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Affine transformations of three-hinged arches: reviving an old method

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Abstract

This paper revives an almost forgotten graphical procedure for drawing bending moment diagrams for three-hinged and related systems and presents a generalization of the superposition method for general (i.e., not only vertical) loading. In the first part of the paper, the superposition method for drawing bending moments for three-hinged systems for vertical loads is described. The final bending moment diagram is obtained by a graphical subtraction of two bending moment diagrams: the first, constructed for a simply supported beam with the same span as the three-hinged system and the second, determined by an affine transformation of the three-hinged system axis. In the second part of the paper generalization of the superposition method for general loading is presented. Two interpretations of the graphical construction of the second bending moment diagram are shown: "pure" projective-geometrical interpretation and mechanical interpretation which explains the relation between the geometrical and structural notions and the procedural steps.

Keywords: three-hinged arch, superposition method, affine transformation, homology, graphic statics

1. Introduction

The superposition procedure for drawing bending moment diagrams for three-hinged and related systems by using the projective-geometric construction of bending moment diagram $M^H = H \cdot h$, which will be described, was brought into lectures on Structural Analysis at Faculty of Civil Engineering, University of Zagreb, in the middle of 1950s, when professor Otto Werner took over those lectures. The procedure is described in a textbook written by Veselin Simović [18]. Although the superposition of moment diagrams itself is relatively often described in literature, especially in the older one, we did not succeed in finding the roots of the projective-geometric construction neither in the earlier books nor in the later. In the early books on graphic statics, such as Culmann [6], Chalmers [2], Föppl [7], Ritter [16], and Wolf [19], thrust lines and influence lines are described when solving arches. Müller-Breslau [13] states the expression

$$M(x) = M^0(x) - H \cdot h(x), \tag{1}$$

on which the superposition procedure is based, but does not describe the procedure for drawing the diagram of moments

$$M^H(x) = H \cdot h(x). \tag{2}$$

A similar state of affairs is with many of the later books from German speaking area in which the tradition of graphic statics has been preserved for the longest: e.g., Chmelka and Melan [4], Hirschfeld [9], Sattler [17]. In the later books, the superposition procedure is less often mentioned; it is described in Karnovsky and Lebed [11], and in Karnovsky [10] it is used in finding the optimal arch shape (the shape in which the bending moments do not exist). The superposition procedure has not been mentioned either in papers or in monographs on the history of structural analysis with the chapters dedicated to graphic statics, such as Carlton [3], Gerhardt et al. [8], Kurrer [12] and Boothby [1].

The method is simple and versatile and therefore it can be used for quick verification of analytic or numeric results. From the educational standpoint, it gives a vivid insight into the behaviour of three-hinged and tied systems. For the practical application, it is interesting that the idea, on which the method is based, as it is shown in Karnovsky [10], can be used in shape optimization (similar to drawing a thrust line).

In the first part of the paper "classic" application of superposition method for drawing bending moment diagrams for three-hinged systems exposed to vertical loads is described, where diagram M^H is drawn by an affine transformation (homology with centre at infinity, Cremona [5]) of the axis of an arch or frame. In the second part of the paper generalization of the superposition method for general (i.e., not only vertical) loading is presented.

2. Superposition method for vertical loading

The "classic" superposition method is used for drawing bending moment diagrams for three-hinged systems loaded by vertical loads (Müller-Breslau [13], Chmelka and Melan [4], Hirschfeld [9], Sattler [17]). Reactions at supports are decomposed into vertical components A^0 and B^0, and components H' acting on the line which connects supports (Figure 1.a). It is easy to demonstrate that the vertical components of the reactions are equal to the reactions of the simply supported beam whose span is the same as the horizontal distance between supports of the three-hinged system, and which is loaded with the same forces at points with the same abscissas. If the forces are vertical, shape of the axis of the substitute beam is not relevant and therefore it is usually taken as straight and horizontal (Figure 1.b).

Comparison of the expressions for the bending moment value $M(x)$ in a section of three-hinged system with abscissa x and for the bending moment value $M^0(x)$ in a corresponding section of the substitute beam shows that we can write the expression (1) where H is horizontal component of the force H' and $h(x)$ is the vertical distance between the point and the line which connects supports of the three-hinged system. Thus, $H{\cdot}h(x)$ are bending moments which are imposed in the arch/frame by the reaction component on the support connecting line of the three-hinged system. These moments do not exist in the beam with straight horizontal axis, but they exist in the simply supported beam whose axis has the same shape as the axis of the three-hinged system, loaded by the force H' acting on the line connecting its supports (Figures 1.e and f.). Using (2) we can write

$$M(x) = M^0(x) - M^H(x). \tag{3}$$

It follows that the diagram of bending moments M can be drawn by subtracting diagram M^H (Figure 1.g) from diagram M^0 (Figure 1.c). The diagram M^H is not drawn on the system axis, but on its projection on the horizontal by lapping over the diagram M^0; by overlapping the diagrams, we can graphically subtract them (Figure 1.d). The final values of moments are read along vertical lines from the line of diagram M^H to the line of diagram M^0; in other words, the line of diagram M^H is considered as the reference line of the final diagram. Lastly, the final diagram can be drawn on the axis of the arch or frame by tracing the values of the moments perpendicularly to the axis (Figure 1.h).

When drawing a bending moment diagram, points of the axis of the structure are mapped into points which determine values of bending moments in these points. Procedures for drawing diagrams of bending moments M^0 in simply supported beams are well known so only the procedure for graphical construction of the diagram of bending moments M^H is described. For this construction, a geometric interpretation of the right-hand side of the expression (2) is needed. As seen in Figure 2, multiplication by the number H means extension (if $H > 1$) or contraction (if $0 < H < 1$) of the length $h(x)$ so that the point with "coordinates" (x, $h(x)$) is assigned to the one with coordinates (x, $H{\cdot}h(x)$). (The quotation marks are used because the heights are not measured from the *x* axis but from a chosen inclined line.) Points on a straight line are mapped into points on (another) straight line.

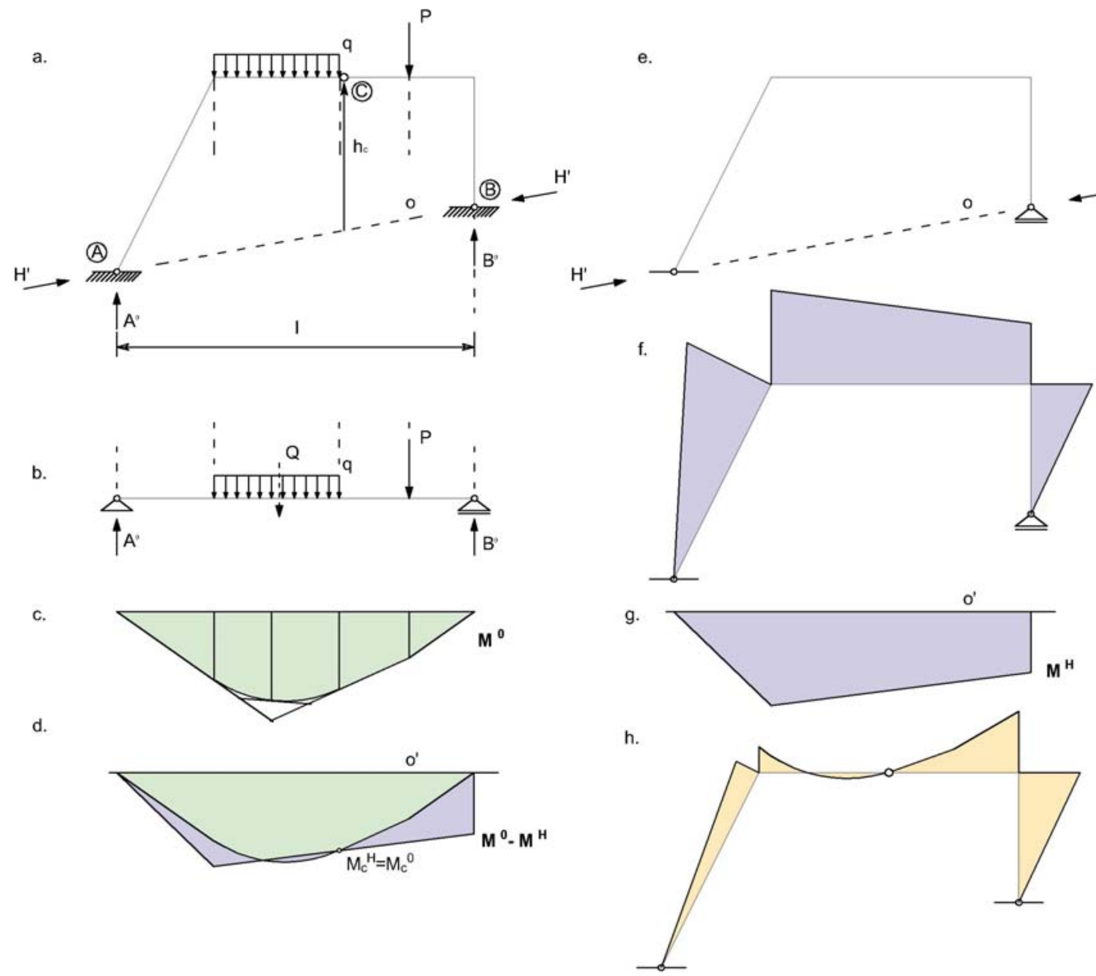

Figure 1: Solving the three-hinged system exposed to vertical loads

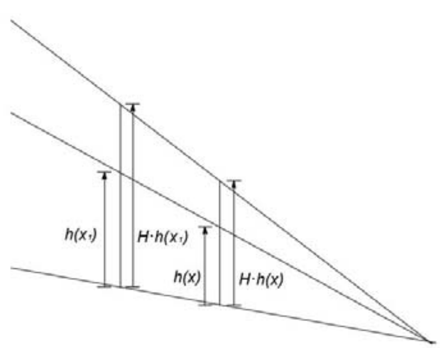

Figure 2: Affinity in plane

As the moment diagram is drawn below the arch in a way that the value of a bending moment at some point of the axis is drawn beneath it, lines through corresponding points are parallel and vertical. At two hinges located at the supports, values of moments are zero. The point on the location of the central hinge has the value of the moment $M_C^0 = M^0(x_C)$, where x_C is the abscissa of the central hinge. Namely, since the value of the moment in the central hinge of three-hinged system is zero, the expression (3) becomes

$$M(x_C) = M^0(x_C) - M^H(x_C) = 0, \tag{4}$$

wherefrom it follows that

$$M^H(x_c) = M^0(x_c). \tag{5}$$

Therefore, three pairs of corresponding points are known. Although $H = M^0(x_C)/h(x_C)$ can be calculated from the expressions (1) and $M(x_C) = 0$, that value is not needed; diagram M^H can be constructed directly by using the collineation. (Superposition procedures for drawing diagrams of moments M^H as shown in Müller-Breslau [13], Chmelka and Melan [4], Hirschfeld [9], Sattler [17] use the values calculated from the expression (2) after the value H is computed.)

A collineation in plane is uniquely determined when four pairs of corresponding points are given. A special type of collineation in which all lines which connect pairs of corresponding points pass through the same point is called perspective collineation or homology in plane. The point of intersection of lines through pairs of corresponding points is its centre. In this case, it is enough to know three pairs of corresponding points (fourth “pair” is the centre, which is a self-corresponding point). If the lines through pairs of corresponding points are parallel, the centre of homology is a point at infinity and the homology is called affinity in plane (Cremona [5], Niče [14]).

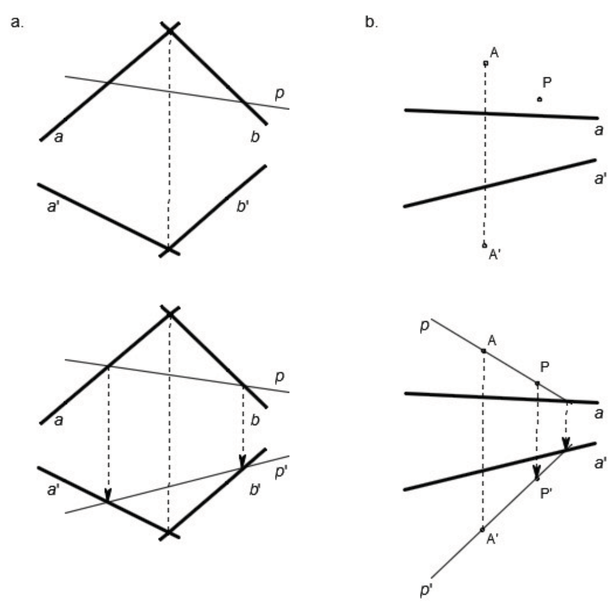

Figure 3: Affine images of a line and a point

In collineation, if a line contains a point, then the corresponding line contains the corresponding point. Therefore, to find the line *p′* corresponding to some line *p*, firstly the intersection points with any two lines *a*, *b* for which corresponding lines *a′*, *b′* are known, must be determined. Then, points which correspond to intersection points are determined on lines a' and b'. The line connecting these corresponding points is p' (Figure 3.a). If line p already contains the point with a known corresponding point, only one more point must be determined. To find the point P' corresponding to some point P, we can connect P and the point A with the known corresponding point A', then determine the line corresponding to the connecting line by using the intersection with some other line a and, on this corresponding line, determine the point P' (Figure 3.b). (Direct connection to the second interpretation of the method is obtained if the line connecting supporting hinges and the axis of the bending moment diagram are taken as known corresponding lines.)

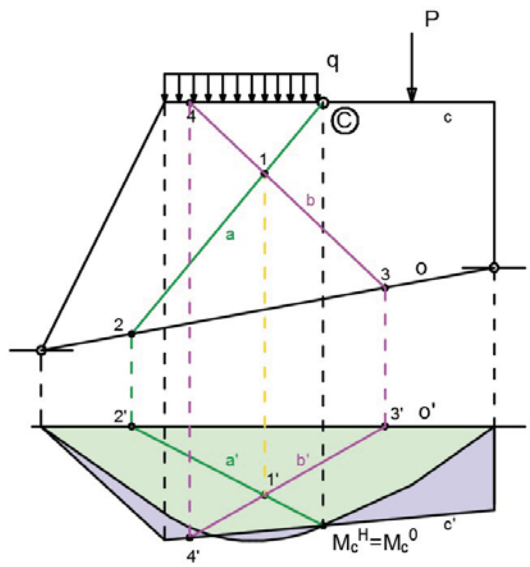

Figure 4: Construction of the affine image of the frame

Procedures described in previous paragraph can be applied to the construction of the diagram M^H as the affine image of the three-hinged arch or frame. Diagram M^H in Figure 1.g is composed of parts of the lines which represent lines corresponding to the axes of elements of the three-hinged frame shown in Figure 1.a. The construction of the affine image of the frame is shown in Figure 4. The line c containing the axis of the horizontal part of the frame passes through the hinge C, so according to the expression (5), the line c' corresponding to the line c has to pass through the point (x_C, M_C^0) corresponding to the hinge C. However, the intersection point of the line c and the line o connecting remaining two points whose corresponding points are known (i.e., the line which connects two hinged supports), lays out of drawing margins. Still, we can find the line c'. First, we draw the line a through the point C and an arbitrarily chosen point 1. The corresponding line a' lays between the point 2' which corresponds to the intersection point 2 of lines a and o and the point (x_C, M_C^0). Next, through the point 1 and an arbitrary point 4 on the line c line b is drawn and the intersection 3 with o is found as well as its corresponding point 3'. Since the line b' is defined by points 3' and 1', the point 4' corresponding to the point 4 can be easily found. With the point 4' and (x_C, M_C^0), the line c' corresponding to the line c is determined. The axes of the remaining two elements of three-hinged frame pass through points with known corresponding points (i.e., supports) and intersect the line c so lines corresponding to them can be drawn immediately.

Homology in plane and, as a special case, affinity in plane are usually defined as plane collineations with the centre (self-corresponding point) and axis (self-corresponding line on which all points are self-corresponding). The centre is the common intersection point of all lines through pairs of corresponding points (these lines are also self-corresponding, but only two of their points are self-corresponding: the centre and the point of intersection with axis). Existence of an axis is ensured by the Desargues' theorem: If two triangles ABC and $A'B'C'$, lying in the same plane, are such that the straight lines AA', BB', CC' meet in the same point O, then the three points of intersection of the sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ lie on the same straight line o (Figure 5). Points of intersection of corresponding lines are obviously self-corresponding points. The projectivity on line is determined when three pairs of corresponding points are given and thus if three points are self-corresponding, projectivity is identity – all other points are self-corresponding too. Accordingly, in homology and in affinity all intersection points of corresponding lines lie on the same line, the axis.

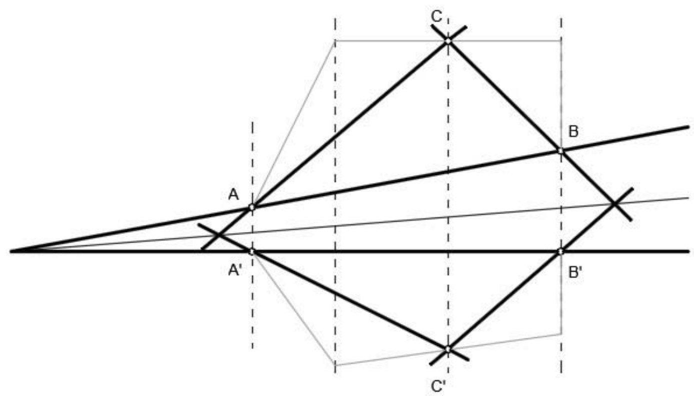

Figure 5: Desargues' theorem (special case: O is a point at infinity)

If the axis of affinity is known, the affinity is determined when a pair of corresponding points is given. We will use this form of determination of the affinity to give a static interpretation to the geometric construction.

Expression (2) defines the affine transformation of points

$$(x, h(x)) \to (x, M^H(x)). \tag{6}$$

which maps the point laying on the axis of the system (or generally, any point in the field called the frame field) to the point at the moment diagram (or generally, the point in the field called the moment field).

As $H' \cdot 0 = 0$, force H' does not impose moments on the line on which it acts. In each example H is a constant, but it can be any value; therefore $h(x) = 0$ is the only number which gives $H \cdot h(x) = h(x)$. We conclude that the axis of affinity is the line on which the force H' acts, thus the line which connects hinged supports of the system. (It should be noted that this axis differs from the one determined in Figure 5.) This line should be also the axis (reference line) of the moment diagram, because of $M^H(x) = H \cdot 0 = 0$. However, moment diagram is usually drawn from some other, distinct, generally horizontal line. Therefore, we perform the procedure of the affine transformation on the drawing where the line which connects supports of the three-hinged system and the axis of the moment diagram are separated (Simović [18]). Formally, this is a composition of affine transformation and "shear deformations" (also a kind of affinity) by which we separate the axis of moment diagram and at the same time bring the axis in a horizontal position (Figure 6).

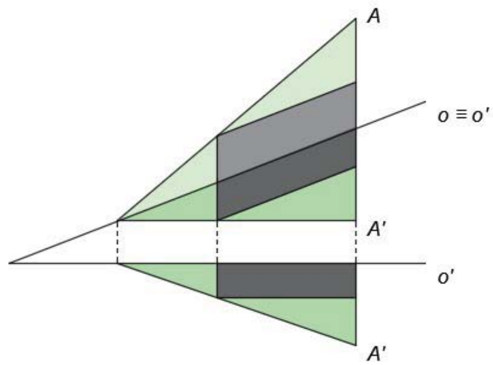

Figure 6: Composition of affinity and shear deformation

The graphical construction is the same as described above, but to some steps a different interpretation is given. The line *o* which connects two supports of the three-hinged frame in Figure 4 represents the axis of affinity in the frame field, and the reference line o' of the moment diagram represents the axis of affinity in the moment field. The known pair of corresponding points is, of course, *C* and (x_C, M_C^0).

The superposition method for vertical loading can be implemented as a "pure" graphical method, where the diagram M^0 is drawn as a funicular polygon in an appropriate scale (in fact, diagram M^0 is an affine image of funicular polygon), or as a combined analytical-graphical method, where the diagram M^0 is drawn by using calculated values. The construction of diagram M^H in both methods is graphical.

Prakash Rao in [15], one of the rare recent books completely dedicated to graphic statics, describes "inverted" superposition method where the diagram M^0 is drawn over the contour of the system which is interpreted as the diagram M^H. That version of the procedure is also mentioned in Sattler [17].

Unmodified, the procedure can be used for loads acting on parallel lines with any orientation. The substitute beam is placed perpendicularly to the lines of action, and the roller support is such that the support reaction is parallel to the forces. There is a limitation, as in the case of vertical forces, that lines of action of individual forces are not allowed to intersect the axis of the arch/frame in more than one point.

3. Generalization of the method

In the case of vertical loading, part of the expression for bending moments in three-hinged arches and frames without M^H is equal to the expression for bending moments in the simply supported beam with the straight horizontal axis. This latter expression is also equal to the expression for bending moments in the simply supported beam with the same axis as the three-hinged system. We used straight horizontal beam as substitute system because it is "simpler" than the polygonal or curved beam. As all forces are vertical, their distances from certain points are measured along horizontal lines.

However, if the forces are horizontal or inclined, the three-hinged system cannot be replaced with a straight horizontal beam, because the heights of their application points are lost and with them the moments of horizontal forces or horizontal components of inclined forces. But if the simply supported beam with the same axis as the three-hinged system is taken as a substitute system, spatial relations remain unchanged. Horizontal components of reactions in supports of the three-hinged system are not equal, and in addition, there is a horizontal component of reaction in the hinged support of substitute beam.

When the left support of the polygonal or curved substitute beam is hinged (and the right support is a horizontally moving roller), the expression for bending moment is

$$M(x) = M_l^0(x) - (A_h - A_h^0) \cdot h(x) \tag{7}$$

where M_l^0 is the expression for bending moments in substitute beam with left hinged support, A_h^0 the horizontal component of reaction in that support and A_h the horizontal component of reaction in the left support of three-hinged system (positive orientation of horizontal components is supposed to be from the left to the right). If the right support is hinged, the expression for bending moments is

$$M(x) = M_d^0(x) - A_h \cdot h(x) \tag{8}$$

where M_d^0 is the expression for bending moment in substitute beam with right hinged support. The expression for M_d^0 is a bit simpler, because reaction in the left support is vertical, while the expression for M_l^0 additionally contains A_h^0. However, there is no need to know the value A_h for the graphical construction of diagrams $M_l^H(x) = (A_h - A_h^0) \cdot h(x)$ and $M_d^H(x) = A_h \cdot h(x)$. A disadvantage of this generalization is the impossibility of drawing diagram M^0 as a funicular polygon. The diagrams M_l^H and M_d^H are, however, drawn as described above, as affine images of the axis of the arch or frame.

For the system shown in Figure 7.a a substitute simply supported polygonal beam with left hinged support and its corresponding diagram M_l^0 drawn on an axis of the system are shown in Figure 7.b. For the implementation of the superposition procedure, the diagram is reduced to the horizontal axis, as shown in lower part of Figure 7.c. In the same figure diagram M_l^H graphically constructed as an affine image of the axis of the system is shown and the graphical subtraction is conducted. (This part of the procedure is the same as in the procedure with vertical loading.) Substitute simply supported polygonal beam with right hinged support and its corresponding diagram M_d^0 drawn on axis of the system are shown in Figure 7.e, and a superposition procedure with the affine construction of diagram M_d^H is shown in Figure 7.f. The final moment diagram, drawn on axis of the system is, of course, same for both options of the substitute beam: Figure 7.d.

4. Concluding remarks

In this paper, we revive an almost forgotten graphical procedure for drawing bending-moment diagrams for three-hinged arches and frames for vertical loads. It is a variant of superposition method in which the bending-moment diagram of a three-hinged system is drawn according to the expression (1) where a diagram of moments $M^H = H \cdot h$ is drawn by affine transformation of the axis of the arch or frame. Two interpretations of this graphical construction are shown: "pure" projective-geometrical interpretation and mechanical interpretation which explains the relation between the geometrical and structural notions and the procedural steps. The second interpretation of the procedure is suitable for solving tied arches and frames. It is clear that the unmodified procedure can be used if one of the hinged supports is replaced by a roller support and a tie is placed in the line which connects; the tie can also be raised above the connecting line. Furthermore, a single tie can be replaced by a system of ties, where different ties define different affine images, so the interpretation of ties as the axes of affinity seems natural.

Generalization of the superposition method for general – vertical and horizontal – loading, described in the second part of the paper, is also applicable to the tied system solving.

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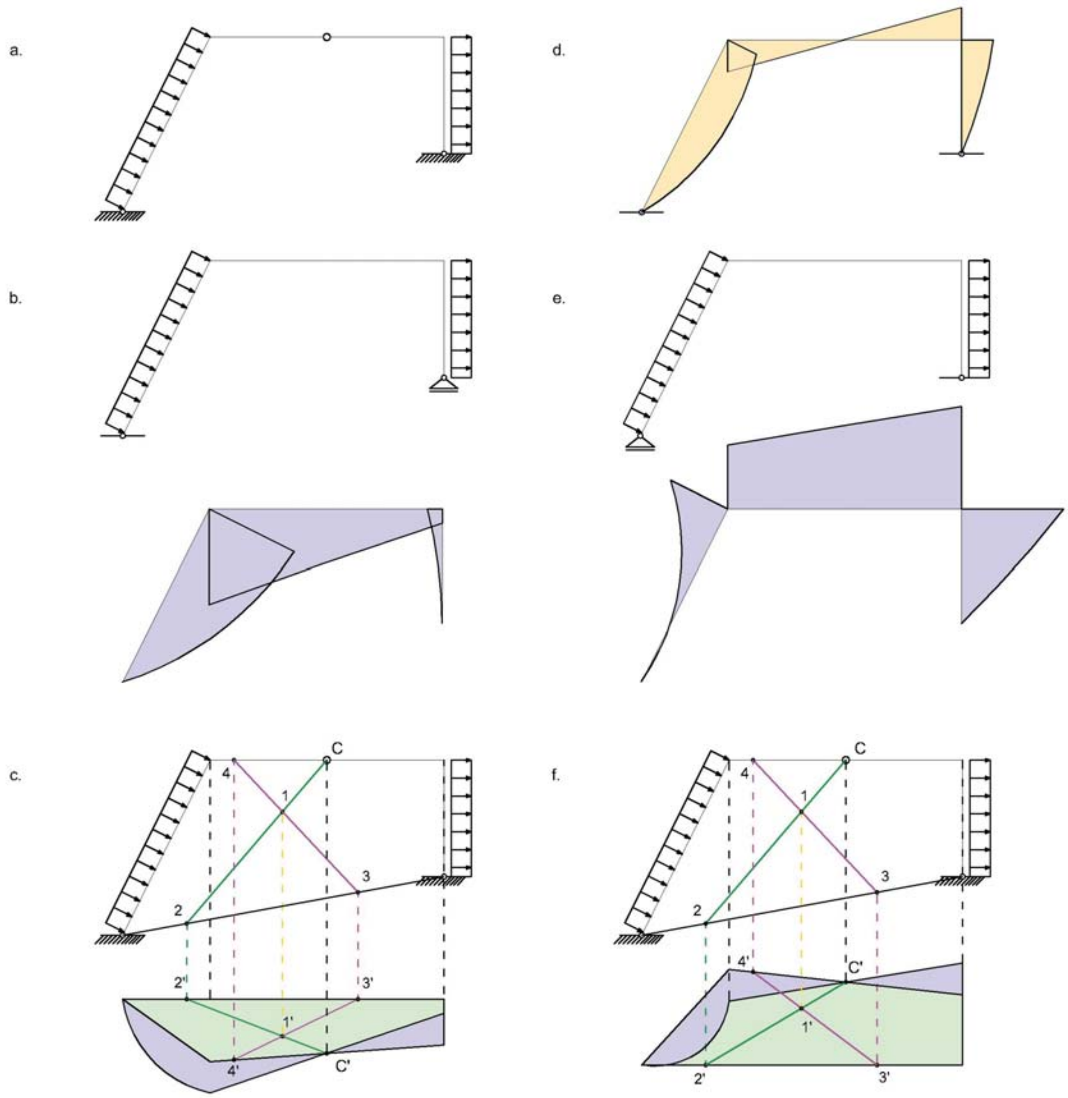

Figure 7: Solving the three-hinged system exposed to general loads

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