NELINEARNA STATIKA
ŠTAPNIH KONSTRUKCIJA

Prof. Mladen Meštrović

SEMINAR

Izradila: Kristina Škrtić, 0082021534
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1.0 ODREĐIVANJE PROGIBA NA SREDINI ZADANOG SUSTAVA PRIMJENOM TEORIJE II. REDA

1.1 Rješenje diferencijalne jednadžbe

Uz pretpostavku malih pomaka i deformacija, za moment savijanja vrijedi odnos iz teorije I. reda:

$$M''(x) = -(EI \cdot w'')'',$$

slijedi diferencijalna veza pomaka i opterećenja

$$(EI \cdot w'')'' - H \cdot w'' + n \cdot w' = q.$$

Diferencijalna jednadžba opisuje geometrijski nelinearnu teoriju u smislu teorije II. reda.
S obzirom da je rješenje jednadžbe u praktičnom smislu presloženo, uvodimo neka ograničenja. Pretpostavljamo da je krutost štapa na savijanje konstantna po cijeloj duljini štapa, EI=const. i da je uzdužno opterećenje n=0.

$$w'''' - \frac{H}{EI} \cdot w'' = \frac{q}{EI}$$

Ako pomnožimo ovu jednadžbu s L^3 dobivamo:

$$w'''' \cdot L^3 - \frac{H \cdot L^2}{EI} \cdot L \cdot w'' = \frac{q \cdot L^3}{EI}$$

$$\overline{H} = \frac{H \cdot L^2}{EI}, \overline{q} = \frac{q \cdot L^3}{EI}$$

Tada dobivamo jednadžbu:

$$L^3 \cdot w'''' - \overline{H} \cdot L \cdot w'' = \overline{q}.$$

Ako je H tlačna sila, H<0, uz uvođenje oznake h, *uzdužne karakteristike štapa*,

$$h^2 = \frac{|H| \cdot L^2}{EI}.$$

Iz toga proizlazi konačna jednadžba:

$$L^3 \cdot w'''' + h^2 \cdot L \cdot w'' = \overline{q}.$$

Ako je:

$$\xi = \frac{x}{L},$$

$$[0.L] \rightarrow [0,1]$$

Tada je homogeno rješenje diferencijalne jednadžbe:

$$w_H = \left[C_1 + C_2 \cdot \xi + C_3 \cdot \sin\left(\mathrm{h} \cdot \xi\right) + C_4 \cdot \cos(h \cdot \xi)\right] \cdot L$$

I pripadno partikularno rješenje:

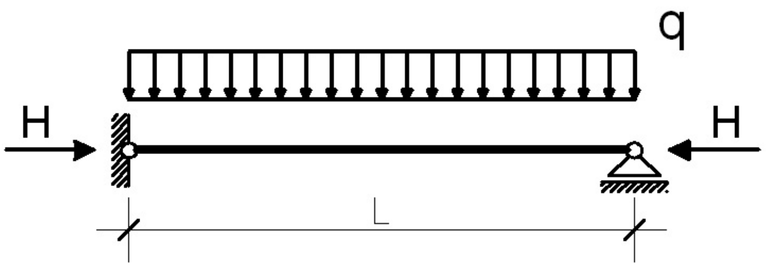
$$w_P = \frac{1}{2} \cdot \frac{\overline{q}}{h^2} \cdot L \cdot \xi^2 .$$

Ukupno rješenje jest:

$$w = \left[C_1 + C_2 \cdot \xi + C_3 \cdot \sin\left(\mathrm{h} \cdot \xi\right) + C_4 \cdot \cos(h \cdot \xi)\right] \cdot L + \frac{1}{2} \cdot \frac{\overline{q}}{h^2} \cdot L \cdot \xi^2 .$$

ZADATAK:

Za zadatak na crtežu odrediti potrebno je odrediti momentni dijagram prema teoriji II. reda.



Izraz za progib:

$$w = \left[C_1 + C_2 \cdot \xi + C_3 \cdot \sin\left(\mathrm{h} \cdot \xi\right) + C_4 \cdot \cos(h \cdot \xi)\right] \cdot L + \frac{1}{2} \cdot \frac{\overline{q}}{h^2} \cdot L \cdot \xi^2 ,$$

ima za prvu derivaciju po ξ:

$$w' = \left[C_2 + h \cdot C_3 \cdot \cos\left(\mathrm{h} \cdot \xi\right) - h \cdot C_4 \cdot \sin(h \cdot \xi)\right] + \frac{\overline{q}}{h^2} \cdot \xi .$$

Druga derivacija po ξ je:

$$w'' = \left[-C_3 \cdot h^2 \cdot \sin\left(\mathrm{h} \cdot \xi\right) - C_4 \cdot h^2 \cdot \cos(h \cdot \xi)\right] \cdot \frac{1}{L} + \frac{\overline{q}}{h^2 \cdot L} .$$

Konstante C_1, C_2, C_3, C_4 dobijemo iz rubnih uvjeta za postavljeni problem.

1) Iz $w_{\xi=0} = 0$ dobijemo:

$$\left[C_1 + C_2 \cdot 0 + C_3 \cdot \sin\left(\mathrm{h} \cdot 0\right) + C_4 \cdot \cos(h \cdot 0)\right] \cdot L + \frac{1}{2} \cdot \frac{\overline{q}}{h^2} \cdot L \cdot 0^2 = 0$$

$$\left[C_1 + C_4\right] \cdot L = 0$$

$$C_1 = -C_4$$

2) Iz $w''_{\xi=0} = 0$ dobijemo:

$$\left[-C_3 \cdot h^2 \cdot \sin\left(h \cdot 0\right) - C_4 \cdot h^2 \cdot \cos\left(h \cdot 0\right)\right] \cdot \frac{1}{L} + \frac{\overline{q}}{h^2 \cdot L} = 0$$

$$-C_4 \cdot \frac{h^2}{L} + \frac{\overline{q}}{h^2 \cdot L} = 0 / \cdot h^2$$

$$-C_4 \cdot h^4 = -\overline{q}$$

$$C_4 = \frac{\overline{q}}{h^4}$$

3) Iz $w''_{\xi=1} = 0$ dobijemo:

$$\left[-C_3 \cdot h^2 \cdot \sinh - C_4 \cdot h^2 \cdot \cosh\right] \cdot \frac{1}{L} + \frac{\overline{q}}{h^2 \cdot L} = 0$$

$$\frac{-C_3 \cdot h^2 \cdot \sinh}{L} - \frac{\overline{q}}{h^4} \cdot \frac{h^2 \cdot \cosh}{L} + \frac{\overline{q}}{h^2 \cdot L} = 0 / \cdot h^2 \cdot L$$

$$-C_3 \cdot h^4 \cdot \sinh - \overline{q} \cdot \cosh + \overline{q} = 0$$

$$-C_3 \cdot h^4 \cdot \sinh = \overline{q} \cdot \cos h - \overline{q}$$

$$C_3 = \frac{\overline{q} \cdot \left(1 - \cosh\right)}{h^4 \cdot \sinh}$$

4) Iz $w'_{\xi=\frac{1}{2}} = 0$ dobijemo:

$$\left[C_2 - C_3 \cdot h \cdot \cos\left(\mathrm{h} \cdot \xi\right) - C_4 \cdot h \cdot \sin(h \cdot \xi)\right] + \frac{\overline{q}}{h^2} \cdot \xi = 0$$

$$C_2 + h \cdot \frac{\overline{q} \cdot \left(1 - \cosh\right)}{h^4 \cdot \sinh} \cdot \cos\left(\mathrm{h} \cdot \frac{1}{2}\right) - h \cdot \frac{\overline{q}}{h^4} \cdot \sin\left(h \cdot \frac{1}{2}\right) + \frac{\overline{q}}{h^2} \cdot \frac{1}{2} = 0$$

$$C_2 + \frac{\overline{q} \cdot \left(1 - \cosh\right)}{h^3 \cdot \sinh} \cdot \cos\left(\mathrm{h} \cdot \frac{1}{2}\right) - \frac{\overline{q}}{h^3} \cdot \sin\left(h \cdot \frac{1}{2}\right) + \frac{\overline{q}}{h^2 \cdot 2} = 0$$

$$C_2 = \frac{\overline{q}}{h^3} \cdot \left(\sin\frac{h}{2} - \frac{1 - \cosh}{\sinh} \cdot \cos\frac{h}{2} - \frac{h}{2}\right) = \frac{\overline{q}}{h^3} \cdot \left(\sin\frac{h}{2} - \frac{1 - \cos^2\frac{h}{2} + \sin^2\frac{h}{2}}{2 \cdot \sin\frac{h}{2} \cdot \cos\frac{h}{2}} \cdot \cos\frac{h}{2} - \frac{h}{2}\right)$$

$$C_2 = \frac{\overline{q}}{h^3} \cdot \left(\sin\frac{h}{2} - \frac{2 \cdot \sin^2\frac{h}{2}}{2 \cdot \sin\frac{h}{2}} - \frac{h}{2}\right) = \frac{\overline{q}}{h^3} \cdot \left(-\frac{h}{2}\right)$$

$$C_2 = -\frac{1}{2} \cdot \frac{\overline{q}}{h^2}$$

Momentna jednadžba $M = -EI \cdot w"$ u bezdimenzionalnom obliku glasi

$$\bar{M} = \frac{M \cdot L}{EI} = -L \cdot w",$$

što povlači konačan izraz za momentnu funkciju u bezdimenzionalnom obliku

$$\bar{M} = \frac{\bar{q}}{h^2} \cdot \left[\frac{1-\cosh}{\sinh} \cdot \sin\left(h \cdot \xi\right) + \cos\left(h \cdot \xi\right) - 1 \right].$$

Momentnu funkciju možemo razviti u red po h oko 0:

$$\bar{M} = \frac{\bar{q}}{2}\left(\xi - \xi^2\right) + \frac{\bar{q} \cdot h^2}{24}\left(\xi - 2 \cdot \xi^3 + \xi^4\right) + 0(h^4),$$

pri čemu je linearni dio jednak rješenju prema teoriji I. reda:

$$\bar{M}_{lin} = \frac{\bar{q}}{2}\left(\xi - \xi^2\right).$$

U $\xi = \frac{1}{2}$, tj. na polovini nosača:

$$\bar{M}_{lin} = \frac{\bar{q}}{2}\left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right)$$

$$\bar{M}_{lin} = \frac{\bar{q}}{8}$$

Ako znamo da je $\bar{M} = \frac{M \cdot L}{EI}$, i ako je $\bar{q} = \frac{q \cdot L^3}{EI}$, dobijemo poznati izraz za moment u polovici raspona proste grede:

$$\frac{M_{lin} \cdot L}{EI} = \frac{q \cdot L^3}{EI} \cdot \frac{1}{8} / \cdot \frac{EI}{L}$$

$$M_{lin} = \frac{q \cdot L^2}{8}$$

Moment u polovici raspona proste grede po teoriji II. reda:

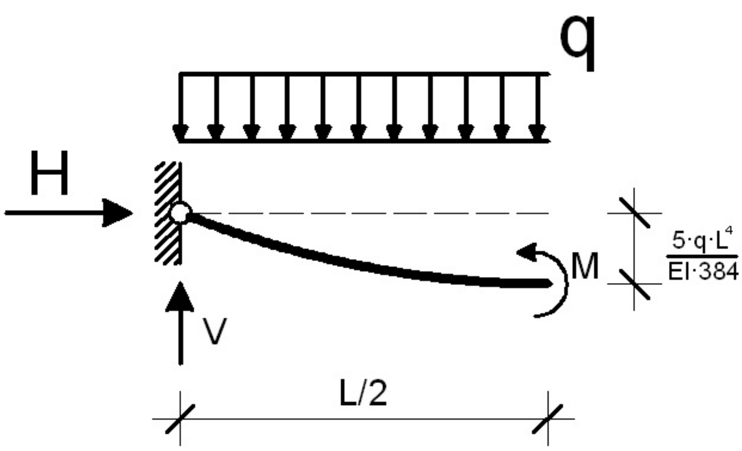
$$\bar{M}_{nelin} = \frac{\bar{q}}{2}\left(\xi - \xi^2\right) + \frac{\bar{q} \cdot h^2}{24}\left(\xi - 2 \cdot \xi^3 + \xi^4\right) + 0(h^4)$$

$$\bar{M}_{nelin} = \frac{\bar{q}}{8} + \frac{5 \cdot \bar{q} \cdot h^2}{384}$$

$$\frac{M_{nelin} \cdot L}{EI} = \frac{q \cdot L^3}{8 \cdot EI} + \frac{5 \cdot q \cdot L^3 \cdot h^2}{384 \cdot EI}$$
$$\frac{M_{nelin} \cdot L}{EI} = \frac{q \cdot L^3}{8 \cdot EI} + \frac{5 \cdot q \cdot L^3}{384 \cdot EI} \cdot \frac{H \cdot L^2}{EI} / \cdot \frac{EI}{L}$$

$$M_{nelin} = \frac{q \cdot L^2}{8} + H \cdot \frac{5}{384} \cdot \frac{q \cdot L^4}{EI}.$$

Ovo možemo pokazati i na crtežu:



Progib u polovici raspona je:

$$w_{x=\frac{L}{2}} = \frac{5 \cdot q \cdot L^4}{EI \cdot 384}$$

Ako napišemo jednadžbu za ravnotežu momenata u točki na polovici raspona dobivamo:

$$\sum M_{x=\frac{L}{2}} = 0$$

$$M + q \cdot \frac{L}{2} \cdot \frac{L}{4} - V \cdot \frac{L}{2} - H \cdot \frac{5 \cdot q \cdot L^4}{EI \cdot 384} = 0$$
$$V = q \cdot \frac{L}{2}$$
$$M + q \cdot \frac{L}{2} \cdot \frac{L}{4} - q \cdot \frac{L}{2} \cdot \frac{L}{2} - H \cdot \frac{5 \cdot q \cdot L^4}{EI \cdot 384} = 0$$

Za moment na polovici raspona tako dobivamo istu vrijednost kao i iz rješenja diferencijalne jednadžbe:

$$M = \frac{q \cdot L^2}{8} + H \cdot \frac{5 \cdot q \cdot L^4}{EI \cdot 384}$$

```
w[x_] := (-q / h^4 + ((-q) / (2 * h^2)) * x / L + ((q * (1 - Cos[h])) / (h^4 * Sin[h])) * Sin[h * x / L] + (q / h^4) * Cos[h * x / L]) * L +
  (1 / 2) * (q / h^2) * L * (x^2 / L^2)
```

```
w[x]
```

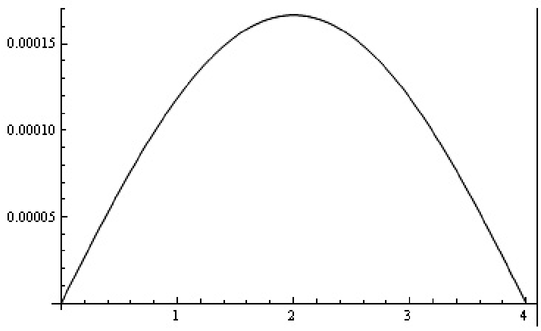
$$\frac{q\,x^2}{2\,h^2\,L} + L\left(-\frac{q}{h^4} - \frac{q\,x}{2\,h^2\,L} + \frac{q\,\mathrm{Cos}\left[\frac{h\,x}{L}\right]}{h^4} + \frac{q\,(1-\mathrm{Cos}[h])\,\mathrm{Csc}[h]\,\mathrm{Sin}\left[\frac{h\,x}{L}\right]}{h^4}\right)$$

$$\texttt{w1[x\_] := w[x] /. }\left\{L \to 4,\ h \to \sqrt{\frac{50 * 16}{200\,000}},\ q \to \frac{10 * 4\,\hat{}\,3}{200\,000}\right\}$$

```
w1[x]
```

$$\frac{x^2}{10} + 4\left(-200 - \frac{x}{10} + 200\,\mathrm{Cos}\left[\frac{x}{20\sqrt{10}}\right] + 200\left(1 - \mathrm{Cos}\left[\frac{1}{5\sqrt{10}}\right]\right)\mathrm{Csc}\left[\frac{1}{5\sqrt{10}}\right]\mathrm{Sin}\left[\frac{x}{20\sqrt{10}}\right]\right)$$

```
Plot[w1[x], {x, 0, 4}]
```




```
w1[x] /. x → 2
```

$$\frac{2}{5} + 4\left(-\frac{1001}{5} + 200\,\mathrm{Cos}\left[\frac{1}{10\sqrt{10}}\right] + 200\left(1 - \mathrm{Cos}\left[\frac{1}{5\sqrt{10}}\right]\right)\mathrm{Csc}\left[\frac{1}{5\sqrt{10}}\right]\mathrm{Sin}\left[\frac{1}{10\sqrt{10}}\right]\right)$$

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N[%]
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0.000166734

2.0 ODREĐIVANJE PROGIBA NA SREDINI ZADANOG SUSTAVA POMOĆU MATRICE KRUTOSTI TLAČNOG ŠTAPA

2.1 Matrica krutosti tlačnog štapa

Diferencijalnu jednadžbu koja opisuje geometrijski nelinearnu teoriju, uz definiranu uzdužnu karakteristiku štapa, za djelovanje tlačne sile možemo zapisati u obliku:

$$w''''+\frac{h^2}{L^2}\cdot w''=\frac{q}{EI},$$

rješenje ove diferencijalne jednadžbe za tlačnu silu iznosi:

$$w=w_H+w_P=\left[C_1+C_2\cdot\xi+C_3\cdot\sin(h\cdot\xi)+C_4\cdot\cos(h\cdot\xi)\right]\cdot L+\frac{1}{2}\cdot\frac{\overline{q}}{h^2}\cdot\xi^2$$

$$w_H=C_1+C_2\cdot x+C_3\cdot\sin\left(h\cdot\frac{x}{L}\right)+C_4\cdot\cos\left(h\cdot\frac{x}{L}\right)$$

Ako uvrstimo x = 0, i x = L dobijemo:

$$w(0)=C_1+C_4\rightarrow\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}$$

$$\varphi(0)=-w_H{}'(0)=-C_2-\frac{h}{L}\cdot C_3\rightarrow\begin{bmatrix}0 & -1 & -\frac{h}{L} & 0\end{bmatrix}$$

$$w(L)=C_1+C_2\cdot L+C_3\cdot\sin(h)+C_4\cdot\cos(h)\rightarrow\begin{bmatrix}1 & L & \sin(h) & \cos(h)\end{bmatrix}$$

$$\varphi(L)=-w_H{}'(L)=-C_2-\frac{h}{L}\cdot\cos(h)\cdot C_3+\frac{h}{L}\cdot\sin(h)\cdot C_4\rightarrow\begin{bmatrix}0 & -1 & -\frac{h}{L}\cdot\cos(h) & \frac{h}{L}\cdot\sin(h)\end{bmatrix}$$

Matrično:

$$\begin{bmatrix}w(0)\\\varphi(0)\\w(L)\\\varphi(L)\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 1\\0 & -1 & -\frac{h}{L} & 0\\1 & L & \sin(h) & \cos(h)\\0 & -1 & -\frac{h}{L}\cdot\cos(h) & \frac{h}{L}\cdot\sin(h)\end{bmatrix}$$

Skraćeno:

$$w=BC\rightarrow C=B^{-1}w$$

$$w_H{}'(x) = C_2 + \frac{h}{L}\cdot\cos\left(\frac{h}{L}\cdot x\right)\cdot C_3 - \frac{h}{L}\cdot\sin\left(\frac{h}{L}\cdot x\right)\cdot C_4$$

$$w_H{}''(x) = -\frac{h^2}{L^2}\cdot\sin\left(\frac{h}{L}\cdot x\right)\cdot C_3 - \frac{h^2}{L^2}\cdot\cos\left(\frac{h}{L}\cdot x\right)\cdot C_4$$

$$w_H{}'''(x) = -\frac{h^3}{L^3}\cdot\cos\left(\frac{h}{L}\cdot x\right)\cdot C_3 + \frac{h^3}{L^3}\cdot\sin\left(\frac{h}{L}\cdot x\right)\cdot C_4$$

Sile na krajevima štapa možemo definirati prema izrazima:

$$T_{ik} = EI\left[w''' + \frac{h^2}{L^2}\cdot w'\right]_{x=0} = EI\left[-\frac{h^3}{L^3}\cdot C_3 + \frac{h^2}{L^2}\cdot\left(C_2 + \frac{h}{L}\cdot C_3\right)\right] = EI\cdot C_2\cdot\frac{h^2}{L^2}$$

$$M_{ik} = EI\cdot w''_{x=0} = -EI\cdot\frac{h^2}{L^2}\cdot C_4$$

$$T_{ki} = -EI\left[w''' + \frac{h^2}{L^2}\cdot w'\right]_{x=L} = EI\left[\begin{array}{l} -\frac{h^3}{L^3}\cdot\cos(h)\cdot C_3 + \frac{h^3}{L^3}\cdot\sin(h)\cdot C_4 + \\ \frac{h^2}{L^2}\cdot\left(C_2 + \frac{h}{L}\cdot\cos(h)\cdot C_3 - \frac{h}{L}\cdot\sin(h)\cdot C_4\right)\end{array}\right]$$

$$T_{ki} = -EI\cdot C_2\cdot\frac{h^2}{L^2}$$

$$M_{ki} = -EI\cdot w''_{x=L} = EI\left[\frac{h^2}{L^2}\cdot\sin(h)\cdot C_3 + \frac{h^2}{L^2}\cdot\cos(h)\cdot C_4\right].$$

Ili u matričnom obliku:

$$\begin{bmatrix} T_{ik} \\ M_{ik} \\ T_{ki} \\ M_{ki}\end{bmatrix} = f_{ik} = EI\begin{bmatrix} 0 & \frac{h^2}{L^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{h^2}{L^2} \\ 0 & -\frac{h^2}{L^2} & 0 & 0 \\ 0 & 0 & \frac{h^2}{L^2}\cdot\sin(h) & \frac{h^2}{L^2}\cdot\cos(h)\end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4\end{bmatrix},$$

Ili kraće zapisano u obliku

$$f_{ik} = GC = GB^{-1}w,$$

pri čemu je vektor $w^T = \begin{bmatrix} w_{ik} & \varphi_i & w_{ki} & \varphi_k\end{bmatrix}$ vektor pomaka kraja štapa, a produkt matrica GB^{-1} predstavlja lokalnu matricu krutosti štapa K_{ik}.

$$K_{ik}=\frac{EI}{-2+2\cdot\cosh+h\cdot\sinh}\cdot\begin{bmatrix}\frac{-h^3\cdot\sinh}{L^3} & -\frac{h^2\cdot(-1+\cosh)}{L^2} & \frac{h^3\cdot\sinh}{L^3} & -\frac{h^2\cdot(-1+\cosh)}{L^2}\\ -\frac{h^2\cdot(-1+\cosh)}{L^2} & \frac{h\cdot(h\cdot\cosh-\sinh)}{L} & \frac{h^2\cdot(-1+\cosh)}{L^2} & -\frac{h\cdot(h-\sinh)}{L}\\ \frac{h^3\cdot\sinh}{L^3} & \frac{h^2\cdot(-1+\cosh)}{L^2} & -\frac{h^3\cdot\sinh}{L^3} & \frac{h^2\cdot(-1+\cosh)}{L^2}\\ -\frac{h^2\cdot(-1+\cosh)}{L^2} & -\frac{h\cdot(h-\sinh)}{L} & \frac{h^2\cdot(-1+\cosh)}{L^2} & \frac{h\cdot(h\cdot\cosh-\sinh)}{L}\end{bmatrix}$$

$$K_{ik}=EI\cdot\begin{bmatrix}\frac{12}{L^3}-\frac{6\cdot h^2}{5\cdot L^3} & -\frac{6}{L^2}+\frac{h^2}{10\cdot L^2} & -\frac{12}{L^3}+\frac{6\cdot h^2}{5\cdot L^3} & -\frac{6}{L^2}+\frac{h^2}{10\cdot L^2}\\ -\frac{6}{L^2}+\frac{h^2}{10\cdot L^2} & \frac{4}{L}-\frac{2\cdot h^2}{15\cdot L} & \frac{6}{L^2}-\frac{h^2}{10\cdot L^2} & \frac{2}{L}+\frac{h^2}{30\cdot L}\\ -\frac{12}{L^3}+\frac{6\cdot h^2}{5\cdot L^3} & \frac{6}{L^2}-\frac{h^2}{10\cdot L^2} & \frac{12}{L^3}-\frac{6\cdot h^2}{5\cdot L^3} & \frac{6}{L^2}-\frac{h^2}{10\cdot L^2}\\ -\frac{6}{L^2}+\frac{h^2}{10\cdot L^2} & \frac{2}{L}+\frac{h^2}{30\cdot L} & \frac{6}{L^2}-\frac{h^2}{10\cdot L^2} & \frac{4}{L}-\frac{2\cdot h^2}{15\cdot L}\end{bmatrix}$$

Razvijenu matricu krutosti možemo izraziti pomoću uzdužne tlačne sile H:

$$K_{ik}=\begin{bmatrix}\frac{12\cdot EI}{L^3}-\frac{6\cdot H}{5\cdot L} & -\frac{6\cdot EI}{L^2}+\frac{H}{10} & -\frac{12\cdot EI}{L^3}+\frac{6\cdot H}{5\cdot L} & -\frac{6\cdot EI}{L^2}+\frac{H}{10}\\ -\frac{6\cdot EI}{L^2}+\frac{H}{10} & \frac{4\cdot EI}{L}-\frac{2\cdot H\cdot L}{15} & \frac{6\cdot EI}{L^2}-\frac{H}{10} & \frac{2\cdot EI}{L}+\frac{H\cdot L}{30}\\ -\frac{12\cdot EI}{L^3}+\frac{6\cdot H}{5\cdot L} & \frac{6\cdot EI}{L^2}-\frac{H}{10} & \frac{12\cdot EI}{L^3}-\frac{6\cdot H}{5\cdot L} & \frac{6\cdot EI}{L^2}-\frac{H}{10}\\ -\frac{6\cdot EI}{L^2}+\frac{H}{10} & \frac{2\cdot EI}{L}+\frac{H\cdot L}{30} & \frac{6\cdot EI}{L^2}-\frac{H}{10} & \frac{4\cdot EI}{L}-\frac{2\cdot H\cdot L}{15}\end{bmatrix}$$

Ukoliko podijelimo štap na dva jednaka dijela, dobit ćemo proširenu matricu krutosti tlačnog štapa, sa L'=L/2.

$$K_{ik}{}^{p} = EI \cdot \begin{bmatrix} \frac{96}{L^3}-\frac{48\cdot h^2}{5\cdot L^3} & -\frac{24}{L^2}+\frac{2\cdot h^2}{5\cdot L^2} & -\frac{96}{L^3}+\frac{48\cdot h^2}{5\cdot L^3} & -\frac{24}{L^2}+\frac{2\cdot h^2}{5\cdot L^2} & 0 & 0 \\ -\frac{24}{L^2}+\frac{2\cdot h^2}{5\cdot L^2} & \frac{8}{L}-\frac{4\cdot h^2}{15\cdot L^2} & \frac{24}{L^2}-\frac{2\cdot h^2}{5\cdot L^2} & \frac{4}{L}+\frac{h^2}{15\cdot L} & 0 & 0 \\ -\frac{96}{L^3}+\frac{48\cdot h^2}{5\cdot L^3} & \frac{24}{L^2}-\frac{2\cdot h^2}{5\cdot L^2} & \frac{192}{L^3}-\frac{96\cdot h^2}{5\cdot L^3} & 0 & -\frac{96}{L^3}+\frac{48\cdot h^2}{5\cdot L^3} & -\frac{24}{L^2}+\frac{2\cdot h^2}{5\cdot L^2} \\ -\frac{24}{L^2}+\frac{2\cdot h^2}{5\cdot L^2} & \frac{4}{L}+\frac{h^2}{15\cdot L} & 0 & \frac{16}{L}+\frac{8\cdot h^2}{15\cdot L^2} & \frac{24}{L^2}-\frac{2\cdot h^2}{5\cdot L^2} & \frac{4}{L}+\frac{h^2}{15\cdot L} \\ 0 & 0 & -\frac{96}{L^3}+\frac{48\cdot h^2}{5\cdot L^3} & \frac{24}{L^2}-\frac{2\cdot h^2}{5\cdot L^2} & \frac{96}{L^3}-\frac{48\cdot h^2}{5\cdot L^3} & \frac{24}{L^2}-\frac{2\cdot h^2}{5\cdot L^2} \\ 0 & 0 & -\frac{24}{L^2}+\frac{2\cdot h^2}{5\cdot L^2} & \frac{4}{L}+\frac{h^2}{15\cdot L} & \frac{24}{L^2}-\frac{2\cdot h^2}{5\cdot L^2} & \frac{8}{L}-\frac{4\cdot h^2}{15\cdot L^2} \end{bmatrix}$$

2.2 Vektor upetosti tlačnog štapa

Jednadžbu uz definiranu uzdužnu karakteristiku štapa možemo zapisati u obliku:

$$w'''' + \frac{h^2}{L^2}\cdot w'' = \frac{q}{EI}.$$

Pri čemu je: $\frac{q}{EI} = \overline{q}$

Rješenje diferencijalne jednadžbe glasi:

$$w(x) = C_1 + C_2\cdot x + C_3\cdot\sin\left(h\cdot\frac{x}{L}\right) + C_4\cdot\cos\left(h\cdot\frac{x}{L}\right) + \frac{\overline{q}\cdot L^2\cdot x^2}{2\cdot h^2}$$

Matrično:

$$\begin{bmatrix} w_0 \\ \varphi_0 \\ w_L \\ \varphi_L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -\frac{h}{L} & 0 \\ 0 & L & \sinh & \cosh \\ 0 & -1 & -\frac{h}{L}\cdot\cosh & \frac{h}{L}\cdot\sinh \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\overline{q}\cdot L^4}{2\cdot h^2} \\ -\frac{\overline{q}\cdot L^3}{h^2} \end{bmatrix}$$

$$w = B\times C + q$$

$$w = 0$$

$$C = -B^{-1}\cdot q$$

$$C = \begin{bmatrix} \dfrac{-\overline{q}\cdot L^4}{2\cdot h^3}\cdot ctg\,\dfrac{h}{2} \\ \dfrac{-\overline{q}\cdot L^3}{2\cdot h^2} \\ \dfrac{\overline{q}\cdot L^3}{2\cdot h^2} \\ \dfrac{\overline{q}\cdot L^4}{2\cdot h^3}\cdot ctg\,\dfrac{h}{2} \end{bmatrix}$$

Progibnu liniju sada mozemo zapisati u obliku:

$$w = \begin{bmatrix} 1 & x & \sin\dfrac{h\cdot x}{L} & \cos\dfrac{h\cdot x}{L} \end{bmatrix}\cdot C + \frac{\overline{q}\cdot L^2\cdot x^2}{2\cdot h^2}$$

$$w = -\begin{bmatrix} 1 & x & \sin\dfrac{h\cdot x}{L} & \cos\dfrac{h\cdot x}{L} \end{bmatrix}\cdot B^{-1}\cdot q + \frac{\overline{q}\cdot L^2\cdot x^2}{2\cdot h^2}$$

$$w = \frac{\overline{q}\cdot L^2}{2\cdot h^2}\cdot x^2 + \frac{\overline{q}\cdot L^3}{2\cdot h^2}\cdot x - \frac{\overline{q}\cdot L^4}{2\cdot h^3}\left(ctg\,\frac{h}{2} - ctg\,\frac{h}{2}\cdot\cos\frac{h\cdot x}{L} - \sin\frac{h\cdot x}{L}\right)$$

Sile na krajevima štapa:

$$\overline{T}_{ik} = EI\left[w''' + \frac{h^2}{L^2}\cdot w'\right]_{x=0} = -\frac{q\cdot L}{2}$$

$$\overline{M}_{ik} = EIw''_{x=0} = \frac{q\cdot L^2\left(2 - h\cdot ctg\,\dfrac{h}{2}\right)}{2\cdot h^2}$$

$$\overline{T}_{ki} = -EI\left[w''' + \frac{h^2}{L^2}\cdot w'\right]_{x=0} = -\frac{q\cdot L}{2}$$

$$\overline{M}_{ki} = -EIw''_{x=L} = -\frac{q\cdot L^2\left(2 - h\cdot\cosh\cdot ctg\,\dfrac{h}{2} + h\cdot\sinh\right)}{2\cdot h^2} = -\frac{q\cdot L^2\left(2 - h\cdot ctg\,\dfrac{h}{2}\right)}{2\cdot h^2}$$

Vektor upetosti tlačnog štapa prema teoriji drugog reda:

$$f_{ik} = \begin{bmatrix} -\dfrac{q\cdot L}{2} \\ \dfrac{q\cdot L^2}{2\cdot h^2}\cdot\left(2 - h\cdot ctg\,\dfrac{h}{2}\right) \\ -\dfrac{q\cdot L}{2} \\ -\dfrac{q\cdot L^2}{2\cdot h^2}\cdot\left(2 - h\cdot ctg\,\dfrac{h}{2}\right) \end{bmatrix}$$

Ukoliko štap dijelimo na dva jednaka dijela, preklapanjem dviju matrica f_{ik} za L'=L/2 dobit ćemo:

$$f_{ik} = \begin{bmatrix} -\frac{q \cdot L}{4} \\ \frac{q \cdot L^2}{8 \cdot h^2}\left(2 - h \cdot ctg\frac{h}{2}\right) \\ -\frac{q \cdot L}{2} \\ 0 \\ -\frac{q \cdot L}{4} \\ -\frac{q \cdot L^2}{8 \cdot h^2}\left(2 - h \cdot ctg\frac{h}{2}\right) \end{bmatrix}$$

2.3 Rješenje

Ako uvrstimo proširenu matricu krutosti u izraz $f_{ik} = K_{ik} \times w$, slijedi:

$$\begin{bmatrix} -\frac{q \cdot L}{4} \\ \frac{q \cdot L^2}{8 \cdot h^2}\left(2 - h \cdot ctg\frac{h}{2}\right) \\ -\frac{q \cdot L}{2} \\ 0 \\ -\frac{q \cdot L}{4} \\ -\frac{q \cdot L^2}{8 \cdot h^2}\left(2 - h \cdot ctg\frac{h}{2}\right) \end{bmatrix} = EI \cdot \begin{bmatrix} \frac{96}{L^3}-\frac{48 \cdot h^2}{5 \cdot L^3} & -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} & -\frac{96}{L^3}+\frac{48 \cdot h^2}{5 \cdot L^3} & -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} & 0 & 0 \\ -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{8}{L}-\frac{4 \cdot h^2}{15 \cdot L^2} & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{4}{L}+\frac{h^2}{15 \cdot L} & 0 & 0 \\ -\frac{96}{L^3}+\frac{48 \cdot h^2}{5 \cdot L^3} & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{192}{L^3}-\frac{96 \cdot h^2}{5 \cdot L^3} & 0 & -\frac{96}{L^3}+\frac{48 \cdot h^2}{5 \cdot L^3} & -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} \\ -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{4}{L}+\frac{h^2}{15 \cdot L} & 0 & \frac{16}{L}-\frac{8 \cdot h^2}{15 \cdot L^2} & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{4}{L}+\frac{h^2}{15 \cdot L} \\ 0 & 0 & -\frac{96}{L^3}+\frac{48 \cdot h^2}{5 \cdot L^3} & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{96}{L^3}-\frac{48 \cdot h^2}{5 \cdot L^3} & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} \\ 0 & 0 & -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{4}{L}+\frac{h^2}{15 \cdot L} & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{8}{L}-\frac{4 \cdot h^2}{15 \cdot L^2} \end{bmatrix} \begin{bmatrix} w(0) \\ \varphi(0) \\ w\left(\frac{L}{2}\right) \\ \varphi\left(\frac{L}{2}\right) \\ w(L) \\ \varphi(L) \end{bmatrix}$$

Iskoristimo sada rubne uvjete: w(0) = 0, w(L) = 0:

$$\begin{bmatrix} -\frac{q \cdot L}{4} \\ \frac{q \cdot L^2}{8 \cdot h^2}\left(2 - h \cdot ctg\frac{h}{2}\right) \\ -\frac{q \cdot L}{2} \\ 0 \\ -\frac{q \cdot L}{4} \\ -\frac{q \cdot L^2}{8 \cdot h^2}\left(2 - h \cdot ctg\frac{h}{2}\right) \end{bmatrix} = EI \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{L}-\frac{4 \cdot h^2}{15 \cdot L^2} & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{4}{L}+\frac{h^2}{15 \cdot L} & 0 & 0 \\ 0 & \frac{24}{L^2}-\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{192}{L^3}-\frac{96 \cdot h^2}{5 \cdot L^3} & 0 & 0 & -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} \\ 0 & \frac{4}{L}+\frac{h^2}{15 \cdot L} & 0 & \frac{16}{L}-\frac{8 \cdot h^2}{15 \cdot L^2} & 0 & \frac{4}{L}+\frac{h^2}{15 \cdot L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{24}{L^2}+\frac{2 \cdot h^2}{5 \cdot L^2} & \frac{4}{L}+\frac{h^2}{15 \cdot L} & 0 & \frac{8}{L}-\frac{4 \cdot h^2}{15 \cdot L^2} \end{bmatrix} \begin{bmatrix} w(0) \\ \varphi(0) \\ w\left(\frac{L}{2}\right) \\ \varphi\left(\frac{L}{2}\right) \\ w(L) \\ \varphi(L) \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{q\cdot L^2\left(2-h\cdot ctg\dfrac{h}{2}\right)}{8\cdot h^2} \\ -\dfrac{q\cdot L}{2} \\ 0 \\ -\dfrac{q\cdot L^2\left(2-h\cdot ctg\dfrac{h}{2}\right)}{8\cdot h^2} \end{bmatrix} = EI\cdot\begin{bmatrix} \dfrac{8}{L}-\dfrac{4\cdot h^2}{15\cdot L} & \dfrac{24}{L^2}-\dfrac{2\cdot h^2}{5\cdot L^2} & \dfrac{4}{L}+\dfrac{h^2}{15\cdot L} & 0 \\ \dfrac{24}{L^2}-\dfrac{2\cdot h^2}{5\cdot L^2} & \dfrac{192}{L^3}-\dfrac{96\cdot h^2}{5\cdot L^3} & 0 & -\dfrac{24}{L^2}+\dfrac{2\cdot h^2}{5\cdot L^2} \\ \dfrac{4}{L}+\dfrac{h^2}{15\cdot L} & 0 & \dfrac{16}{L}-\dfrac{8\cdot h^2}{15\cdot L} & \dfrac{4}{L}+\dfrac{h^2}{15\cdot L} \\ 0 & -\dfrac{24}{L^2}+\dfrac{2\cdot h^2}{5\cdot L^2} & \dfrac{4}{L}+\dfrac{h^2}{15\cdot L} & \dfrac{8}{L}-\dfrac{4\cdot h^2}{15\cdot L} \end{bmatrix}\begin{bmatrix} \varphi(0) \\ w\left(\dfrac{L}{2}\right) \\ \varphi\left(\dfrac{L}{2}\right) \\ \varphi(L) \end{bmatrix}$$

2.3.1 Kondenzacija matrice krutosti:

Lijeva strana:

$$K_{ik}^{lijevo} = EI\cdot\begin{bmatrix} \dfrac{12}{L^3}-\dfrac{6\cdot h^2}{5\cdot L^3} & -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} & -\dfrac{12}{L^3}+\dfrac{6\cdot h^2}{5\cdot L^3} & -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} \\ -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} & \dfrac{4}{L}-\dfrac{2\cdot h^2}{15\cdot L} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{2}{L}+\dfrac{h^2}{30\cdot L} \\ -\dfrac{12}{L^3}+\dfrac{6\cdot h^2}{5\cdot L^3} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{12}{L^3}-\dfrac{6\cdot h^2}{5\cdot L^3} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} \\ -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} & \dfrac{2}{L}+\dfrac{h^2}{30\cdot L} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{4}{L}-\dfrac{2\cdot h^2}{15\cdot L} \end{bmatrix}$$

$$\begin{bmatrix} -\dfrac{q\cdot L}{2} \\ \dfrac{q\cdot L^2}{2\cdot h^2}\cdot\left(2-h\cdot ctg\dfrac{h}{2}\right) \\ -\dfrac{q\cdot L}{2} \\ -\dfrac{q\cdot L^2}{2\cdot h^2}\cdot\left(2-h\cdot ctg\dfrac{h}{2}\right) \end{bmatrix} = EI\cdot\begin{bmatrix} \dfrac{12}{L^3}-\dfrac{6\cdot h^2}{5\cdot L^3} & -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} & -\dfrac{12}{L^3}+\dfrac{6\cdot h^2}{5\cdot L^3} & -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} \\ -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} & \dfrac{4}{L}-\dfrac{2\cdot h^2}{15\cdot L} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{2}{L}+\dfrac{h^2}{30\cdot L} \\ -\dfrac{12}{L^3}+\dfrac{6\cdot h^2}{5\cdot L^3} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{12}{L^3}-\dfrac{6\cdot h^2}{5\cdot L^3} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} \\ -\dfrac{6}{L^2}+\dfrac{h^2}{10\cdot L^2} & \dfrac{2}{L}+\dfrac{h^2}{30\cdot L} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{4}{L}-\dfrac{2\cdot h^2}{15\cdot L} \end{bmatrix}\begin{bmatrix} w(0) \\ \varphi(0) \\ w\left(\dfrac{L}{2}\right) \\ \varphi\left(\dfrac{L}{2}\right) \end{bmatrix}$$

Uvrštavanjem rubnih uvjeta dobije se:

$$\begin{bmatrix} 0 \\ \dfrac{q\cdot L^2}{2\cdot h^2}\cdot\left(2-h\cdot ctg\dfrac{h}{2}\right) \\ -\dfrac{q\cdot L}{2} \\ -\dfrac{q\cdot L^2}{2\cdot h^2}\cdot\left(2-h\cdot ctg\dfrac{h}{2}\right) \end{bmatrix} = EI\cdot\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{4}{L}-\dfrac{2\cdot h^2}{15\cdot L} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{2}{L}+\dfrac{h^2}{30\cdot L} \\ 0 & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{12}{L^3}-\dfrac{6\cdot h^2}{5\cdot L^3} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} \\ 0 & \dfrac{2}{L}+\dfrac{h^2}{30\cdot L} & \dfrac{6}{L^2}-\dfrac{h^2}{10\cdot L^2} & \dfrac{4}{L}-\dfrac{2\cdot h^2}{15\cdot L} \end{bmatrix}\begin{bmatrix} w(0) \\ \varphi(0) \\ w\left(\dfrac{L}{2}\right) \\ \varphi\left(\dfrac{L}{2}\right) \end{bmatrix}$$

Budući da je moment u x = 0 jednak nuli, matricu krutosti možemo kondenzirati da bismo ovaj sustav još više reducirali.

$$k_{ij}{}^{c} = k_{ij} - \frac{k_{ik} \cdot k_{kj}}{k_{kk}}$$

Potrebni su nam kondenzirani izrazi za $k_{33}, k_{34}, k_{43}, k_{44}$ u proširenoj matrici krutosti 6x6, a da bismo njih dobili trebamo odrediti članove $k_{33}, k_{34}, k_{43}, k_{44}$ u matrici krutosti 4x4.

$$k_{33}{}^{c} = k_{33} - \frac{k_{32} \cdot k_{23}}{k_{22}} = \frac{12}{L^3} - \frac{6 \cdot h^2}{5 \cdot L^3} - \frac{\left(\frac{6}{L^2} - \frac{h^2}{10 \cdot L^2}\right) \cdot \left(\frac{6}{L^2} - \frac{h^2}{10 \cdot L^2}\right)}{\frac{4}{L} - \frac{2 \cdot h^2}{15 \cdot L}} = \frac{3}{L^3} - \frac{6 \cdot h^2}{5 \cdot L^3}$$

$$k_{34}{}^{c} = k_{43}{}^{c} = k_{34} - \frac{k_{32} \cdot k_{24}}{k_{22}} = \frac{6}{L^2} - \frac{h^2}{10 \cdot L^3} - \frac{\left(\frac{6}{L^2} - \frac{h^2}{10 \cdot L^2}\right) \cdot \left(\frac{2}{L} + \frac{h^2}{30 \cdot L^2}\right)}{\frac{4}{L} - \frac{2 \cdot h^2}{15 \cdot L}} = \frac{3}{L^3} - \frac{h^2}{5 \cdot L}$$

$$k_{44}{}^{c} = k_{44} - \frac{k_{42} \cdot k_{24}}{k_{22}} = \frac{4}{L} - \frac{2 \cdot h^2}{15 \cdot L} - \frac{\left(\frac{2}{L} + \frac{h^2}{30 \cdot L}\right) \cdot \left(\frac{2}{L} + \frac{h^2}{30 \cdot L}\right)}{\frac{4}{L} - \frac{2 \cdot h^2}{15 \cdot L}} = \frac{3}{L} - \frac{h^2}{5 \cdot L}$$

Dobili smo matricu:

$$K_{ik}{}^{c,lijevo} = EI \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{L^3} - \frac{6 \cdot h^2}{5 \cdot L^3} & \frac{3}{L^2} - \frac{h^2}{5 \cdot L^2} \\ 0 & 0 & \frac{3}{L^2} - \frac{h^2}{5 \cdot L^2} & \frac{3}{L} - \frac{h^2}{5 \cdot L} \end{bmatrix},$$

odnosno,

$$K_{ik}{}^{c,lijevo} = EI \cdot \begin{bmatrix} \frac{3}{L^3} - \frac{6 \cdot h^2}{5 \cdot L^3} & \frac{3}{L^2} - \frac{h^2}{5 \cdot L^2} \\ \frac{3}{L^2} - \frac{h^2}{5 \cdot L^2} & \frac{3}{L} - \frac{h^2}{5 \cdot L} \end{bmatrix}$$

Desna strana:

$$K_{ik}^{desno} = EI \cdot \begin{bmatrix} \frac{12}{L^3} - \frac{6\cdot h^2}{5\cdot L^3} & -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & -\frac{12}{L^3} + \frac{6\cdot h^2}{5\cdot L^3} & -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} \\ -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & \frac{4}{L} - \frac{2\cdot h^2}{15\cdot L} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} & \frac{2}{L} + \frac{h^2}{30\cdot L} \\ -\frac{12}{L^3} + \frac{6\cdot h^2}{5\cdot L^3} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} & \frac{12}{L^3} - \frac{6\cdot h^2}{5\cdot L^3} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} \\ -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & \frac{2}{L} + \frac{h^2}{30\cdot L} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} & \frac{4}{L} - \frac{2\cdot h^2}{15\cdot L} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{q\cdot L}{2} \\ \frac{q\cdot L^2}{2\cdot h^2}\cdot\left(2 - h\cdot ctg\frac{h}{2}\right) \\ -\frac{q\cdot L}{2} \\ -\frac{q\cdot L^2}{2\cdot h^2}\cdot\left(2 - h\cdot ctg\frac{h}{2}\right) \end{bmatrix} = EI \cdot \begin{bmatrix} \frac{12}{L^3} - \frac{6\cdot h^2}{5\cdot L^3} & -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & -\frac{12}{L^3} + \frac{6\cdot h^2}{5\cdot L^3} & -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} \\ -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & \frac{4}{L} - \frac{2\cdot h^2}{15\cdot L} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} & \frac{2}{L} + \frac{h^2}{30\cdot L} \\ -\frac{12}{L^3} + \frac{6\cdot h^2}{5\cdot L^3} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} & \frac{12}{L^3} - \frac{6\cdot h^2}{5\cdot L^3} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} \\ -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & \frac{2}{L} + \frac{h^2}{30\cdot L} & \frac{6}{L^2} - \frac{h^2}{10\cdot L^2} & \frac{4}{L} - \frac{2\cdot h^2}{15\cdot L} \end{bmatrix} \begin{bmatrix} w(0) \\ \varphi(0) \\ w\left(\frac{L}{2}\right) \\ \varphi\left(\frac{L}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} -\frac{q\cdot L}{2} \\ \frac{q\cdot L^2}{2\cdot h^2}\cdot\left(2 - h\cdot ctg\frac{h}{2}\right) \\ 0 \\ -\frac{q\cdot L^2}{2\cdot h^2}\cdot\left(2 - h\cdot ctg\frac{h}{2}\right) \end{bmatrix} = EI \cdot \begin{bmatrix} \frac{12}{L^3} - \frac{6\cdot h^2}{5\cdot L^3} & -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & 0 & -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} \\ -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & \frac{4}{L} - \frac{2\cdot h^2}{15\cdot L} & 0 & \frac{2}{L} + \frac{h^2}{30\cdot L} \\ 0 & 0 & 0 & 0 \\ -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} & \frac{2}{L} + \frac{h^2}{30\cdot L} & 0 & \frac{4}{L} - \frac{2\cdot h^2}{15\cdot L} \end{bmatrix} \begin{bmatrix} w\left(\frac{L}{2}\right) \\ \varphi\left(\frac{L}{2}\right) \\ w(L) \\ \varphi(L) \end{bmatrix}$$

Budući da je moment u x = L jednak nuli, matricu krutosti možemo kondenzirati da bismo ovaj sustav još više reducirali.

$$k_{ij}^{\,c} = k_{ij} - \frac{k_{ik}\cdot k_{kj}}{k_{kk}}$$

Potrebni su nam kondenzirani izrazi za $k_{33}, k_{34}, k_{43}, k_{44}$ u proširenoj matrici krutosti 6x6, a da bismo njih dobili trebamo odrediti članove $k_{11}, k_{12}, k_{21}, k_{22}$ u matrici krutosti 4x4.

$$k_{11}^{\,c} = k_{11} - \frac{k_{14}\cdot k_{41}}{k_{44}} = \frac{12}{L^3} - \frac{6\cdot h^2}{5\cdot L^3} - \frac{\left(-\frac{6}{L^2} + \frac{h^2}{10\cdot L^2}\right)\cdot\left(-\frac{6}{L^2} + \frac{h^2}{10\cdot L^2}\right)}{\frac{4}{L} - \frac{2\cdot h^2}{15\cdot L}} = \frac{3}{L^3} - \frac{6\cdot h^2}{5\cdot L^3}$$

$$k_{12}{}^{c} = k_{21}{}^{3} = k_{12} - \frac{k_{14}\cdot k_{42}}{k_{44}} = -\frac{6}{L^2} + \frac{h^2}{10\cdot L^2} - \frac{\left(\frac{2}{L}+\frac{h^2}{30\cdot L^2}\right)\cdot\left(-\frac{6}{L^2}+\frac{h^2}{10\cdot L^2}\right)}{\frac{4}{L}-\frac{2\cdot h^2}{15\cdot L}} = -\frac{3}{L^2}+\frac{h^2}{5\cdot L^2}$$

$$k_{22}{}^{c} = k_{22} - \frac{k_{42}\cdot k_{24}}{k_{22}} = \frac{4}{L} - \frac{2\cdot h^2}{15\cdot L} - \frac{\left(\frac{2}{L}+\frac{h^2}{30\cdot L}\right)\cdot\left(\frac{2}{L}+\frac{h^2}{30\cdot L}\right)}{\frac{4}{L}-\frac{2\cdot h^2}{15\cdot L}} = \frac{3}{L}-\frac{h^2}{5\cdot L}$$

Dobili smo matricu:

$$K_{ik}{}^{c,lijevo} = EI\cdot\begin{bmatrix} \frac{3}{L^3}-\frac{6\cdot h^2}{5\cdot L^3} & -\frac{3}{L^2}+\frac{h^2}{5\cdot L^2} & 0 & 0 \\ -\frac{3}{L^2}+\frac{h^2}{5\cdot L^2} & \frac{3}{L}-\frac{h^2}{5\cdot L} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

odnosno,

$$K_{ik}{}^{c,desno} = EI\cdot\begin{bmatrix} \frac{3}{L^3}-\frac{6\cdot h^2}{5\cdot L^3} & -\frac{3}{L^2}+\frac{h^2}{5\cdot L^2} \\ -\frac{3}{L^2}+\frac{h^2}{5\cdot L^2} & \frac{3}{L}-\frac{h^2}{5\cdot L} \end{bmatrix}$$

Kada preklopimo matrice:

$$K_{ik}{}^{c,p} = EI\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{L^3}-\frac{6\cdot h^2}{5\cdot L^3}+\frac{3}{L^3}-\frac{6\cdot h^2}{5\cdot L^3} & \frac{3}{L^2}-\frac{h^2}{5\cdot L^2}-\frac{3}{L^2}+\frac{h^2}{5\cdot L^2} & 0 & 0 \\ 0 & 0 & \frac{3}{L^2}-\frac{h^2}{5\cdot L^2}-\frac{3}{L^3}+\frac{h^2}{5\cdot L^2} & \frac{3}{L}-\frac{h^2}{5\cdot L}+\frac{3}{L}-\frac{h^2}{5\cdot L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

za L' = L/2 vrijedi:

$$K_{ik}^{\;c,p} = EI \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{6}{\left(\dfrac{L}{2}\right)^3} - \dfrac{12 \cdot h^2}{5 \cdot \left(\dfrac{L}{2}\right)^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{6}{\dfrac{L}{2}} - \dfrac{2 \cdot h^2}{5 \cdot \dfrac{L}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{ik}^{\;c,p} = EI \cdot \begin{bmatrix} \dfrac{48}{L^3} - \dfrac{96 \cdot h^2}{5 \cdot L^3} & 0 \\ 0 & \dfrac{12}{L} - \dfrac{4 \cdot h^2}{5 \cdot L} \end{bmatrix}$$

Ako uvrstimo da je $h^2 = \dfrac{H \cdot L^2}{EI}$, dobijemo konačnu matricu krutosti:

$$K_{ik}^{\;c,p} = \begin{bmatrix} \dfrac{48EI}{L^3} - \dfrac{96 \cdot H}{5 \cdot L} & 0 \\ 0 & \dfrac{12EI}{L} - \dfrac{4 \cdot H \cdot L}{5} \end{bmatrix}$$

2.3.2 Kondenzacija vektora upetosti

$$f_{ik} = \begin{bmatrix} -\dfrac{q \cdot L}{2} \\ \dfrac{q \cdot L^2}{2 \cdot h^2} \cdot \left(2 - h \cdot ctg\,\dfrac{h}{2}\right) \\ -\dfrac{q \cdot L}{2} \\ -\dfrac{q \cdot L^2}{2 \cdot h^2} \cdot \left(2 - h \cdot ctg\,\dfrac{h}{2}\right) \end{bmatrix}$$

Ako razvijemo vektor upetosti u Taylorov red dobivamo:

$$f_{ik}=\begin{bmatrix} -\frac{q\cdot L}{2} \\ \frac{q\cdot L^2}{12}+\frac{L^2\cdot q\cdot h^2}{720} \\ -\frac{q\cdot L}{2} \\ -\frac{q\cdot L^2}{12}-\frac{L^2\cdot q\cdot h^2}{720} \end{bmatrix}$$

Kondenzacija lijevo:

$$\bar{T}_{12}{}^{c}=\bar{T}_{12}+\frac{3}{2\cdot L}\bar{M}_{12}=-\frac{q\cdot L}{2}+\frac{3}{2\cdot L}\left(\frac{q\cdot L^2}{12}+\frac{L^2\cdot q\cdot h^2}{720}\right)=-\frac{3\cdot q\cdot L}{8}+\frac{q\cdot L\cdot h^2}{480}$$

$$\bar{T}_{21}{}^{c}=\bar{T}_{21}+\frac{3}{2\cdot L}\bar{M}_{12}=-\frac{q\cdot L}{2}-\frac{3}{2\cdot L}\left(\frac{q\cdot L^2}{12}+\frac{L^2\cdot q\cdot h^2}{720}\right)=-\frac{5\cdot q\cdot L}{8}-\frac{q\cdot L\cdot h^2}{480}$$

$$\bar{M}_{21}{}^{c}=\bar{M}_{21}+\frac{1}{2}\bar{M}_{12}=-\frac{q\cdot L^2}{12}-\frac{L^2\cdot q\cdot h^2}{720}-\frac{1}{2}\cdot\left(\frac{q\cdot L^2}{12}+\frac{L^2\cdot q\cdot h^2}{720}\right)=-\frac{q\cdot L^2}{8}-\frac{L^2\cdot q\cdot h^2}{480}$$

$$f_{ik}{}^{c,lijevo}=\begin{bmatrix} -\frac{3\cdot q\cdot L}{8}+\frac{q\cdot L\cdot h^2}{480} \\ 0 \\ -\frac{5\cdot q\cdot L}{8}-\frac{q\cdot L\cdot h^2}{480} \\ -\frac{q\cdot L^2}{8}-\frac{L^2\cdot q\cdot h^2}{480} \end{bmatrix}$$

Kondenzacija desno:

$$\bar{T}_{12}{}^{c}=\bar{T}_{12}+\frac{3}{2\cdot L}\bar{M}_{21}=-\frac{q\cdot L}{2}+\frac{3}{2\cdot L}\left(-\frac{q\cdot L^2}{12}-\frac{L^2\cdot q\cdot h^2}{720}\right)=-\frac{5\cdot q\cdot L}{8}-\frac{q\cdot L\cdot h^2}{480}$$

$$\bar{T}_{21}{}^{c}=\bar{T}_{21}+\frac{3}{2\cdot L}\bar{M}_{21}=-\frac{q\cdot L}{2}-\frac{3}{2\cdot L}\left(-\frac{q\cdot L^2}{12}-\frac{L^2\cdot q\cdot h^2}{720}\right)=-\frac{3\cdot q\cdot L}{8}+\frac{q\cdot L\cdot h^2}{480}$$

$$\bar{M}_{12}{}^{c}=\bar{M}_{12}+\frac{1}{2}\bar{M}_{21}=\frac{q\cdot L^2}{12}+\frac{L^2\cdot q\cdot h^2}{720}-\frac{1}{2}\cdot\left(-\frac{q\cdot L^2}{12}-\frac{L^2\cdot q\cdot h^2}{720}\right)=\frac{q\cdot L^2}{8}+\frac{L^2\cdot q\cdot h^2}{480}$$

$$f_{ik}^{c,desno} = \begin{bmatrix} -\frac{5\cdot q\cdot L}{8} - \frac{q\cdot L\cdot h^2}{480} \\ \frac{q\cdot L^2}{8} + \frac{L^2\cdot q\cdot h^2}{480} \\ -\frac{3\cdot q\cdot L}{8} + \frac{q\cdot L\cdot h^2}{480} \\ 0 \end{bmatrix}$$

Preklopljeni vektor upetosti:

$$f_{ik}^{c,p} = \begin{bmatrix} -\frac{3\cdot q\cdot L}{8} + \frac{q\cdot L\cdot h^2}{480} \\ 0 \\ -\frac{5\cdot q\cdot L}{8} - \frac{q\cdot L\cdot h^2}{480} - \frac{5\cdot q\cdot L}{8} - \frac{q\cdot L\cdot h^2}{480} \\ -\frac{q\cdot L^2}{8} - \frac{L^2\cdot q\cdot h^2}{480} + \frac{q\cdot L^2}{8} + \frac{L^2\cdot q\cdot h^2}{480} \\ \frac{3\cdot q\cdot L}{8} + \frac{q\cdot L\cdot h^2}{480} \\ 0 \end{bmatrix}$$

$$f_{ik}^{c,p} = \begin{bmatrix} 0 \\ 0 \\ -\frac{5\cdot q\cdot L}{4} - \frac{q\cdot L\cdot h^2}{240} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$f_{ik}^{c,p} = \begin{bmatrix} -\frac{5\cdot q\cdot L}{4} - \frac{q\cdot L\cdot h^2}{240} \\ 0 \end{bmatrix}$$

Ako uvrstimo da je $h^2 = \frac{H\cdot L^2}{EI}$ i $L = \frac{L}{2}$ dobijemo konačni vektor upetosti:

$$f_{ik}^{c,p} = \begin{bmatrix} -\frac{5\cdot q\cdot L}{8} - \frac{q\cdot H\cdot L^3}{EI\cdot 1920} \\ 0 \end{bmatrix}$$

2.3.3 Rješenje sustava

$$\begin{bmatrix} -\frac{5\cdot q\cdot L}{8}-\frac{q\cdot H\cdot L^3}{EI\cdot 1920} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{48EI}{L^3}-\frac{96\cdot H}{5\cdot L} & 0 \\ 0 & \frac{12EI}{L}-\frac{4\cdot H\cdot L}{5} \end{bmatrix} \begin{bmatrix} w\left(\frac{L}{2}\right) \\ \varphi\left(\frac{L}{2}\right) \end{bmatrix}$$

```
k = {{48 EI / (L^3) - (96 * H) / (5 * L), 0}, {0, (12 * EI) / L + (4 * L * H) / (5)}}
```

$$\left\{\left\{\frac{48\,EI}{L^3}-\frac{96\,H}{5\,L},\ 0\right\},\ \left\{0,\ \frac{12\,EI}{L}+\frac{4\,H\,L}{5}\right\}\right\}$$

```
MatrixForm[k]
```

$$\begin{pmatrix} \frac{48\,EI}{L^3}-\frac{96\,H}{5\,L} & 0 \\ 0 & \frac{12\,EI}{L}+\frac{4\,H\,L}{5} \end{pmatrix}$$

```
f = {{(-5 * q * L) / 8 - (q * L^3 * H) / (1920 * EI)}, {0}}
```

$$\left\{\left\{-\frac{5\,L\,q}{8}-\frac{H\,L^3\,q}{1920\,EI}\right\},\ \{0\}\right\}$$

```
MatrixForm[f]
```

$$\begin{pmatrix} -\frac{5\,L\,q}{8}-\frac{H L^3\,q}{1920\,EI} \\ 0 \end{pmatrix}$$

```
LinearSolve[k, f]
```

$$\left\{\left\{-\frac{L^4\left(1200\,EI\,q+H\,L^2\,q\right)}{18\,432\,EI\left(5\,EI-2\,H\,L^2\right)}\right\},\ \{0\}\right\}$$

$$\mathtt{w[x\_] := } -\frac{L^4\left(1200\,EI\,q+H\,L^2\,q\right)}{18\,432\,EI\left(5\,EI-2\,H\,L^2\right)}\ \mathtt{/.\ \{EI \to 200\,000,\ q \to 10,\ H \to 50,\ L \to 4\}}$$

```
w[x]
```

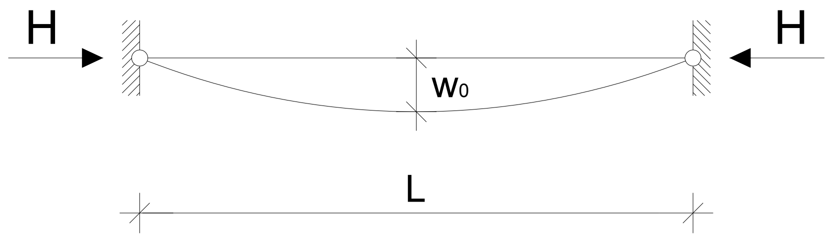
$$-\frac{23\,077}{138\,240\,000}$$

```
N[%]
```

-0.000166934

3.0 ODREĐIVANJE PROGIBNE LINIJE SLOBODNO OSLONJENOG TLAČNOG ŠTAPA SA ZADANOM POČETNOM IMPERFEKCIJOM

Slobodno oslonjeni vlačni štap izveden je s početnom imperfekcijom w_0 .Promatramo slobodno oslonjeni tlačni štap, H < 0. Potrebno je odrediti progibnu liniju sustava na slici.



Jednadžbu progibne linije štapa možemo napisati u obliku:

$$w''+\frac{h^2}{L^2}\cdot w=-\frac{h^2}{L^2}\cdot w^p$$

Geometrijske i strukturalne imperfekcije ćemo prikazati pomoću funkcije početne deformacije:

$$w^P = w_0 \cdot \left(1-\frac{1}{4}\cdot x^2\right)$$

Izgled početne imperfekcije uz $w_0 = 0.01$:

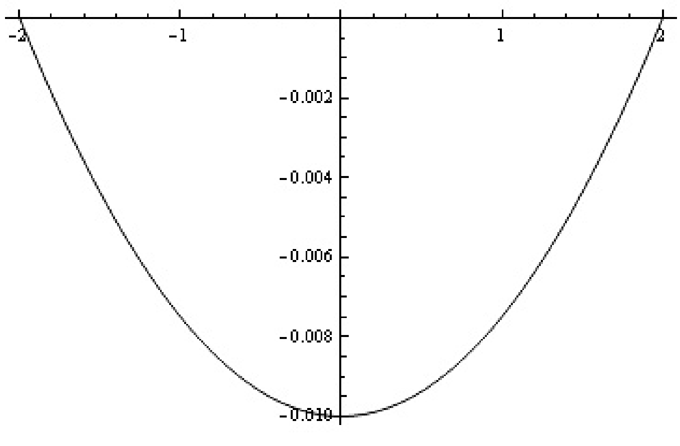
```
In[1]:= wpoc[x_] := w0 * (1 - 1/4 * x^2)

In[2]:= wpoc[x]
```

Out[2]= $w0\left(1-\frac{x^2}{4}\right)$

```
In[4]:= Plot[wpoc[x] /. {w0 → -0.01}, {x, -2, 2}]
```

Out[4]=



Homogeno rješenje diferencijalne jednadžbe za tlačnu horizontalnu silu glasi

$$w_H = c_1 \cdot \sin\left(\frac{h \cdot x}{L}\right) + c_2 \cdot \cos\left(\frac{h \cdot x}{L}\right),$$

odnosno:

$$w_H = A \cdot \sin\left(\frac{h \cdot x}{L}\right) + B \cdot \cos\left(\frac{h \cdot x}{L}\right).$$

Partikularno rješenje ovisi o obliku funkcije početne deformacije. U promatranom slučaju partikularno rješenje je oblika:

$$w_P = C \cdot \frac{x^2}{L^2} + D \cdot \frac{x}{L} + F$$

Prva derivacija jest:

$$w'_P = \frac{2 \cdot C \cdot x}{L^2} + \frac{D}{L}$$

Druga derivacija jest:

$$w_P = \frac{2 \cdot C}{L^2}$$

Ako u diferencijalnu jednadžbu štapa uvrstimo partikularno rješenje i izraz za funkciju početne deformacije štapa, dobit ćemo slijedeću jednakost:

$$\frac{2 \cdot C}{L^2} + \frac{h^2}{L^2} \cdot \left(C \cdot \frac{x^2}{L^2} + D \cdot \frac{x}{L} + F \right) = -\frac{h^2}{L^2} \cdot w_0 \cdot \left(1 - \frac{1}{4} \cdot x^2 \right)$$

Izjednačavanjem koeficijenata uz pojedine potencije uz x dolazimo do izraza za C, D i F:

$$C = \frac{w_0 \cdot L^2}{4}$$

$$D = 0$$

$$F = -w_0 - \frac{w_0 \cdot L^2}{2 \cdot h^2}$$

In[9]:= $\text{ExpandAll}\left[\text{wp}''[x] + \frac{h^2}{L^2} * \text{wp}[x]\right]$

Out[9]= $\frac{2\,C}{L^2} + \frac{F\,h^2}{L^2} + \frac{D\,h^2\,x}{L^3} + \frac{C\,h^2\,x^2}{L^4}$

In[10]:= $\text{ExpandAll}\left[-\frac{h^2}{L^2} * \text{wpoc}[x]\right]$

Out[10]= $-\frac{h^2\,w0}{L^2} + \frac{h^2\,w0\,x^2}{4\,L^2}$

In[12]:= $\text{Solve}\left[\left\{\frac{2*C}{L\hat{}2} + \frac{F*h\hat{}2}{L\hat{}2} == -\frac{h\hat{}2*w0}{L\hat{}2},\ \frac{D*h\hat{}2}{L\hat{}3} == 0,\ \frac{C*h\hat{}2}{L\hat{}4} == \frac{h\hat{}2*w0}{4*L\hat{}2}\right\}, \{C, D, F\}\right]$

Out[12]= $\left\{\left\{F \to -w0 - \frac{L^2\,w0}{2\,h^2},\ D \to 0,\ C \to \frac{L^2\,w0}{4}\right\}\right\}$

In[13]:= $\text{wp}[x\_] := C * \left(\frac{x}{L}\right)^2 + D * \frac{x}{L} + F\ /.\ \left\{F \to -w0 - \frac{L^2\,w0}{2\,h^2},\ D \to 0,\ C \to \frac{L^2\,w0}{4}\right\}$

In[14]:= $\text{wp}[x]$

Out[14]= $-w0 - \frac{L^2\,w0}{2\,h^2} + \frac{w0\,x^2}{4}$

Konačni oblik partikularnog rješenja

$$w_P = -w_0 - \frac{L^2 \cdot w_0}{2 \cdot h^2} + \frac{w_0 \cdot x^2}{4},$$

odnosno kad se razvije u Taylorov red:

$$w_P = w_0\left(\frac{x^2}{4} - 1 - \frac{L^2}{2 \cdot h^2}\right)$$

Konačno rješenje:

$$w(x) = w_H + w_P$$

$$w(x) = A \cdot \sin\left(\frac{h \cdot x}{L}\right) + B \cdot \cos\left(\frac{h \cdot x}{L}\right) + w_0\left(\frac{x^2}{4} - 1 - \frac{L^2}{2 \cdot h^2}\right)$$

Pomoću rubnih uvjeta da je progib na početku i na kraju sustava jednak nuli dobijemo rješenje sustava:

$$A = 0$$

$$B=\frac{L^2\cdot w_0\cdot \sec\left(\frac{2\cdot h}{L}\right)}{2\cdot h^2},\qquad \frac{1}{\cos x}=\sec x\,.$$

Jednadžba za progibnu liniju je:

$$w(x)=\frac{L^2\cdot w_0\cdot \sec\left(\frac{2\cdot h}{L}\right)}{2\cdot h^2}\cdot\cos\left(\frac{h\cdot x}{L}\right)+w_0\left(\frac{x^2}{4}-\frac{L^2}{2\cdot h^2}-1\right)$$

```
In[14]:= w[x_] := A*Sin[h*x/L] + B*Cos[h*x/L] + (-w0 - L^2 w0/(2 h^2) + w0 x^2/4)

In[15]:= w[x]

Out[15]= -w0 - L^2 w0/(2 h^2) + w0 x^2/4 + B Cos[h x/L] + A Sin[h x/L]

In[16]:= Solve[{w[-2] == 0, w[2] == 0}, {A, B}]

Out[16]= {{A -> 0, B -> L^2 w0 Sec[2 h/L]/(2 h^2)}}

In[17]:= w[x] := B Cos[h x/L] + A Sin[h x/L] + (-w0 - L^2 w0/(2 h^2) + w0 x^2/4) /. {B -> L^2 Sec[2 h/L] w0/(2 h^2), A -> 0}

In[18]:= w[x]

Out[18]= -w0 - L^2 w0/(2 h^2) + w0 x^2/4 + L^2 w0 Cos[h x/L] Sec[2 h/L]/(2 h^2)

In[19]:= Simplify[w[x]]

Out[19]= w0 (-2 L^2 + h^2 (-4 + x^2) + 2 L^2 Cos[h x/L] Sec[2 h/L])/(4 h^2)
```

Primjer:

$L=4\ m$

$q=10\ kN/m$

$H=50\ kN$

$EI=200\,000\ kN/m^2$

Konačni progib je zbroj $w^P(x)+w(x)$.

$$w_{kon}=w_0\cdot\left(1-\frac{1}{4}\cdot x^2\right)+\frac{L^2\cdot w_0\cdot \sec\left(\frac{2\cdot h}{L}\right)}{2\cdot h^2}\cdot\cos\left(\frac{h\cdot x}{L}\right)+w_0\left(\frac{x^2}{4}-\frac{L^2}{2\cdot h^2}-1\right)$$

In[20]:= $$\text{wkon}[x\_] := \frac{\text{w0}\left(-2L^2 + h^2\left(-4 + x^2\right) + 2L^2\,\text{Cos}\left[\frac{h\,x}{L}\right]\text{Sec}\left[\frac{2h}{L}\right]\right)}{4h^2} + \text{w0} * \left(1 - \frac{1}{4} * x\text{^}2\right)$$

In[21]:= wkon[x]

Out[21]= $$\text{w0}\left(1 - \frac{x^2}{4}\right) + \frac{\text{w0}\left(-2L^2 + h^2\left(-4 + x^2\right) + 2L^2\,\text{Cos}\left[\frac{h\,x}{L}\right]\text{Sec}\left[\frac{2h}{L}\right]\right)}{4h^2}$$

In[22]:= $$\text{wkon}[x]\ /.\ \left\{h \to \sqrt{\frac{L^2 H}{EI}}\right\}$$

Out[22]= $$\text{w0}\left(1 - \frac{x^2}{4}\right) + \frac{EI\,\text{w0}\left(-2L^2 + \frac{H L^2\left(-4+x^2\right)}{EI} + 2L^2\,\text{Cos}\left[\frac{\sqrt{\frac{HL^2}{EI}}\,x}{L}\right]\text{Sec}\left[\frac{2\sqrt{\frac{HL^2}{EI}}}{L}\right]\right)}{4HL^2}$$

In[23]:= $$\text{wkon}[x] := \text{w0}\left(1 - \frac{x^2}{4}\right) + \frac{EI\,\text{w0}\left(-2L^2 + \frac{H L^2\left(-4+x^2\right)}{EI} + 2L^2\,\text{Cos}\left[\frac{\sqrt{\frac{HL^2}{EI}}\,x}{L}\right]\text{Sec}\left[\frac{2\sqrt{\frac{HL^2}{EI}}}{L}\right]\right)}{4HL^2}$$

In[24]:= wkon[x]

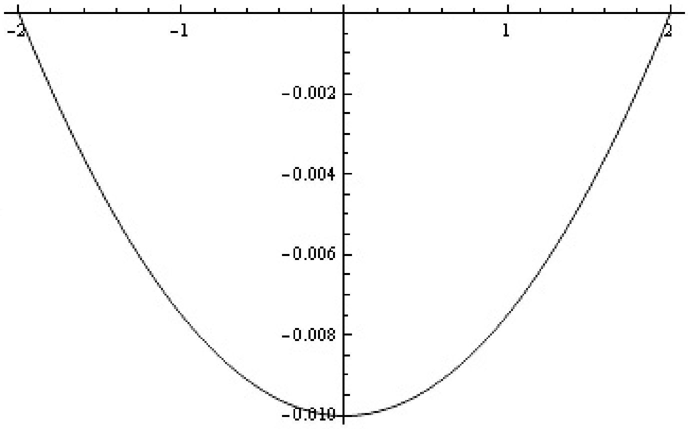
Out[24]= $$\text{w0}\left(1 - \frac{x^2}{4}\right) + \frac{EI\,\text{w0}\left(-2L^2 + \frac{H L^2\left(-4+x^2\right)}{EI} + 2L^2\,\text{Cos}\left[\frac{\sqrt{\frac{HL^2}{EI}}\,x}{L}\right]\text{Sec}\left[\frac{2\sqrt{\frac{HL^2}{EI}}}{L}\right]\right)}{4HL^2}$$

In[28]:= wkon[x] /. {H → 50, L → 4, EI → 200 000, w0 → -0.01}

Out[28]= $$-0.01\left(1 - \frac{x^2}{4}\right) - 0.625\left(-32 + \frac{1}{250}\left(-4 + x^2\right) + 32\,\text{Cos}\left[\frac{x}{20\sqrt{10}}\right]\text{Sec}\left[\frac{1}{10\sqrt{10}}\right]\right)$$

In[30]:= Plot[wkon[x] /. {H → 50, L → 4, EI → 200 000, w0 → -0.01}, {x, -2, 2}]

Out[30]=

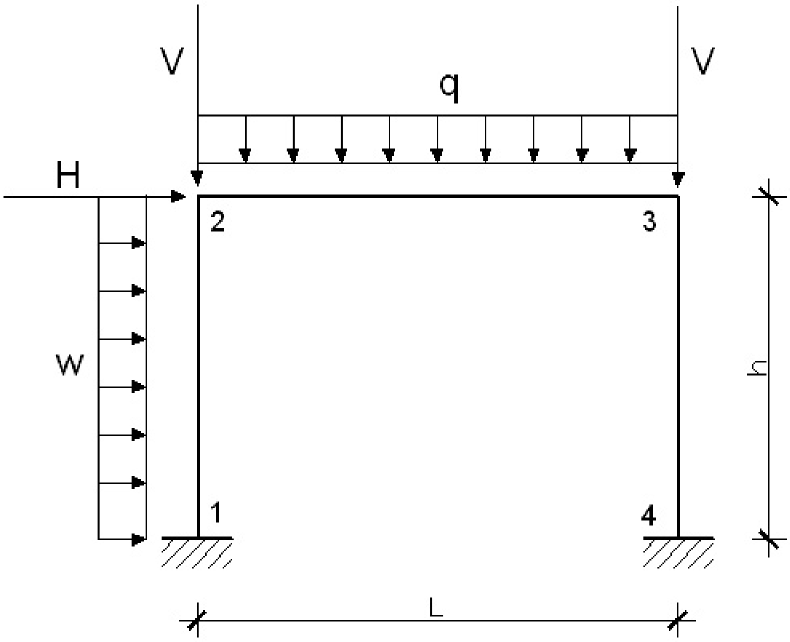


In[31]:= N[wkon[0]]

Out[31]= -0.0100042

4.0 POSTUPAK P – DELTA

P – delta postupkom je potrebno odrediti momentni dijagram okvira koji je opterećen prema slici:



a = 5 m
L = 10 m
h = 6 m

$q = 2 \cdot a = 2 \cdot 5 = 10\ kN/m$

$H = 0.1 \cdot q \cdot L = 0.1 \cdot 10 \cdot 10 = 10\ kN$

V = 400 kN

$w = 0.45 \cdot 1.2 \cdot a = 0.45 \cdot 1.2 \cdot 5 = 2.7\ kN/m$

$E = 2.8 \cdot 10^8\ kN/m^2$

STUP IPE 330:

$A = 62.61\,cm^2$

$I_y = 11770\,cm^4$

GREDA IPE 400:

$A = 84.46\,cm^2$

$I_y = 23130\,cm^4$

4.1 Izrazi za momente u čvorovima

$$M_{12} = \left(\frac{2EI_s}{h} + \frac{hN_{s1}}{30} \right)\varphi_2 + \left(\frac{6EI_s}{h^2} - \frac{N_{s1}}{10} \right)u + \left(\frac{wh^2}{12} + \frac{wh^4 N_{s1}}{720E_{s1}} \right)$$

$$M_{21} = \left(\frac{4EI_s}{h} - \frac{2hN_{s1}}{15} \right)\varphi_2 + \left(\frac{6EI_s}{h^2} - \frac{N_{s1}}{10} \right)u - \left(\frac{wh^2}{12} + \frac{wh^4 N_{s1}}{720EI_s} \right)$$

$$M_{23} = \left(\frac{4EI_g}{L} - \frac{2LN_g}{15} \right)\varphi_2 + \left(\frac{2EI_g}{L} + \frac{LN_g}{30} \right)\varphi_3 + \left(\frac{qL^2}{12} + \frac{qL^4 N_g}{720EI_g} \right)$$

$$M_{32} = \left(\frac{4EI_g}{L} - \frac{2LN_g}{15} \right)\varphi_3 + \left(\frac{2EI_g}{L} + \frac{LN_g}{30} \right)\varphi_2 - \left(\frac{qL^2}{12} + \frac{qL^4 N_g}{720EI_g} \right)$$

$$M_{34} = \left(\frac{4EI_s}{h} - \frac{2hN_{s2}}{15} \right)\varphi_3 + \left(\frac{6EI_s}{h^2} - \frac{N_{s2}}{10} \right)u$$

$$M_{43} = \left(\frac{2EI_s}{h} + \frac{hN_{s2}}{30} \right)\varphi_3 + \left(\frac{6EI_s}{h^2} - \frac{N_{s2}}{10} \right)u$$

$$EI_s = 2.8 \cdot 10^8 \cdot 11770 \cdot 10^{-8} = 32956\, kNm^2$$
$$EI_g = 2.8 \cdot 10^8 \cdot 23130 \cdot 10^{-8} = 64\,764\, kNm^2$$
$$h = 6\ m$$

Degrees of Freedom: 6

Nodal displacements:

nd	u_i	v_i	phi_i
1:	0	0	0
2:	0.00655817	-0.0015263	-0.00280061
3:	0.00646736	-0.00155399	0.00175407
4:	0	0	0

Element end forces:

el	N_ij	T_ij	M_ij	N_ji	T_ji	M_ji
1:	445.955	4.72444	13.3562	-445.955	11.4756	-33.6095
2:	21.4756	45.9548	33.6095	-21.4756	54.0452	-74.0612
3:	454.045	21.4756	54.7921	-454.045	-21.4756	74.0612

Reactions:

nd	R_x	R_y	M
1:	-4.72444	445.955	13.3562
4:	-21.4756	454.045	54.7921

Uzdužne sile dobijemo proračunom zadanog okvira prema linearnoj teoriji:

$$N_{s1} = 445.955\ kN$$
$$N_{s2} = 454.045\ kN$$
$$N_g = 21.4756\ kN$$

Iz toga slijede izrazi za momente:

$$M_{12} = \left(\frac{2\cdot 32956}{6} + \frac{6\cdot 445.955}{30}\right)\varphi_2 + \left(\frac{6\cdot 32956}{6^2} - \frac{445.955}{10}\right)u + \left(\frac{2.7\cdot 6^2}{12} + \frac{2.7\cdot 6^4\cdot 445.955}{720\cdot 32956}\right)$$
$$M_{21} = \left(\frac{4\cdot 32956}{6} - \frac{2\cdot 6\cdot 445.955}{15}\right)\varphi_2 + \left(\frac{6\cdot 32956}{6^2} - \frac{445.955}{10}\right)u + \left(\frac{2.7\cdot 6^2}{12} + \frac{2.7\cdot 6^4\cdot 445.955}{720\cdot 32956}\right)$$
$$M_{23} = \left(\frac{4\cdot 64764}{10} - \frac{2\cdot 10\cdot 21.4756}{15}\right)\varphi_2 + \left(\frac{2\cdot 64764}{10} + \frac{10\cdot 21.4756}{30}\right)\varphi_3 + \left(\frac{10\cdot 10^2}{12} + \frac{10\cdot 10^4\cdot 21.4756}{720\cdot 64764}\right)$$
$$M_{23} = \left(\frac{2\cdot 64764}{10} + \frac{10\cdot 21.4756}{30}\right)\varphi_2 + \left(\frac{4\cdot 64764}{10} - \frac{2\cdot 10\cdot 21.4756}{15}\right)\varphi_3 - \left(\frac{10\cdot 10^2}{12} + \frac{10\cdot 10^4\cdot 21.4756}{720\cdot 64764}\right)$$
$$M_{34} = \left(\frac{4\cdot 32956}{6} - \frac{2\cdot 6\cdot 454.045}{15}\right)\varphi_3 + \left(\frac{6\cdot 32956}{6^2} - \frac{454.045}{10}\right)u$$
$$M_{43} = \left(\frac{2\cdot 32956}{6} + \frac{6\cdot 454.045}{30}\right)\varphi_3 + \left(\frac{6\cdot 32956}{6^2} - \frac{454.045}{10}\right)u$$

$$M_{12} = 11074.5243\cdot\varphi_2 + 5448.0712\cdot u + 8.1658$$
$$M_{21} = 21613.9027\cdot\varphi_2 + 5448.0712\cdot u - 8.1658$$
$$M_{23} = 25876.9659\cdot\varphi_2 + 12959.9585\cdot\varphi_3 + 83.3794$$
$$M_{32} = 12959.9585\cdot\varphi_2 + 25876.9659\cdot\varphi_3 - 83.3794$$
$$M_{34} = 21607.4307\cdot\varphi_3 + 5447.2622\cdot u$$
$$M_{43} = 11076.1423\cdot\varphi_3 + 5447.2622\cdot u$$

4.2 Rješenje sustava

Nepoznanice φ_2, φ_3 i u se odrede iz sljedećih uvjeta:

a) $M_{21} + M_{23} = 0$

b) $M_{32} + M_{34} = 0$

c) jednadžba virtualnog rada

$$\left(M_{12} + M_{21}\right)\cdot\psi_{12} + \left(M_{34} + M_{43}\right)\cdot\psi_{34} + H + w\cdot h\cdot 0.5 = 0$$

$$47490.8686\cdot\varphi_2+12959.9585\cdot\varphi_3+5448.0712\cdot u+75.2114=0 \qquad (1)$$

$$12959.9585\cdot\varphi_2+47484.3966\cdot\varphi_3+5447.2622\cdot u-83.3794=0 \qquad (2)$$

$$\begin{pmatrix}11074.5243\cdot\varphi_2+5448.0712\cdot u+8.1658+\\ 21613.9027\cdot\varphi_2+5448.0712\cdot u-8.1658\end{pmatrix}\cdot\frac{-1}{6}+\begin{pmatrix}21607.4307\cdot\varphi_3+5447.2622\cdot u+\\ 11076.1423\cdot\varphi_3+5447.2622\cdot u\end{pmatrix}\cdot\frac{-1}{6}+$$
$$10+2.7\cdot 6\cdot 0.5=0$$

$$-5448.0712\cdot\varphi_2-1816.0237\cdot u-5447.2622\cdot\varphi_3-1815.7541\cdot u+18.1=0 \qquad (3)$$
$$-5448.0712\cdot\varphi_2-5447.2622\cdot\varphi_3-3631.7778\cdot u+18.1=0$$

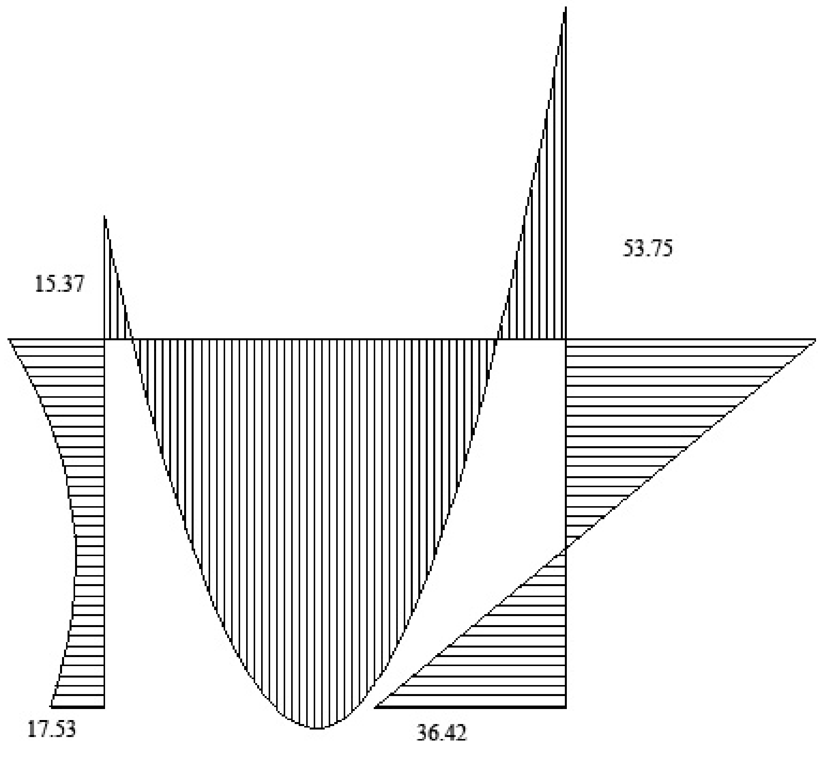
$$47490.8686\cdot\varphi_2+12959.9585\cdot\varphi_3+5448.0712\cdot u+75.2114=0 \qquad (1)$$

$$12959.9585\cdot\varphi_2+47484.3966\cdot\varphi_3+5447.2622\cdot u-83.3794=0 \qquad (2)$$

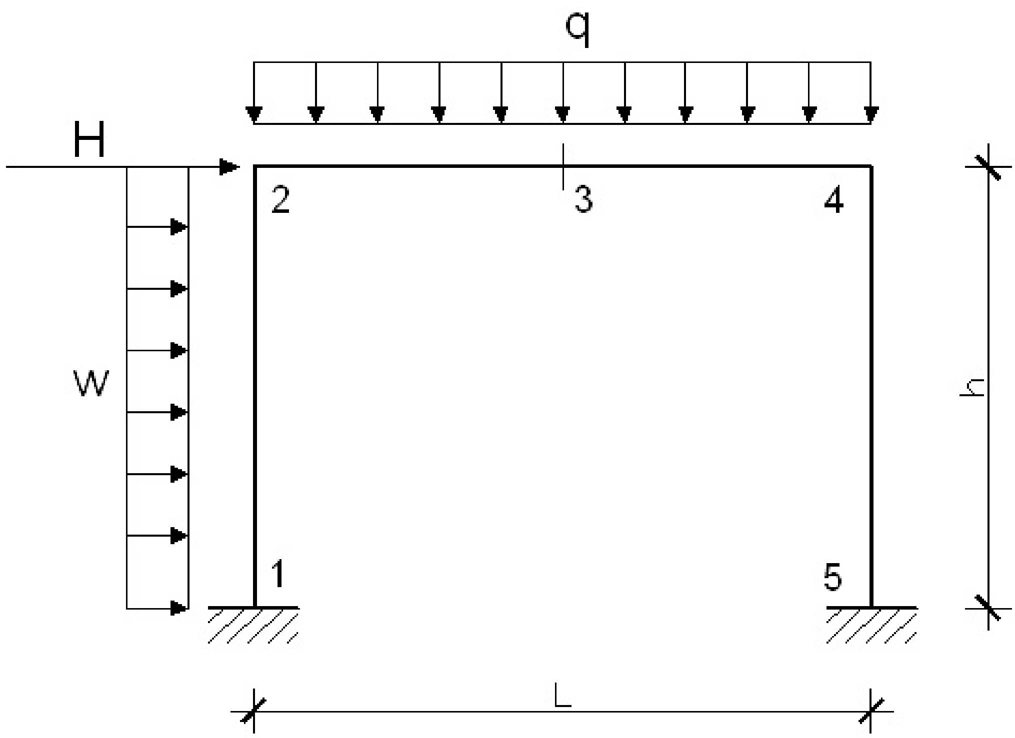
$$5448.0712\cdot\varphi_2+5447.2622\cdot\varphi_3+3631.7778\cdot u-18.1=0 \qquad (3)$$

$\varphi_2 =$ -0.000497085
$\varphi_3 = 0.0173018$
u = 0.0316857

Momentni dijagram okvira:



5.0 STABILNOST OKVIRA



a = 5 m
L = 10 m
h = 6 m
$q = 2 \cdot a = 2 \cdot 5 = 10\ kN/m$
$H = 0.1 \cdot q \cdot L = 0.1 \cdot 10 \cdot 10 = 10\ kN$
V = 400 kN
$w = 0.45 \cdot 1.2 \cdot a = 0.45 \cdot 1.2 \cdot 5 = 2.7\ kN/m$
$E = 2.8 \cdot 10^8\ kN/m^2$

Stupovi i greda IPE 300:

$A = 84.46\ cm^2$

$I_y = 23130\ cm^4$

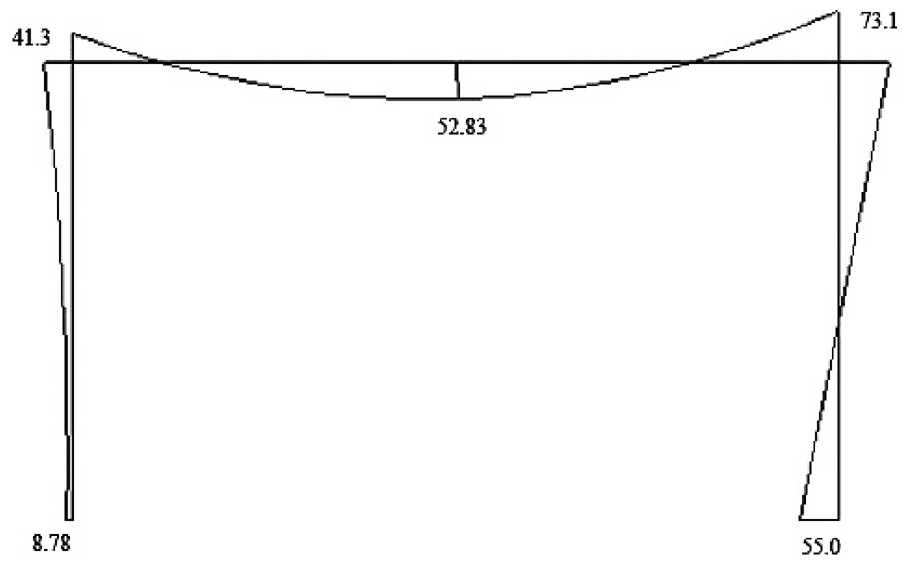
$W_y = 557\ cm^3$

$S255 \rightarrow \sigma_F = 255\ N/mm^2$
$M_{pl} = 142\ kNm$

Potrebno je odrediti koeficijent α kojim je moguće uvećati opterećenje okvira na slici, a da okvir zadrži stabilnost.

5.1 Momentni dijagram početnog sustava

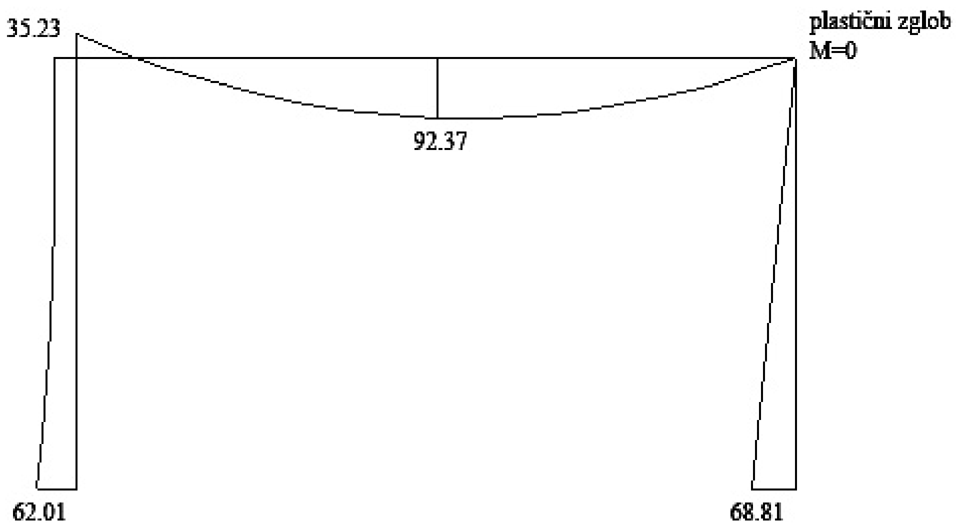
Iz momentnog dijagrama početnog sustava uzmemo najveći moment kako bismo dobili koeficijent α_1.



$$\alpha_1 = \frac{142}{73.1} = 1.94254$$

5.2 Momentni dijagram istog okvira kojemu je u treći čvor umetnut zglob

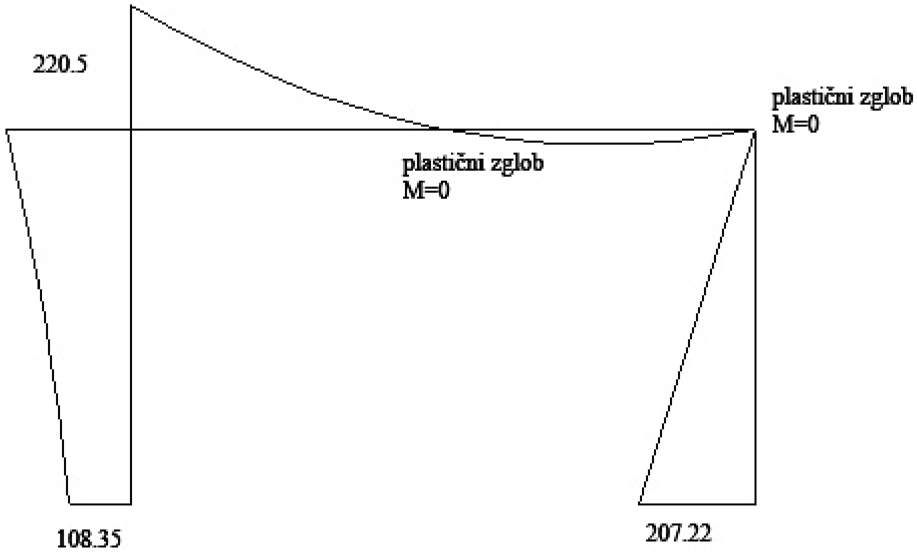
Zglob se umeće u četvrti čvor upravo zato jer je u njemu najveći moment koji će povećanjem za izračunati koeficijent α_1 stvoriti u tom čvor plastični zglob.



$$\alpha_2 = \frac{142 - \alpha_1 \cdot M_1}{M_2} = \frac{142 - 1.94254 \cdot 52.83}{92.37} = 0.4263$$

5.3 Momentni dijagram istog okvira kojemu je i u treći i četvrti čvor umetnut zglob

Sada stavljamo zglob u čvor u kojemu je u prethodnom dijagramu najveći moment savijanja, jer će se na tome mjestu stvoriti slijedeći plastični zglob.



$$\alpha_3 = \frac{142 - \alpha_1 \cdot M_1 - \alpha_2 \cdot M_2}{M_3} = \frac{142 - 1.94254 \cdot 41.3 - 0.4263 \cdot 35.23}{220.5} = 0.2120$$

5.4 Dosegnuta granica stabilnosti

Pojavljivanjem plastičnog zgloba u sljedećem kritičnom čvoru (čvor 2) dosegnuta je granica stabilnosti jer sistem postaje mehanizam.

Ukupni koeficijent α :

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 = 1.94254 + 0.4263 + 0.2120 = 2.58084$$