Chapter XXII

Geometric Nonlinearity

SAP2000 is capable of considering geometric nonlinearity in the form of either P-delta effects or large-displacement/rotation effects. Strains within the elements are assumed to be small. Geometric nonlinearity can be considered on a step-by-step basis in nonlinear static and direct-integration time-history analysis, and incorporated in the stiffness matrix for linear analyses.

Advanced Topics

- Overview
- Nonlinear Load Cases
- The P-Delta Effect
- Initial P-Delta Analysis
- Large Displacements

Overview

When the load acting on a structure and the resulting deflections are small enough, the load-deflection relationship for the structure is linear. For the most part, SAP2000 analyses assume such linear behavior. This permits the program to form the equilibrium equations using the original (undeformed) geometry of the structure. Strictly speaking, the equilibrium equations should actually refer to the geometry of the structure after deformation.

The linear equilibrium equations are independent of the applied load and the resulting deflection. Thus the results of different static and/or dynamic loads can be superposed (scaled and added), resulting in great computational efficiency.

If the load on the structure and/or the resulting deflections are large, then the load-deflection behavior may become nonlinear. Several causes of this nonlinear behavior can be identified:

- **P-delta (large-stress) effect:** when large stresses (or forces and moments) are present within a structure, equilibrium equations written for the original and the deformed geometries may differ significantly, even if the deformations are very small.
- **Large-displacement effect:** when a structure undergoes large deformation (in particular, large strains and rotations), the usual engineering stress and strain measures no longer apply, and the equilibrium equations must be written for the deformed geometry. This is true even if the stresses are small.
- **Material nonlinearity:** when a material is strained beyond its proportional limit, the stress-strain relationship is no longer linear. Plastic materials strained beyond the yield point may exhibit history-dependent behavior. Material nonlinearity may affect the load-deflection behavior of a structure even when the equilibrium equations for the original geometry are still valid.
- **Other effects:** Other sources of nonlinearity are also possible, including nonlinear loads, boundary conditions and constraints.

The large-stress and large-displacement effects are both termed geometric (or kinematic) nonlinearity, as distinguished from material nonlinearity. Kinematic nonlinearity may also be referred to as second-order geometric effects.

This Chapter deals with the geometric nonlinearity effects that can be analyzed using SAP2000. For each nonlinear static and nonlinear direct-integration time-history analysis, you may choose to consider:

- No geometric nonlinearity
- P-delta effects only
- Large displacement and P-delta effects

The large displacement effect in SAP2000 includes only the effects of large translations and rotations. The strains are assumed to be small in all elements.

Material nonlinearity is discussed in Chapters “The Frame Element” (page 81), “Frame Hinge Properties” (page 121), and “The Link/Support Element—Basic” (page 213). Since small strains are assumed, material nonlinearity and geometric nonlinearity effects are independent.

Once a nonlinear analysis has been performed, its final stiffness matrix can be used for subsequent linear analyses. Any geometric nonlinearity considered in the nonlinear analysis will affect the linear results. In particular, this can be used to include relatively constant P-delta effects in buildings or the tension-stiffening effects in cable structures into a series of superposable linear analyses.

For more information:

- See Chapter “Load Cases” (page 289)
- See Chapter “Nonlinear Static Analysis” (page 359)
- See Chapter “Nonlinear Time-History Analysis” (page 381)

Nonlinear Load Cases

For nonlinear static and nonlinear direct-integration time-history analysis, you may choose the type of geometric nonlinearity to consider:

- **None:** All equilibrium equations are considered in the undeformed configuration of the structure
- **P-delta only:** The equilibrium equations take into partial account the deformed configuration of the structure. Tensile forces tend to resist the rotation of elements and stiffen the structure, and compressive forces tend to enhance the rotation of elements and destabilize the structure. This may require a moderate amount of iteration.
- **Large displacements:** All equilibrium equations are written in the deformed configuration of the structure. This may require a large amount of iteration; Newton-Raphson iterations are usually most effective. Although large displacement and large rotation effects are modeled, all strains are assumed to be small. P-delta effects are included.

When continuing one nonlinear Load Case from another, it is recommended that they both have the same geometric-nonlinearity settings.

The large displacement option should be used for any structures undergoing significant deformation; and for buckling analysis, particularly for snap-through buckling and post-buckling behavior. Cables (modeled by frame elements) and other el-

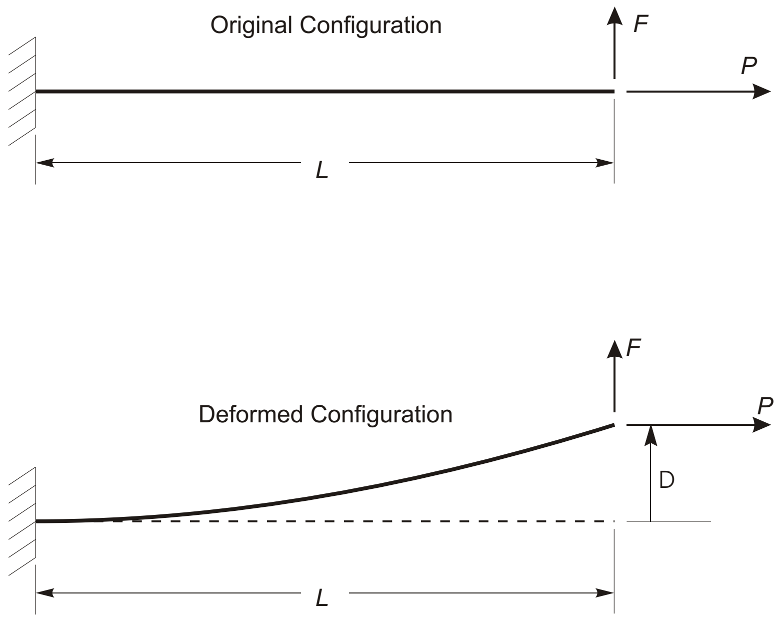

Figure 73
Geometry for Cantilever Beam Example

ements that undergo significant relative rotations within the element should be divided into smaller elements to satisfy the requirement that the strains and relative rotations within an element are small.

For most other structures, the P-delta option is adequate, particularly when material nonlinearity dominates.

If reasonable, it is recommended that the analysis be performed first without geometric nonlinearity, adding P-delta, and possibly large-displacement effects later.

Geometric nonlinearity is not available for nonlinear modal time-history (FNA) analyses, except for the fixed effects that may have been included in the stiffness matrix used to generate the modes.

Note that the catenary Cable element does not require P-delta or Large Displacements to exhibit its internal geometric nonlinearity. The choice should be determined by the rest of the structure.

The P-Delta Effect

The **P-Delta effect** refers specifically to the nonlinear geometric effect of a large tensile or compressive direct stress upon transverse bending and shear behavior. A compressive stress tends to make a structural member more flexible in transverse bending and shear, whereas a tensile stress tends to stiffen the member against transverse deformation.

This option is particularly useful for considering the effect of gravity loads upon the lateral stiffness of building structures, as required by certain design codes (ACI 2002; AISC 2003). It can also be used for the analysis of some cable structures, such as suspension bridges, cable-stayed bridges, and guyed towers. Other applications are possible.

The basic concepts behind the P-Delta effect are illustrated in the following example. Consider a cantilever beam subject to an axial load P and a transverse tip load F as shown in Figure 73 (page 346). The internal axial force throughout the member is also equal to P.

If equilibrium is examined in the original configuration (using the undeformed geometry), the moment at the base is M FL, and decreases linearly to zero at the loaded end. If, instead, equilibrium is considered in the deformed configuration, there is an additional moment caused by the axial force P acting on the transverse tip displacement, . The moment no longer varies linearly along the length; the variation depends instead upon the deflected shape. The moment at the base is now M FL P . The moment diagrams for various cases are shown in Figure 74 (page 348).

Note that only the transverse deflection is considered in the deformed configuration. Any change in moment due to a change in length of the member is neglected here.

If the beam is in tension, the moment at the base and throughout the member is reduced, hence the transverse bending deflection, , is also reduced. Thus the member is effectively stiffer against the transverse load F.

Conversely, if the beam is in compression, the moment throughout the member, and hence the transverse bending deflection, , are now increased. The member is effectively more flexible against the load F.

If the compressive force is large enough, the transverse stiffness goes to zero and hence the deflection tends to infinity; the structure is said to have buckled. The

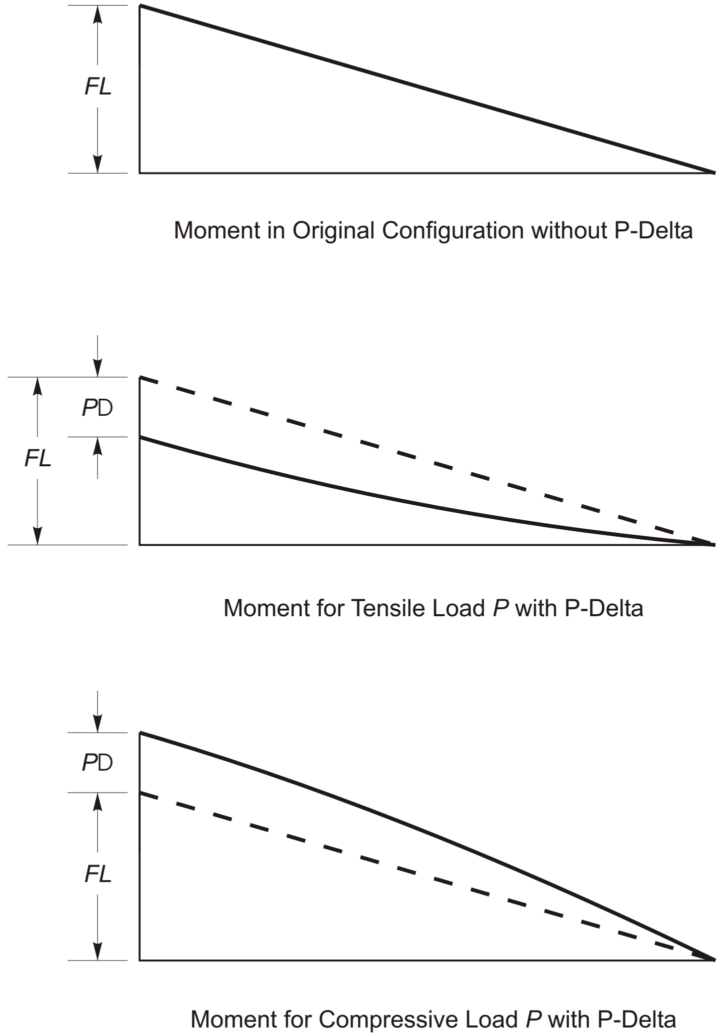

Figure 74
Moment Diagrams for Cantilever Beam Examples

theoretical value of P at which this occurs is called the Euler buckling load for the beam; it is denoted by P_{cr} and is given by the formula

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

where EI is the bending stiffness of the beam section.

The exact P-Delta effect of the axial load upon the transverse deflection and stiffness is a rather complicated function of the ratio of the force P to the buckling load P_{cr}. The true deflected shape of the beam, and hence the effect upon the moment diagram, is described by cubic functions under zero axial load, hyperbolic functions under tension, and trigonometric functions under compression.

The P-Delta effect can be present in any other beam configuration, such as simply-supported, fixed-fixed, etc. The P-Delta effect may apply locally to individual members, or globally to the structural system as a whole.

The key feature is that a large axial force, acting upon a small transverse deflection, produces a significant moment that affects the behavior of the member or structure. If the deflection is small, then the moment produced is proportional to the deflection.

P-Delta Forces in the Frame Element

The implementation of the P-Delta effect in the Frame element is described in the following subtopics.

Cubic Deflected Shape

The P-Delta effect is integrated along the length of each Frame element, taking into account the deflection within the element. For this purpose the transverse deflected shape is assumed to be cubic for bending and linear for shear between the rigid ends of the element. The length of the rigid ends is the product of the rigid-end factor and the end offsets, and is usually zero. See Topic “End Offsets” (page 102) in Chapter “The Frame Element” for more information.

The true deflected shape may differ somewhat from this assumed cubic/linear deflection in the following situations:

- The element has non-prismatic Section properties. In this case the P-Delta deflected shape is computed as if the element were prismatic using the average of the properties over the length of the element
- Loads are acting along the length of the element. In this case the P-Delta deflected shape is computed using the equivalent fixed-end forces applied to the ends of the element.

- A large P-force is acting on the element. The true deflected shape is actually described by trigonometric functions under large compression, and by hyperbolic functions under large tension.

The assumed cubic shape is usually a good approximation to these shapes except under a compressive P-force near the buckling load with certain end restraints. Excellent results, however, can be obtained by dividing any structural member into two or more Frame elements. See the *SAP2000 Software Verification Manual* for more detail.

Computed P-Delta Axial Forces

The P-Delta axial force in each Frame element is determined from the axial displacements computed in the element. For meaningful results, it is important to use realistic values for the axial stiffness of these elements. The axial stiffness is determined from the Section properties that define the cross-sectional area and the modulus of elasticity. Using values that are too small may underestimate the P-Delta effect. Using values that are too large may make the P-Delta force in the element very sensitive to the iteration process.

Elements that have an axial force release, or that are constrained against axial deformation by a Constraint, will have a zero P-Delta axial force and hence no P-Delta effect.

The P-Delta axial force also includes loads that act within the element itself. These may include Self-Weight and Gravity Loads, Concentrated and Distributed Span Loads, Prestress Load, and Temperature Load.

The P-Delta axial force is assumed to be constant over the length of each Frame element. If the P-Delta load combination includes loads that cause the axial force to vary, then the average axial force is used for computing the P-Delta effect. If the difference in axial force between the two ends of an element is small compared to the average axial force, then this approximation is usually reasonable. This would normally be the case for the columns in a building structure. If the difference is large, then the element should be divided into many smaller Frame elements wherever the P-Delta effect is important. An example of the latter case could be a flagpole under self-weight.

For more information:

- See Topic “Section Properties” (page 90) in Chapter “The Frame Element.”
- See Topic “End Releases” (page 106) in Chapter “The Frame Element.”
- See Chapter “Constraints and Welds” (page 49).

Prestress

When Prestress Load is included in the P-Delta load combination, the combined tension in the prestressing cables tends to stiffen the Frame elements against transverse deflections. This is true regardless of any axial end releases. Axial compression of the Frame element due to Prestress Load may reduce this stiffening effect, perhaps to zero.

See Topic “Prestress Load” (page 114) in Chapter “The Frame Element” for more information.

Directly Specified P-delta Axial Forces

You may directly specify P-delta forces known to be acting on Frame elements. This is an old-fashioned feature that can be used to model cable structures where the tensions are large and well-known. No iterative analysis is required to include the effect of directly specified P-Delta axial forces.

Use of this feature is not usually recommended! The program does not check if the forces you specify are in equilibrium with any other part of the structure. The directly specified forces apply in *all* analyses and are *in addition to* any P-delta affects calculated in a nonlinear analysis.

We recommend instead that you perform a nonlinear analysis including P-delta or large-displacement effects.

If you use directly specified P-delta forces, you should treat them as if they were a section property that always affects the behavior of the element.

You can assign directly specified P-Delta force to any Frame element using the following parameters:

- The P-Delta axial force, **p**
- A fixed coordinate system, **csys** (the default is zero, indicating the global coordinate system)
- The projection, **px**, of the P-Delta axial force upon the X axis of **csys**
- The projection, **py**, of the P-Delta axial force upon the Y axis of **csys**
- The projection, **pz**, of the P-Delta axial force upon the Z axis of **csys**

Normally only one of the parameters **p**, **px**, **py**, or **pz** should be given for each Frame element. If you do choose to specify more than one value, they are additive:

$$P_0 \quad \mathbf{p} \quad \frac{\mathbf{px}}{c_x} \quad \frac{\mathbf{py}}{c_y} \quad \frac{\mathbf{pz}}{c_z}$$

where P_0 is the P-Delta axial force, and c_x, c_y, and c_z are the cosines of the angles between the local 1 axis of the Frame element and the X, Y, and Z axes of coordinate system **csys**, respectively. To avoid division by zero, you may not specify the projection upon any axis of **csys** that is perpendicular to the local 1 axis of the element.

The use of the P-delta axial force projections is convenient, for example, when specifying the tension in the main cable of a suspension bridge, since the horizontal component of the tension is usually the same for all elements.

It is important when directly specifying P-Delta axial forces that you include all significant forces in the structure. The program does not check for equilibrium of the specified P-Delta axial forces. In a suspension bridge, for example, the cable tension is supported at the anchorages, and it is usually sufficient to consider the P-Delta effect only in the main cable (and possibly the towers). On the other hand, the cable tension in a cable-stayed bridge is taken up by the deck and tower, and it is usually necessary to consider the P-Delta effect in all three components.

P-Delta Forces in the Link/Support Element

P-delta effects can only be considered in a Link/Support element if there is stiffness in the axial (U1) degree of freedom to generate an axial force. A transverse displacement in the U2 or U3 direction creates a moment equal to the axial force (P) times the amount of the deflection (delta).

The total P-delta moment is distributed to the joints as the sum of:

- A pair of equal and opposite shear forces at the two ends that cause a moment due to the length of the element
- A moment at End I
- A moment at End J

The shear forces act in the same direction as the shear displacement (delta), and the moments act about the respectively perpendicular bending axes.

For each direction of shear displacement, you can specify three corresponding fractions that indicate how the total P-delta moment is to be distributed between the three moments above. These fractions must sum to one.

For any element that has zero length, the fraction specified for the shear forces will be ignored, and the remaining two fractions scaled up so that they sum to one. If both of these fractions are zero, they will be set to 0.5.

You must consider the physical characteristics of the device being modeled by a Link/Support element in order to determine what fractions to specify. Long brace or link objects would normally use the shear force. Short stubby isolators would normally use moments only. A friction-pendulum isolator would normally take all the moment on the dish side rather than on the slider side.

Other Elements

For element types other than the Frame and Link/Support, the stresses in the each element are first determined from the displacements computed in the previous iteration. These stresses are then integrated over the element, with respect to the derivatives of the isoparametric shape functions for that element, to compute a standard geometric stiffness matrix that represents the P-delta effect. This is added to the original elastic stiffness matrix of the element. This formulation produces only forces, no moments, at each joint in the element.

Shell elements that are modeling only plate bending will not produce any P-delta effects, since no in-plane stresses will be developed.

Initial P-Delta Analysis

For many applications, it is adequate to consider the P-delta effect on the structure under one set of loads (usually gravity), and to consider all other analyses as linear using the stiffness matrix developed for this one set of P-delta loads. This enables all analysis results to be superposed for the purposes of design.

To do this, define a nonlinear static Load Case that has, at least, the following features:

- Set the name to, say, “PDELTA”
- Start from zero initial conditions
- Apply the Load Patterns that will cause the P-delta effect; often this will be dead load and a fraction of live load
- For geometric nonlinearity, choose P-delta effects

Other parameters include the number of saved steps, the number of iterations allowed per step, and the convergence tolerance. If the P-delta effect is reasonably small, the default values are adequate. We are not considering staged construction here, although that could be added.

We will refer to this nonlinear static case as the **initial P-delta case.** You can then define or modify other linear Load Cases so that they use the stiffness from case PDELTA:

- Linear static cases
- A modal Load Cases, say called "PDMODES"
- Linear direct-integration time-history cases
- Moving-Load Load Cases

Other linear Load Cases can be defined that are based on the modes from case PDMODES:

- Response-spectrum cases
- Modal time-history cases

Results from all of these cases are superposable, since they are linear and are based upon the same stiffness matrix.

You may also want to define a buckling Load Case that applies the same loads as does case PDELTA, and that starts from zero conditions (not from case PDELTA). The resulting buckling factors will give you an indication of how far from buckling are the loads that cause the P-delta effect.

Below are some additional guidelines regarding practical use of the P-Delta analysis option. See also the *SAP2000 Software Verification Manual* for example problems.

Building Structures

For most building structures, especially tall buildings, the P-Delta effect of most concern occurs in the columns due to gravity load, including dead and live load. The column axial forces are compressive, making the structure more flexible against lateral loads.

Building codes (ACI 2002; AISC 2003) normally recognize two types of P-Delta effects: the first due to the overall sway of the structure and the second due to the deformation of the member between its ends. The former effect is often significant; it can be accounted for fairly accurately by considering the total vertical load at a story level, which is due to gravity loads and is unaffected by any lateral loads. The latter effect is significant only in very slender columns or columns bent in single curvature (not the usual case); this requires consideration of axial forces in the members due to both gravity and lateral loads.

SAP2000 can analyze both of these P-Delta effects. However, it is recommended that the former effect be accounted for in the SAP2000 analysis, and the latter effect be accounted for in design by using the applicable building-code moment-magnification factors (White and Hajjar 1991). This is how the SAP2000 design processors for steel frames and concrete frames are set up.

The P-Delta effect due to the sway of the structure can be accounted for accurately and efficiently, even if each column is modeled by a single Frame element, by using the factored dead and live loads in the initial P-delta Load Case. The iterative P-Delta analysis should converge rapidly, usually requiring few iterations.

As an example, suppose that the building code requires the following load combinations to be considered for design:

(1) 1.4 dead load

(2) 1.2 dead load + 1.6 live load

(3) 1.2 dead load + 0.5 live load + 1.3 wind load

(4) 1.2 dead load + 0.5 live load – 1.3 wind load

(5) 0.9 dead load + 1.3 wind load

(6) 0.9 dead load + 1.3 wind load

For this case, the P-Delta effect due to overall sway of the structure can usually be accounted for, conservatively, by specifying the load combination in the initial P-delta Load Case to be 1.2 times the dead load plus 0.5 times the live load. This will accurately account for this effect in load combinations 3 and 4 above, and will conservatively account for this effect in load combinations 5 and 6. This P-delta effect is not generally important in load combinations 1 and 2 since there is no lateral load.

The P-Delta effect due to the deformation of the member between its ends can be accurately analyzed only when separate nonlinear Load Cases are run for each load combination above. Six cases would be needed for the example above. Also, at least two Frame elements per column should be used. Again, it is recommended that this effect be accounted for instead by using the SAP2000 design features.

Cable Structures

The P-Delta effect can be a very important contributor to the stiffness of suspension bridges, cable-stayed bridges, and other cable structures. The lateral stiffness of cables is due almost entirely to tension, since they are very flexible when unstressed.

In many cable structures, the tension in the cables is due primarily to gravity load, and it is relatively unaffected by other loads. If this is the case, it is appropriate to define an initial P-delta Load Case that applies a realistic combination of the dead load and live load. It is important to use realistic values for the P-delta load combination, since the lateral stiffness of the cables is approximately proportional to the P-delta axial forces.

P-delta effects are inherent in any nonlinear analysis of Cable elements. P-delta analysis of the whole structure should be considered if you are concerned about compression in the tower, or in the deck of a cable-stayed bridge.

Because convergence tends to be slower for stiffening than softening structures, the nonlinear P-delta analysis may require many iterations. Twenty or more iterations would not be unusual.

Guyed Towers

In guyed towers and similar structures, the cables are under a large tension produced by mechanical methods that shorten the length of the cables. These structures can be analyzed by the same methods discussed above for cabled bridges.

A Strain or Deformation load can be used to produce the requisite shortening. The P-delta load combination should include this load, and may also include other loads that cause significant axial force in the cables, such as gravity and wind loads. Several analyses may be required to determine the magnitude of the length change needed to produce the desired amount of cable tension.

Large Displacements

Large-displacements analysis considers the equilibrium equations in the deformed configuration of the structure. Large displacements and rotations are accounted for, but strains are assumed to be small. This means that if the position or orientation of an element changes, its effect upon the structure is accounted for. However, if the element changes significantly in shape or size, this effect is ignored.

The program tracks the position of the element using an updated Lagrangian formulation. For Frame, Shell, and Link/Support elements, rotational degrees of freedom are updated assuming that the change in rotational displacements between steps is small. This requires that the analysis use smaller steps than might be required for a P-delta analysis. The accuracy of the results of a large-displacement analysis should be checked by re-running the analysis using a smaller step size and comparing the results.

Large displacement analysis is also more sensitive to convergence tolerance than is P-delta analysis. You should always check your results by re-running the analysis using a smaller convergence tolerance and comparing the results.

Applications

Large-displacement analysis is well suited for the analysis of some cable or membrane structures. Cable structures can be modeled with Frame elements, and membrane structures with full Shell elements (you could also use Plane stress elements). Be sure to divide the cable or membrane into sufficiently small elements so that the relative rotations within each element are small.

The catenary Cable element does not require large-displacements analysis. For most structures with cables, P-delta analysis is sufficient unless you expect significant deflection or rotation of the structure supporting or supported by the cables.

Snap-through buckling problems can be considered using large-displacement analysis. For nonlinear static analysis, this usually requires using displacement control of the load application. More realistic solutions can be obtained using nonlinear direct-integration time-history analysis.

Initial Large-Displacement Analysis

The discussion in Topic “Initial P-Delta Analysis” (page 353) in this Chapter applies equally well for an initial large-displacement analysis. Define the initial nonlinear static Load Case in the same way, select large-displacement effects instead of P-delta effects, and make sure the convergence tolerance is small enough. This case can be used as the basis for all subsequent linear analyses.