3 \cdot n - 2 \cdot z_1 = 3 \cdot 1 - 2 \cdot 1 = 1)$ i ima jednu (vertikalnu) os simetrije.

Za odabir osnovnog sistema iskorištavamo os simetrije, promatramo samo lijevu polovicu prstena, a na osi simetrije zadovoljavamo rubne uvjete.

Zbog simetrije presjek **C** se ne zaokreće tijekom deformiranja prstena (vidi oblik elastične linije), pa lijevu polovicu prstena možemo razmatrati kao da je **upeta u presjeku C**.

Na drugoj strani simetrije u presjeku **A**, poprečna sila jednaka je polovici sile F_1, moment je jednak nuli zbog zgloba, pa kao prekobrojna veličina ostaje samo uzdužna sila X. Na toj strani zadovoljavamo rubni uvjet: **zbog osi simetrije horizontalni pomak točke A je jednak nuli**, pa prekobrojnu veličinu, uzdužnu silu X, određujemo iz uvjeta da je pomak na mjestu i u smjeru te sile jednak nuli.

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Prema drugom Castiglianovom teoremu parcijalna derivacija potencijalne energije deformacije po sili jednaka je pomaku na mjestu i u smjeru te sile:

$$\delta_{AH} = \frac{\partial U}{\partial X} = 0 \qquad \delta_{AH} = \frac{\partial U}{\partial X} = \int_0^s \frac{M}{E \cdot I} \cdot \frac{\partial M}{\partial X} \cdot ds = 0 \qquad U = U_I + U_{II}$$

Utjecaje uzdužne i poprečne sile na ukupnu potencijalnu energiju deformacije smo zanemarili.

Dio **I**: $M_{\varphi 1} = -\frac{F_1}{2} \cdot r \cdot \sin\varphi_1 + X \cdot r \cdot (1 - \cos\varphi_1) \qquad \frac{\partial M_{\varphi_1}}{\partial X} = +r \cdot (1 - \cos\varphi_1)$

Dio **II**: $M_{\varphi 2} = -\frac{F_1}{2} \cdot r \cdot \cos\varphi_2 + X \cdot r \cdot (1 + \sin\varphi_2) - F_2 \cdot r \cdot \sin\varphi_2 \qquad \frac{\partial M_{\varphi_2}}{\partial X} = +r \cdot (1 + \sin\varphi_2)$

$$\delta_{AH} = \frac{\partial U}{\partial X} = \frac{1}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} M_{\varphi_1} \cdot \frac{\partial M_{\varphi_1}}{\partial X} \cdot r \cdot d\varphi_1 + \frac{1}{E \cdot I} \cdot \int_0^{\frac{\pi}{2}} M_{\varphi_2} \cdot \frac{\partial M_{\varphi_2}}{\partial X} \cdot r \cdot d\varphi_2 = 0 \;\Big|\cdot (E \cdot I) \qquad r \cdot d\varphi = ds$$

$$\int_0^{\frac{\pi}{2}} M_{\varphi_1} \cdot \frac{\partial M_{\varphi_1}}{\partial X} \cdot r \cdot d\varphi_1 + \int_0^{\frac{\pi}{2}} M_{\varphi_2} \cdot \frac{\partial M_{\varphi_2}}{\partial X} \cdot r \cdot d\varphi_2 = 0$$

$$\int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot r \cdot \sin\varphi_1 + X \cdot r \cdot (1 - \cos\varphi_1) \right] \cdot [r \cdot (1 - \cos\varphi_1)] \cdot r \cdot d\varphi_1 + \int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot r \cdot \cos\varphi_2 + X \cdot r \cdot (1 + \sin\varphi_2) - F_2 \cdot r \cdot \sin\varphi_2 \right] \cdot [r \cdot (1 + \sin\varphi_2)] \cdot r \cdot d\varphi_2 = 0 \;\Big|: r^3$$

$$\int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot (\sin\varphi_1 - \sin\varphi_1 \cdot \cos\varphi_1) + X \cdot (1 - 2 \cdot \cos\varphi_1 + \cos^2\varphi_1) \right] d\varphi_1 + \int_0^{\frac{\pi}{2}} \left[-\frac{F_1}{2} \cdot (\cos\varphi_2 + \sin\varphi_2 \cdot \cos\varphi_2) + X \cdot (1 + 2 \cdot \sin\varphi_2 + \sin^2\varphi_2) - F_2 \cdot (\sin\varphi_2 + \sin^2\varphi_2) \right] \cdot d\varphi_2 = 0$$

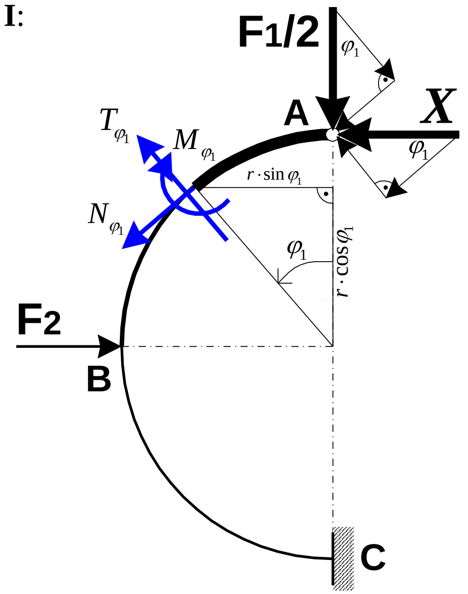
$$X \cdot \left(\frac{\pi}{2} - 2 \cdot 1 + \frac{\pi}{4} \right) + X \cdot \left(\frac{\pi}{2} + 2 \cdot 1 + \frac{\pi}{4} \right) = \frac{F_1}{2} \cdot \left(1 - \frac{1}{2} \right) + \frac{F_1}{2} \cdot \left(1 + \frac{1}{2} \right) + F_2 \cdot \left(1 + \frac{\pi}{4} \right) \qquad F_2 = F_1 = F$$

$$X \cdot (0{,}356194 + 4{,}356194) = F \cdot (0{,}25 + 0{,}75 + 1{,}785398)$$

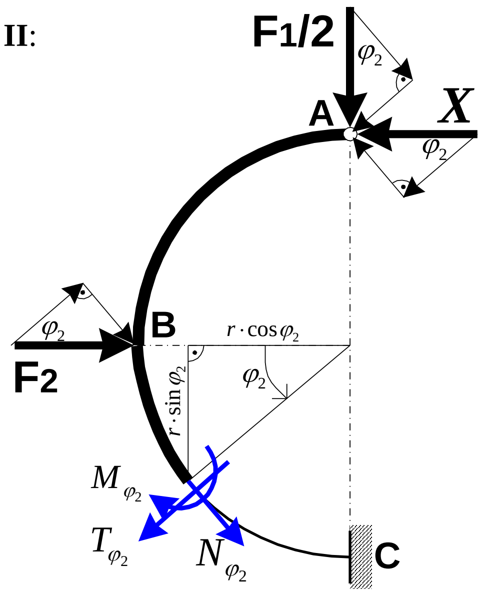
$$X \cdot 4{,}712388 = F \cdot 2{,}785398 \qquad \Rightarrow \qquad X = 0{,}59108 \cdot F = \underline{11{,}824 \;\; kN}$$

Kad je određena prekobrojna veličina uzdužne sile $X = 11{,}824\ kN$, izračunat ćemo veličine momenata savijanja, poprečnih i uzdužnih sila na karakterističnim mjestima i nacrtati dijagrame.

Dio **I**:



Dio **II**:



$$\sum M = 0 \quad \Rightarrow \quad M_{\varphi_1} = -\frac{F_1}{2} \cdot r \cdot \sin\varphi_1 + X \cdot r \cdot (1 - \cos\varphi_1)$$

$$\sum F_t = 0 \quad \Rightarrow \quad T_{\varphi_1} = +\frac{F_1}{2} \cdot \cos\varphi_1 - X \cdot \sin\varphi_1$$

$$\sum F_n = 0 \quad \Rightarrow \quad N_{\varphi_1} = -\frac{F_1}{2} \cdot \sin\varphi_1 - X \cdot \cos\varphi_1$$

$$M_{\varphi_2} = -\frac{F_1}{2} \cdot r \cdot \cos\varphi_2 + X \cdot r \cdot (1 + \sin\varphi_2) - F_2 \cdot r \cdot \sin\varphi_2$$

$$T_{\varphi_2} = -\frac{F_1}{2} \cdot \sin\varphi_2 - X \cdot \cos\varphi_2 + F_2 \cdot \cos\varphi_2$$

$$N_{\varphi_2} = -\frac{F_1}{2} \cdot \cos\varphi_2 + X \cdot \sin\varphi_2 - F_2 \cdot \sin\varphi_2$$

Momenti savijanja

Dio **I**: $M_A = 0$

$$M_{\varphi_1=\frac{\pi}{4}} = -\frac{F_1}{2}\cdot r\cdot\sin\frac{\pi}{4} + X\cdot r\cdot\left(1-\cos\frac{\pi}{4}\right) = -3{,}61\ kNm$$

$$M_{\varphi_1=\frac{\pi}{2}} = M_B = -\frac{F_1}{2}\cdot r + X\cdot r = -10+11{,}824 = +1{,}824\ kNm$$

Dio **II**:

$$M_{\varphi_2=\frac{\pi}{4}} = -\frac{F_1}{2}\cdot r\cdot\cos\frac{\pi}{4} + X\cdot r\cdot\left(1+\sin\frac{\pi}{4}\right) - F_2\cdot r\cdot\sin\frac{\pi}{4} = -1{,}028\ kNm$$

$$M_{\varphi_2=\frac{\pi}{2}} = M_C = +X\cdot 2\cdot r - F_2\cdot r = +23{,}648-20 = +3{,}648\ kNm$$

Poprečne sile

Dio **I**:

$$T_A^{lijevo} = +\frac{F_1}{2} = +10\ kN$$

$$T_{\varphi_1=\frac{\pi}{4}} = +\frac{F_1}{2}\cdot\cos\frac{\pi}{4} - X\cdot\sin\frac{\pi}{4} = -1{,}30\ kN$$

$$T_{\varphi_1=\frac{\pi}{2}} = T_B^{gore} = -X = -11{,}824\ kN$$

Dio **II**:

$$T_B^{dolje} = -X + F = +8{,}176\ kN$$

$$T_{\varphi_2=\frac{\pi}{4}} = -\frac{F_1}{2}\cdot\sin\frac{\pi}{4} - X\cdot\cos\frac{\pi}{4} + F_2\cdot\cos\frac{\pi}{4} = -1{,}30\ kN$$

$$T_{\varphi_2=\frac{\pi}{2}} = T_C^{lijevo} = -\frac{F_1}{2} = -10\ kN$$

Uzdužne sile

$$N_A = -X = -11{,}824\ kN$$

$$N_{\varphi_1=\frac{\pi}{4}} = -\frac{F_1}{2}\cdot\sin\frac{\pi}{4} - X\cdot\cos\frac{\pi}{4} = -15{,}43\ kN$$

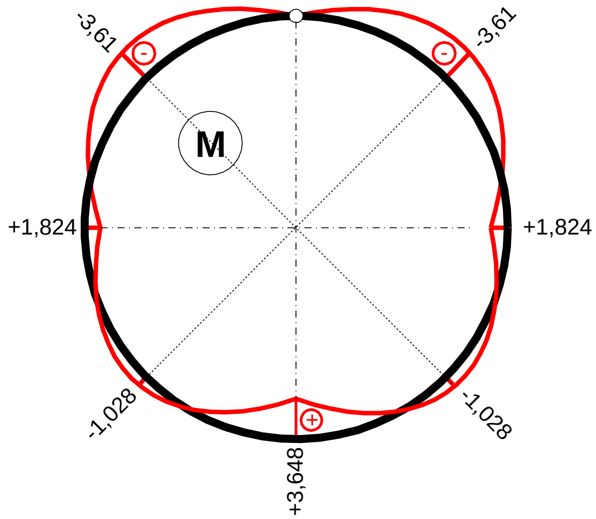
$$N_{\varphi_1=\frac{\pi}{2}} = N_B^{gore} = -\frac{F_1}{2} = -10\ kN$$

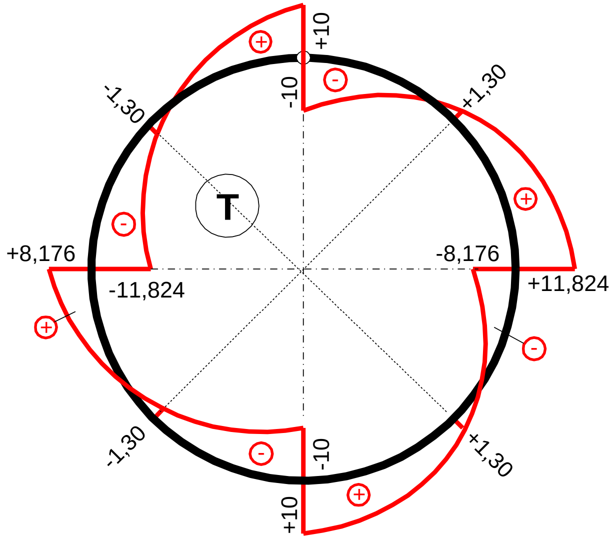
$$N_B^{dolje} = -\frac{F_1}{2} = -10\ kN$$

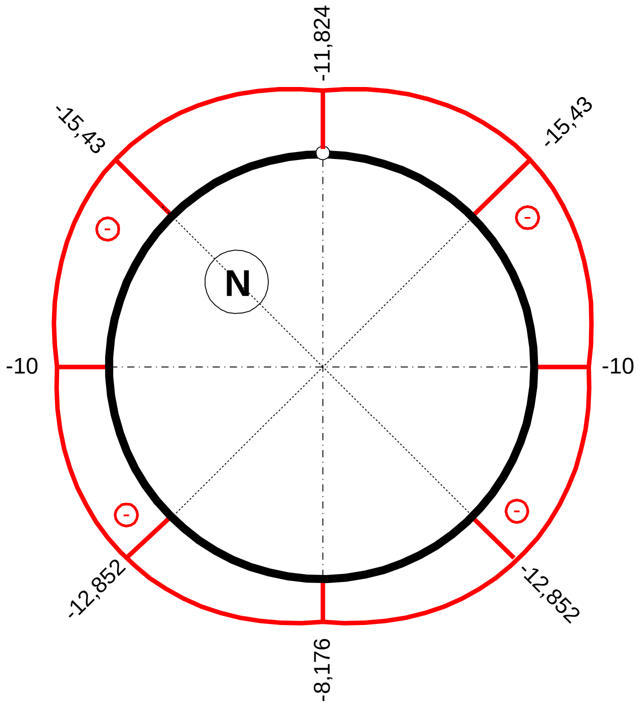
$$N_{\varphi_2=\frac{\pi}{4}} = -\frac{F_1}{2}\cdot\cos\frac{\pi}{4} + X\cdot\sin\frac{\pi}{4} - F_2\cdot\sin\frac{\pi}{4} = -12{,}852\ kN$$

$$N_{\varphi_2=\frac{\pi}{2}} = N_C^{lijevo} = +X - F_2 = -8{,}176\ kN$$

<u>DIJAGRAMI UNUTARNJIH SILA</u>







POMAK TOČKE **A**

Prema drugom Castiglianovom teoremu parcijalna derivacija potencijalne energije deformacije po sili jednaka je pomaku na mjestu i u smjeru te sile:

$$\delta_{AV}=\frac{\partial U}{\partial F_1} \qquad \delta_{AV}=\frac{\partial U}{\partial F_1}=\int_0^s \frac{M}{E\cdot I}\cdot\frac{\partial M}{\partial F_1}\cdot ds \qquad U=2\cdot(U_I+U_{II})$$

Utjecaje uzdužne i poprečne sile na ukupnu potencijalnu energiju deformacije smo zanemarili.

Dio **I**: $$M_{\varphi_1}=-\frac{F_1}{2}\cdot r\cdot\sin\varphi_1+X\cdot r\cdot(1-\cos\varphi_1) \qquad \frac{\partial M_{\varphi_1}}{\partial F_1}=-\frac{1}{2}\cdot r\cdot\sin\varphi_1$$

Dio **II**: $$M_{\varphi_2}=-\frac{F_1}{2}\cdot r\cdot\cos\varphi_2+X\cdot r\cdot(1+\sin\varphi_2)-F_2\cdot r\cdot\sin\varphi_2 \qquad \frac{\partial M_{\varphi_2}}{\partial F_1}=-\frac{1}{2}\cdot r\cdot\cos\varphi_2$$

$$\delta_{AV}=\frac{\partial U}{\partial F_1}=\frac{2}{E\cdot I}\cdot\int_0^{\frac{\pi}{2}} M_{\varphi_1}\cdot\frac{\partial M_{\varphi_1}}{\partial F_1}\cdot r\cdot d\varphi_1+\frac{2}{E\cdot I}\cdot\int_0^{\frac{\pi}{2}} M_{\varphi_2}\cdot\frac{\partial M_{\varphi_2}}{\partial F_1}\cdot r\cdot d\varphi_2 \qquad r\cdot d\varphi=ds$$

$$\delta_{AV}=\frac{2\cdot r^3}{E\cdot I}\cdot\int_0^{\frac{\pi}{2}}\left[-\frac{F_1}{2}\cdot\sin\varphi_1+X\cdot(1-\cos\varphi_1)\right]\cdot\left(-\frac{1}{2}\cdot\sin\varphi_1\right)+\frac{2\cdot r^3}{E\cdot I}\cdot\int_0^{\frac{\pi}{2}}\left[-\frac{F_1}{2}\cdot\cos\varphi_2+X\cdot(1+\sin\varphi_2)-F_2\cdot\sin\varphi_2\right]\cdot\left(-\frac{1}{2}\cdot\cos\varphi_2\right)$$

$$\delta_{AV}=\frac{2\cdot r^3}{E\cdot I}\cdot\int_0^{\frac{\pi}{2}}\left[\frac{F_1}{4}\sin^2\varphi_1-\frac{X}{2}\cdot(\sin\varphi_1-\sin\varphi_1\cdot\cos\varphi_1)\right]\cdot d\varphi_1+\frac{2\cdot r^3}{E\cdot I}\cdot\int_0^{\frac{\pi}{2}}\left[\frac{F_1}{4}\cos^2\varphi_2-\frac{X}{2}\cdot(\cos\varphi_2+\sin\varphi_2\cdot\cos\varphi_2)+\frac{F_2}{2}\cdot\sin\varphi_2\cdot\cos\varphi_2\right]\cdot d\varphi_2$$

$$\delta_{AV}=\frac{2\cdot r^3}{E\cdot I}\cdot\left[\frac{F_1}{4}\cdot\frac{\pi}{4}-\frac{X}{2}\cdot\left(1-\frac{1}{2}\right)\right]+\frac{2\cdot r^3}{E\cdot I}\cdot\left(\frac{F_1}{4}\cdot\frac{\pi}{4}-\frac{X}{2}\cdot\left(1+\frac{1}{2}\right)+\frac{F_2}{2}\cdot\frac{1}{2}\right)$$

$$\delta_{AV}=\frac{2\cdot r^3}{E\cdot I}\cdot[3926{,}99-2956{,}0]+\frac{2\cdot r^3}{E\cdot I}\cdot(3926{,}99-8868{,}0+5000{,}0)=\frac{2\cdot r^3}{E\cdot I}\cdot 1029{,}98 \quad \left(\frac{mm^3}{\frac{N}{mm^2}\cdot mm^4}\cdot N=mm\right)$$

$$\delta_{AV}=\frac{2\cdot r^3}{E\cdot I}\cdot 1029{,}98=\frac{2\cdot 1000^3}{2{,}1\cdot 10^5\cdot 2{,}0\cdot 10^6}\cdot 1029{,}98=\underline{+4{,}904\ mm}$$