Sveučilište u Zagrebu

Građevinski fakultet

Matematika 1
Vježbe

Zagreb, rujan 2020.

Predgovor

Poštovani čitatelji, pred vama se nalazi nastavni materijal sa vježbi za kolegij *Matematika 1* koji se predaje u prvom semestru Preddiplomskog studija na Građevinskom fakultetu Sveučilišta u Zagrebu. Ova zbirka zadataka je u nastajanju, nije gotova niti recenzirana, pa molim studente koji će je koristiti, ako uoče neku pogrešku, da mi jave na e-mail: luka.podrug@grad.unizg.hr.

Neke slike su ujedno i animacije koje se pokreću klikom. Za korištenje ove opcije, sadržaj se mora čitati u programu koji to podržava (npr. Adobe Acrobat Reader ili Foxit Reader).

Sadržaj

Poglavlje 1

Vektori

1.1 Osnovno o vektorima

Skalarne veličine - one veličine određene jednim brojem (npr. duljina, površina, masa, toplina...)
Vektori - za silu, akceleraciju je osim broja potrebno znati i smjer.

Orijentirana (usmjerena) dužina $\overrightarrow{AB}$ je dužina za koju se zna početna točka A i završna točka B.

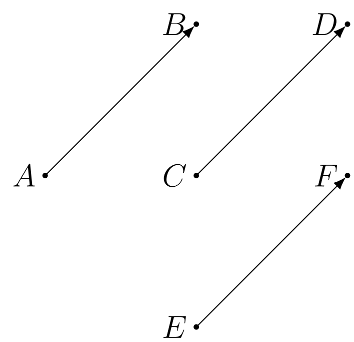
Geometrijski, usmjerena dužina je zadana:

1. **duljinom** ili modulom
2. pravcem nosiocem na kojem usmjerena dužina leži, tj. **smjerom**
3. **orijentacijom**

Za dvije usmjerene dužine kažemo da imaju isti smjer ako leže na istom ili na paralelnim pravcima. O orjentaciji dviju usmjerenih dužina ima smisla govoriti samo ako imaju isti smjer.

Definicija 1.1. Za usmjerene dužine kažemo da su ekvivalentne ako imaju istu duljinu, smjer i orijentaciju. Skup svih međusobno ekvivalentnih usmjerenih dužina nazivamo **vektorom**.

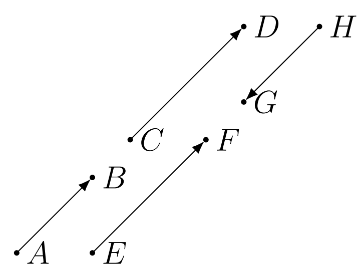
Slika (1.1) prikazuje tri ekvivalentne usmjerene dužine.



Slika 1.1: $\vec{AB} = \vec{CD} = \vec{EF}$ su ekvivalentne usmjerene dužine.

Primjer 1.1. Za vektore $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$ i $\overrightarrow{GH}$ prikazane na slici (1.4) vrijedi:

- $|\overrightarrow{AB}| = |AB|$
- $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}, \overrightarrow{GH}$ su vektori istog smjera jer leže na istom ili na paralelnim pravcima.
- $\overrightarrow{AB}, \overrightarrow{CD}$ i $\overrightarrow{EF}$ su iste orijentacije.
- $\overrightarrow{AB}$ i $\overrightarrow{GH}$ su suprotne orijentacije.



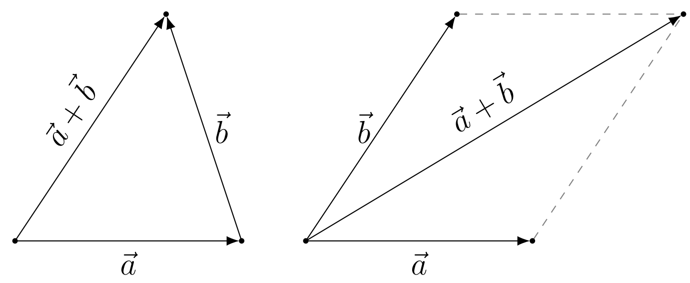
Slika 1.2: Primjer vektora istog smjera i suprotne orijentacije.

Dva su vektora **jednaka** ako imaju istu duljinu, smjer i orijentaciju.

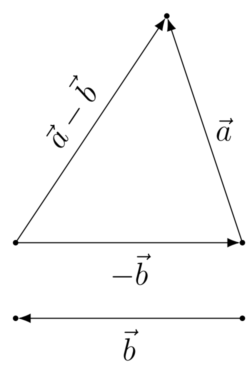
Vektor kojemu su početna i krajnja točka iste nazivamo nulvektor. Npr., $\overrightarrow{AA} = \overrightarrow{BB} = \overrightarrow{0}$.

Vektore možemo zbrajati pravilom paralelograma ili pravilom trokuta.

Vektor $\vec{b}$ oduzimamo od vektora $\vec{a}$ tako da vektoru $\vec{a}$ dodamo suprotni vektor vektora $\vec{b}$.



Slika 1.3: Zbrajanje vektora pravilom trokuta i pravilom paralelograma.



Slika 1.4: Oduzimanje vektora. Vrijedi $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

Svojstva:

1. $\overrightarrow{a}+\overrightarrow{b}=\overrightarrow{b}+\overrightarrow{a}$
2. $\overrightarrow{a}+(\overrightarrow{-a})=\overrightarrow{-a}+\overrightarrow{a}=\overrightarrow{0}$
3. $\overrightarrow{a}+\overrightarrow{0}=\overrightarrow{0}+\overrightarrow{a}=\overrightarrow{a}$
4. $(\overrightarrow{a}+\overrightarrow{b})+\overrightarrow{c}=\overrightarrow{a}+(\overrightarrow{b}+\overrightarrow{c})$

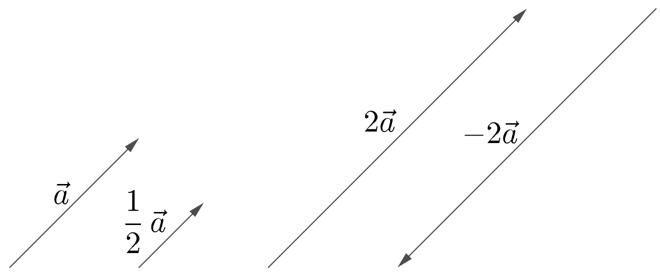
Množenje vektora skalarom

Neka je $\overrightarrow{a}$ vektor i $\lambda\in\mathbb{R}$.

1. $\overrightarrow{a}$ i $\lambda\overrightarrow{a}$ su kolinearni.
2. $|\lambda\overrightarrow{a}|=|\lambda||\overrightarrow{a}|$
3. Ako je

(a) $\lambda > 0$, tada su $\overrightarrow{a}$ i $\lambda\overrightarrow{a}$ iste orijentacije.

(b) $\lambda < 0$, tada su $\overrightarrow{a}$ i $\lambda\overrightarrow{a}$ suprotne orijentacije.



Svojstva:

1. $\lambda\left(\overrightarrow{a}+\overrightarrow{b}\right)=\lambda\overrightarrow{a}+\lambda\overrightarrow{b}$
2. $(\lambda+\mu)\overrightarrow{a}=(\lambda\overrightarrow{a})+\mu\overrightarrow{a}$
3. $(\lambda\mu)\overrightarrow{a}=\lambda(\mu\overrightarrow{a})=\lambda\mu\overrightarrow{a}$
4. $1\cdot\overrightarrow{a}=\overrightarrow{a},(-1)\cdot\overrightarrow{a}=-\overrightarrow{a},0\cdot\overrightarrow{a}=\overrightarrow{0}$

Prikaz vektora u koordinatnom sustavu

$\vec{i},\vec{j},\vec{k}$...jedinični vektori na koordinatnim osima

$$\begin{aligned}\vec{a} &= \overrightarrow{AB}\\ &= a_x\vec{i}+a_y\vec{j}\\ &= (x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}\end{aligned}$$

$$|\vec{a}| = \sqrt{a_x^2+a_y^2}$$

$a_x = x_B - x_A$ i $a_y = y_B - y_A$...skalarne komponente vektora

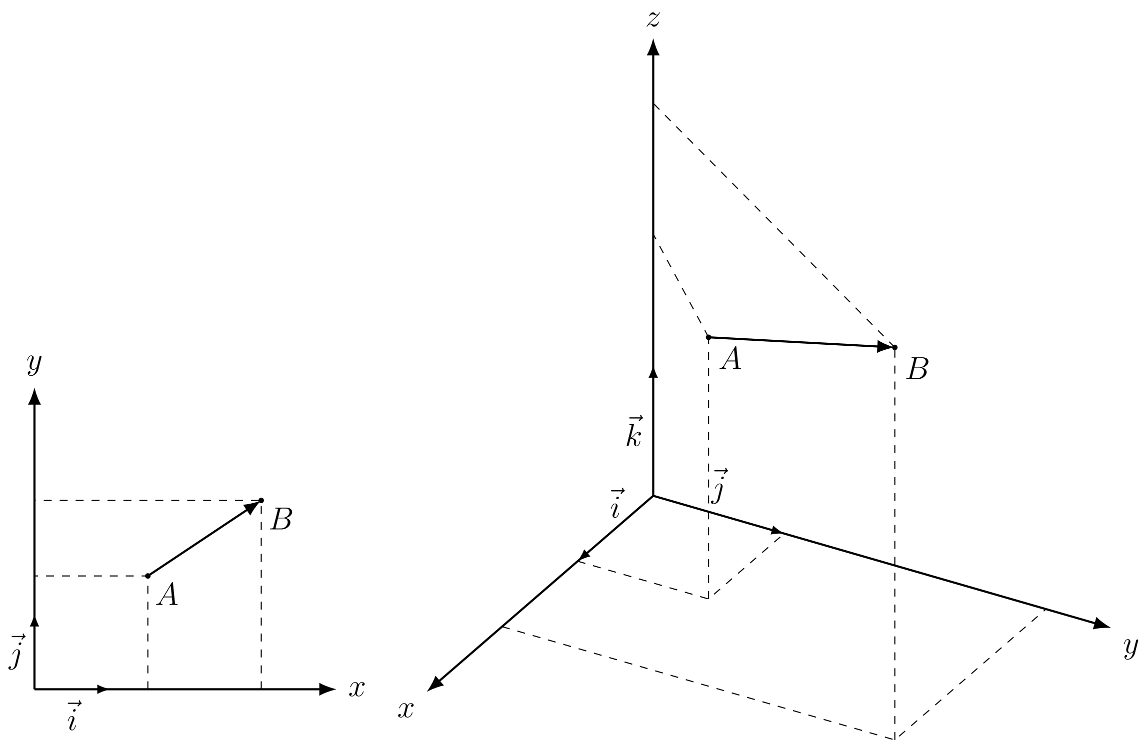
$\vec{i},\vec{j}$...vektorske komponente vektora

$\vec{a}=\overrightarrow{AB}=(x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}+(z_B-z_A)\vec{k}$

$|\vec{a}|=\sqrt{a_x^2+a_y^2+a_z^2}$

Zadatak 1.1. Odredite $\overrightarrow{AB}$ i $|\overrightarrow{AB}|$ ako je:

a) $A(-1,1),B(2,3)$



Slika 1.5: Prikaz vektora u koordinatnom sustavu.

b) $A(1,1,1), B(4,5,7)$

Rješenje:

a)
$$\begin{aligned}\vec{a} &= \overrightarrow{AB} \\ &= (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} \\ &= 3\vec{i} + 2\vec{j}\end{aligned}$$

$$\begin{aligned}|\vec{a}| &= \sqrt{a_x^2 + a_y^2} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13}\end{aligned}$$

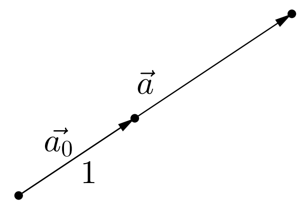
b)
$$\begin{aligned}\vec{a} &= \overrightarrow{AB} \\ &= (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k} \\ &= 3\vec{i} + 4\vec{j} + 6\vec{k}\end{aligned}$$

$$\begin{aligned}|\vec{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ &= \sqrt{3^2 + 4^2 + 6^2} \\ &= \sqrt{61}\end{aligned}$$

□

Jedinični vektor vektora $\vec{a}$ je vektor $\vec{a_0}$...vektor istog smjera i orijentacije kao $\vec{a}$, a duljine 1.

$$\vec{a_0} = \frac{\vec{a}}{|\vec{a}|}$$



Zadatak 1.2. Dan je vektor $\vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}$. Odredite $\vec{a_0}$.

Rješenje:

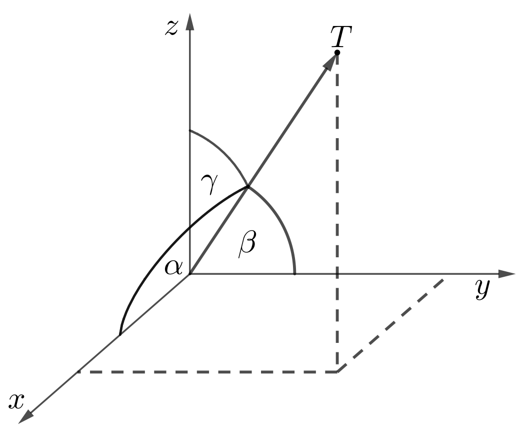
$$\begin{aligned}
|\vec{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\
&= \sqrt{2^2 + 2^2 + 1^2} \\
&= \sqrt{9} = 3
\end{aligned}$$

$$\begin{aligned}
\vec{a_0} &= \frac{\vec{a}}{|\vec{a}|} \\
&= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}
\end{aligned}$$

□

Za vektor $\vec{a_0} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$ iz zadatka **??**, skalari uz $\vec{i}, \vec{j}$ i $\vec{k}$ su kosinusi smjera tj.

$$\cos\alpha = \frac{2}{3}, \cos\beta = \frac{2}{3}, \cos\gamma = \frac{1}{3}.$$



Zadatak 1.3. Ako je $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}, \vec{b} = \vec{i} - \vec{j} + 2\vec{k}$ izračunajte $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{a}$, $|\vec{a}|$, $\vec{a_0}$, te kosinuse smjera od $\vec{a}$.

Rješenje:

$$\begin{aligned}
\vec{a} + \vec{b} &= 2\vec{i} + \vec{j} + 3\vec{k} \\
\vec{a} - \vec{b} &= 3\vec{j} - \vec{k}
\end{aligned}$$

$$
\begin{aligned}
3\vec{a} &= 3\vec{i} + 6\vec{j} + 3\vec{k} \\
|\vec{a}| &= \sqrt{1+4+1} \\
&= \sqrt{6} \\
\vec{a_0} &= \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k} \\
&= \frac{\sqrt{6}}{6}\vec{i} + \frac{\sqrt{6}}{3}\vec{j} + \frac{\sqrt{6}}{6}\vec{k}
\end{aligned}
$$

Kosinusi smjera su: $\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}$ i $\frac{\sqrt{6}}{6}$.

$\square$

Zadatak 1.4. Dane su točke $A(2,2,0)$ i $B(0,-2,5)$. Pronađite $\overrightarrow{AB}$, $|\overrightarrow{AB}|$ i $\overrightarrow{AB}_0$.

Rješenje:

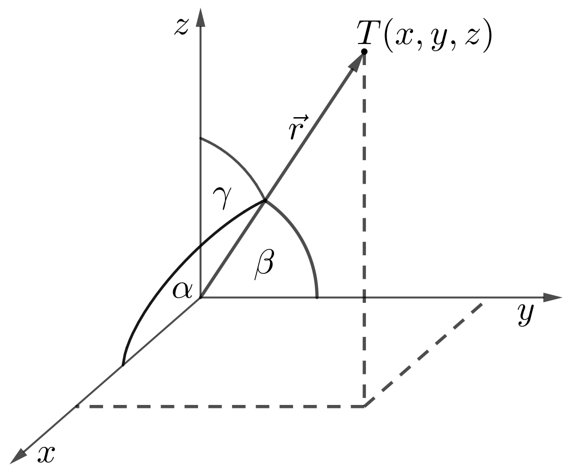
$$
\begin{aligned}
\overrightarrow{AB} &= -2\vec{i} - 4\vec{j} + 5\vec{k} \\
|\overrightarrow{AB}| &= \sqrt{4+16+25} \\
&= \sqrt{45} \\
&= 3\sqrt{5} \\
\overrightarrow{AB}_0 &= \frac{-2\sqrt{5}}{15}\vec{i} - \frac{4\sqrt{5}}{15}\vec{j} + \frac{\sqrt{5}}{3}\vec{k}
\end{aligned}
$$

$\square$

Radij-vektor je vektor s početnom točkom u ishodištu.

$\vec{r} = \overrightarrow{OT} = x\vec{i} + y\vec{j} + z\vec{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$...duljina radij-vektora



Kosinusi smjera radijvektora

$\alpha = \angle(\vec{r}, \vec{i}), \beta = \angle(\vec{r}, \vec{j}), \gamma = \angle(\vec{r}, \vec{k})$

$$\cos\alpha = \frac{x}{|\vec{r}|}, \cos\beta = \frac{y}{|\vec{r}|}, \cos\gamma = \frac{z}{|\vec{r}|}$$

Zadatak 1.5. Odredite duljinu vektora $\vec{a} = 20\vec{i} + 30\vec{j} - 60\vec{k}$ i kosinuse smjera tog radij-vektora.

Rješenje:

$$\begin{aligned} |\vec{a}| &= \sqrt{20^2 + 30^2 + 60^2} \\ &= \sqrt{400 + 900 + 3600} \\ &= \sqrt{4900} \\ &= 70 \end{aligned}$$

$\cos\alpha = \dfrac{2}{7} \Rightarrow \alpha = 73.4°$

$\cos\beta = \dfrac{3}{7} \Rightarrow \beta = 64.6°$

$\cos\gamma = \dfrac{6}{7} \Rightarrow \alpha = 149°$

□

Zadatak 1.6. Radij-vektor točke M zatvara sa osi y kut od $60°$, a sa osi z kut od $45°$. Njegova duljina iznosi 8. Odredite koordinate točke M, ako je apscisa negativna.

Rješenje:

$\beta = \angle(\overrightarrow{OM}, \vec{j}) = 60^\circ$

$\gamma = \angle(\overrightarrow{OM}, \vec{k}) = 45^\circ$

$|\overrightarrow{OM}| = 8$

$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k},\ \ x < 0$

$$\cos 60^\circ = \frac{y}{|\overrightarrow{OM}|} \quad \Rightarrow \quad \frac{1}{2} = \frac{y}{8} \quad \Rightarrow \quad y = 4$$
$$\cos 45^\circ = \frac{z}{|\overrightarrow{OM}|} \quad \Rightarrow \quad \frac{\sqrt{2}}{2} = \frac{z}{8} \quad \Rightarrow \quad z = 4\sqrt{2}$$

$$\begin{aligned}
|\overrightarrow{OM}| &= \sqrt{x^2+y^2+z^2} \ \ /^2 \\
x^2+y^2+z^2 &= 8^2 \\
x^2+4^2+(4\sqrt{2})^2 &= 64 \\
x^2+16+32 &= 64 \\
x^2 &= 16 \\
x &= -4
\end{aligned}$$

$\overrightarrow{OM} = -4\vec{i} + 4\vec{j} + 4\sqrt{2}\vec{k} \ \ \Rightarrow \ \ M(-4, 4, 4\sqrt{2})$

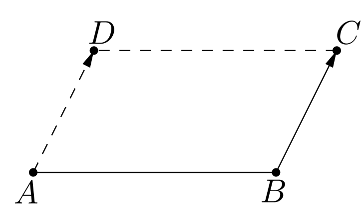
$\square$

Zadatak 1.7. Dana su redom tri uzastopna vrha paralelograma ABCD: $A(1,-2,0)$, $B(2,1,3)$ i $C(-2,0,5)$. Odredite vrh D i duljinu dijagonale $\overrightarrow{BD}$.

Rješenje: Označimo s $D(x,y,z)$ nepoznati vrh paralelograma. Kako je $\overrightarrow{AD} = \overrightarrow{BC}$, koristimo jednakost vektora da bi izračunali nepoznate koordinate točke D:

$$\overrightarrow{AD} = \overrightarrow{BC}$$
$$(x-1)\vec{i} + (y+2)\vec{j} + z\vec{k} = -4\vec{i} - \vec{j} + 2\vec{k}$$

$$x - 1 = -4 \qquad y + 2 = -1 \qquad z = 2$$
$$x = -3 \qquad y = -3$$



Tražene koordinate su $D(-3,-3,2)$

Odredimo još i duljinu dijagonale $\overrightarrow{BD}$.

$\overrightarrow{BD} = -5\vec{i} - 4\vec{j} - \vec{k}$

$|\overrightarrow{BD}| = \sqrt{25+16+1} = \sqrt{42}$

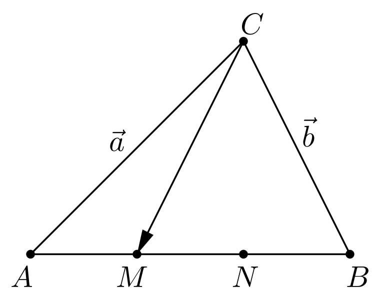
$\square$

Zadatak 1.8. U trokutu ABC stranica $\overline{AB}$ je točkama M i N podijeljena na 3 jednaka dijela tako da je $|AM| = |MN| = |NB|$. Odredite vektor $\overrightarrow{CM}$ ako je $\overrightarrow{CA} = \vec{a}$ i $\overrightarrow{CB} = \vec{b}$.

Rješenje:

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AC} + \overrightarrow{CB} \\ &= \overrightarrow{CB} + \overrightarrow{AC} \\ &= \overrightarrow{CB} - \overrightarrow{CA} \\ &= \vec{b} - \vec{a} \end{aligned}$$

$$\begin{aligned} \overrightarrow{AM} &= \frac{1}{3}\overrightarrow{AB} \\ &= \frac{\vec{b}-\vec{a}}{3} \end{aligned}$$



$$\begin{aligned} \overrightarrow{CM} &= \overrightarrow{CA} + \overrightarrow{AM} \\ &= \vec{a} + \frac{\vec{b}-\vec{a}}{3} \\ &= \frac{2\vec{a}+\vec{b}}{3} \end{aligned}$$

$\square$

1.2 Linearna nezavisnost vektora

Neka su $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ vektori i $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ skalari.
Za relaciju $\alpha_1\overrightarrow{a_1} + \alpha_2\overrightarrow{a_2} + ... + \alpha_n\overrightarrow{a_n} = \overrightarrow{0}$ uvijek postoji trivijalno rješenje $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. Ako je to ujedno i jedino rješenje, onda su $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ linearno nezavisni.

Definicija 1.2. Neka su $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ vektori i $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ skalari. Ako vrijedi: $\alpha_1\overrightarrow{a_1} + \alpha_2\overrightarrow{a_2} + ... + \alpha_n\overrightarrow{a_n} = \vec{0} \iff \alpha_1 = \alpha_2 = ... = \alpha_n = 0$, onda kažemo su vektori $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ **linearno nezavisni**.

1. $n = 1$: $\vec{a}$ je linearno nezavisan $\iff$ $\vec{a} \neq \vec{0}$.
2. $n = 2$: $\vec{a}$ i $\vec{b}$ su linearno zavisni $\iff$ $\exists \lambda \in \mathbb{R}, \lambda \neq 0$ tako da je $\vec{a} = \lambda \vec{b}$. $\vec{a}$ i $\vec{b}$ su kolinearni.
3. $n = 3$: $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno zavisni $\iff$ $\exists \alpha, \beta \in \mathbb{R}$, barem jedan $\neq 0$ tako da je $\vec{c} = \alpha\vec{a} + \beta\vec{b}$. $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni.

Geometrijski:

1. 2 vektora su linearno nezavisna akko **ne** leže na istom niti na paralelnim pravcima.
2. 3 vektora su linearno nezavisna akko **ne** leže u istoj ravnini.

Zadatak 1.9. Odredite $\lambda \in \mathbb{R}$ takav da vektori $\vec{a} = 2\vec{i} - 4\vec{j}$ i $\vec{b} = \lambda\vec{i} + 2\vec{j}$ budu kolinearni.

Rješenje: Za kolinearne vektore $\vec{a}$ i $\vec{b}$ vrijedi da su linearno zavisni pa iz toga zaključujemo da mora postojati $\alpha \in \mathbb{R}$ takav da je $\vec{a} = \alpha\vec{b}$.

$$\begin{aligned} \vec{a} &= \alpha\vec{b} \\ 2\vec{i} - 4\vec{j} &= \alpha\lambda\vec{i} + 2\alpha\vec{j} \end{aligned}$$

$$\begin{aligned} 2 &= \alpha\lambda \\ -4 &= 2\alpha \end{aligned}$$

$(2) \Rightarrow \alpha = -2$

$(1) \Rightarrow \lambda = -1$

$$\left.\begin{aligned} \vec{a} &= 2\vec{i} - 4\vec{j} \\ \vec{b} &= -\vec{i} + 2\vec{j} \end{aligned}\right\} \implies \vec{a} = -2\vec{b}$$

$\square$

Zadatak 1.10. Ispitajte linearnu (ne)zavisnost vektora:

a) $\vec{a} = 4\vec{i} + 2\vec{j} + 5\vec{k}$
$\vec{b} = -\vec{j} + \vec{k}$
$\vec{c} = 5\vec{k}$

b) $\vec{a} = \vec{i} + \vec{j}$
$\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$
$\vec{c} = 3\vec{j} + \vec{k}$

Rješenje:

a)
$$\begin{aligned} \vec{0} &= \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} \\ &= \alpha(4\vec{i} + 2\vec{j} + 5\vec{k}) + \beta(-\vec{j} + \vec{k}) + \gamma(5\vec{k}) = \\ &= 4\alpha\vec{i} + (2\alpha - \beta)\vec{j} + (5\alpha + \beta + 5\gamma)\vec{k} \end{aligned}$$

$$\begin{aligned} 4\alpha &= 0 \\ 2\alpha - \beta &= 0 \\ 5\alpha + \beta + 5\gamma &= 0 \end{aligned}$$

$(1) \Rightarrow\ \alpha = 0$

$(2) \Rightarrow\ \beta = 0$

$(3) \Rightarrow\ \gamma = 0$

$\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno nezavisni.

b)
$$\begin{aligned} \vec{0} &= \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} \\ &= \alpha(\vec{i} + \vec{j}) + \beta(-\vec{i} + 2\vec{j} + \vec{k}) + \gamma(3\vec{j} + \vec{k}) \\ &= (\alpha - \beta)\vec{i} + (\alpha + 2\beta + 3\gamma)\vec{j} + (\beta + \gamma)\vec{k} \end{aligned}$$

$$\begin{aligned} \alpha - \beta &= 0 \\ \alpha + 2\beta + 3\gamma &= 0 \\ \beta + \gamma &= 0 \end{aligned}$$

$$\begin{aligned} (1) &\quad\Rightarrow\quad \alpha = \beta \\ (2) &\quad\Rightarrow\quad 3\beta + 3\gamma = 0 \\ (2) \text{ i } (3) &\quad\Rightarrow\quad \beta = -\gamma \end{aligned}$$

$\alpha = \beta = -\gamma,\ \gamma \in \mathbb{R}$

Sustav ima beskonačno mnogo rješenja, pa su vektori linearno zavisni. ($\vec{c} = \vec{a} + \vec{b}$).

□

Zadatak 1.11. Zadani su vektori:

a) $\vec{a} = \vec{i} + (2\lambda + 1)\vec{j}$

$\vec{b} = 2\vec{i} + \lambda\vec{j} + 2\vec{k}$

$\vec{c} = \vec{i} + \vec{j} + \vec{k}$

b) $\vec{a} = 5\vec{i} + 2\vec{j} + \lambda\vec{k}$
$\vec{b} = 3\vec{i} + 2\vec{j} + 4\vec{k}$
$\vec{c} = 11\vec{i} + 2\vec{j} + (\lambda - 3)\vec{k}$

Odredite $\lambda \in \mathbb{R}$ tako da zadani vektori budu komplanarni i prikažite vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rješenje: Za komplanarne vektore $\vec{a}$, $\vec{b}$ i $\vec{c}$ vrijedi da su linearno zavisni pa iz toga zaključujemo da moraju postojati $\alpha, \beta \in \mathbb{R}$ takvi da je $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

a)
$$\begin{aligned} \vec{c} &= \alpha\vec{a} + \beta\vec{b} \\ \vec{i} + \vec{j} + \vec{k} &= \alpha\left(\vec{i} + (2\lambda + 1)\vec{j}\right) + \beta\left(2\vec{i} + \lambda\vec{j} + 2\vec{k}\right) \\ &= \alpha\vec{i} + 2\alpha\lambda\vec{j} + \alpha\vec{j} + 2\beta\vec{i} + \lambda\beta\vec{j} + 2\beta\vec{k} \\ &= (\alpha + 2\beta)\vec{i} + (2\alpha\lambda + \alpha + \lambda\beta)\vec{j} + 2\beta\vec{k} \end{aligned}$$

$$\begin{aligned} \alpha + 2\beta &= 1 \\ 2\alpha\lambda + \alpha + \lambda\beta &= 1 \\ 2\beta &= 1 \end{aligned}$$

$(3) \Rightarrow \; \beta = \frac{1}{2}$

$(1) \Rightarrow \; \alpha = 0$

$(2) \Rightarrow \; \lambda = 2$

$$\left.\begin{aligned} \vec{a} &= \vec{i} + 5\vec{j} \\ \vec{b} &= 2\vec{i} + 2\vec{j} + 2\vec{k} \\ \vec{c} &= \vec{i} + \vec{j} + \vec{k} \end{aligned}\right\} \Longrightarrow \vec{c} = \frac{1}{2}\vec{b}$$

b)
$$\begin{aligned} \vec{c} &= \alpha\vec{a} + \beta\vec{b} \\ 11\vec{i} + 2\vec{j} + (\lambda - 3)\vec{k} &= \alpha\left(5\vec{i} + 2\vec{j} + \lambda\vec{k}\right) + \beta\left(3\vec{i} + 2\vec{j} + 4\vec{k}\right) \\ &= 5\alpha\vec{i} + 2\alpha\vec{j} + \alpha\lambda\vec{k} + 3\beta\vec{i} + 2\beta\vec{j} + 4\beta\vec{k} \\ &= (5\alpha + 3\beta)\vec{i} + (2\alpha + 2\beta)\vec{j} + (\lambda\alpha + 4\beta)\,\vec{k} \end{aligned}$$

$$\begin{array}{rcll} 5\alpha + 3\beta &=& 11 & /\cdot(-2) \\ 2\alpha + 2\beta &=& 2 & /\cdot 3 \\ \lambda\alpha + 4\beta &=& \lambda - 3 & \\ \hline -4\alpha &=& -16 & \end{array}$$

$$
\begin{array}{rcl}
\lambda\alpha + 4\beta & = & \lambda - 3 \\
\hline
\alpha & = & 4 \\
\beta & = & -3 \\
\hline
4\lambda + 4\cdot(-3) & = & \lambda - 3 \\
3\lambda & = & 9 \\
\lambda & = & 3
\end{array}
$$

$$
\left.\begin{array}{rcl}
\vec{a} & = & 5\vec{i} + 2\vec{j} + 3\vec{k} \\
\vec{b} & = & 3\vec{i} + 2\vec{j} + 4\vec{k} \\
\vec{c} & = & 11\vec{i} + 2\vec{j}
\end{array}\right\} \Longrightarrow \vec{c} = 4\vec{a} - 3\vec{b}
$$

$\square$

1.3 Skalarni produkt vektora

Definicija 1.3. Neka su $\vec{a}$ i $\vec{b}$ vektori. Za funkciju $f : V \times V \to \mathbb{R}$ definiranu s

$$\vec{a} \cdot \vec{b} := |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi,$$

gdje je $\varphi = \angle(\vec{a}, \vec{b})$ kažemo da je **skalarni produkt vektora** $\vec{a}$ i $\vec{b}$.

Svojstva:

1. *Nenegativnost*: $\vec{a} \cdot \vec{a} \geqslant 0$

   Skalarni produkt vektora $\vec{a}$ samog sa sobom može biti 0 ako i samo ako je vektor $\vec{a}$ nulvektor $\left(\vec{a} \cdot \vec{a} = 0 \iff \vec{a} = \vec{0}\right)$.

2. *Homogenost*: $\lambda(\vec{a} \cdot \vec{b}) = (\lambda\vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda\vec{b})$

3. *Komutativnost*: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

4. *Distributivnost*: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Uvjet okomitosti: Skalarni produkt nenul vektora $\vec{a}$ i $\vec{b}$ je 0 ako i samo ako su vektori $\vec{a}$ i $\vec{b}$ međusobno okomiti $\left(\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0\right)$.

Promotrimo vektore $\vec{a}$ i $\vec{b}$ u koordinatnom sustavu prikazani kao linearna kombinacija vektora baze $\vec{i}$, $\vec{j}$ i $\vec{k}$.

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$
$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

Želimo odrediti njihov skalarni produkt. No, prije toga izračunajmo skalarne produkte vektora baze.

$$\begin{aligned}
\vec{i}\cdot\vec{i} &= \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = |1|\cdot|1|\cdot\cos 0^\circ = 1\\
\vec{i}\cdot\vec{j} &= \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = |1|\cdot|1|\cdot\cos 90^\circ = 0\\
\vec{a}\cdot\vec{b} &= \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right)\cdot\left(x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\right)\\
&= x_1x_2\underbrace{\vec{i}\cdot\vec{i}}_{1} + x_1y_2\underbrace{\vec{i}\cdot\vec{j}}_{0} + x_1z_2\underbrace{\vec{i}\cdot\vec{k}}_{0} + y_1x_2\underbrace{\vec{j}\cdot\vec{i}}_{0} + y_1y_2\underbrace{\vec{j}\cdot\vec{j}}_{1} +\\
&\quad + y_1z_2\underbrace{\vec{j}\cdot\vec{k}}_{0} + z_1x_2\underbrace{\vec{k}\cdot\vec{i}}_{0} + z_1y_2\underbrace{\vec{k}\cdot\vec{j}}_{0} + z_1z_2\underbrace{\vec{k}\cdot\vec{k}}_{1}\\
&= x_1x_2 + y_1y_2 + z_1z_2
\end{aligned}$$

Dakle, za vektore $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ i $\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$, skalarni produkt je:

$$\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$$

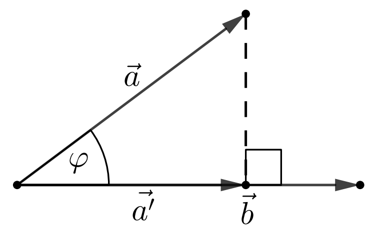
Iz toga slijedi sljedeće:

$$\vec{a}\cdot\vec{a} = x_1^2 + y_1^2 + z_1^2 = |\vec{a}|^2$$

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{(x_1^2 + y_1^2 + z_1^2)\cdot(x_2^2 + y_2^2 + z_2^2)}}, \varphi \in [0, \pi]$$

Ortogonalna projekcija vektora $\vec{a}$ na vektor $\vec{b}$

$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$
$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$
$|\vec{a'}| = \cos\varphi \cdot |\vec{a}| = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} \cdot |\vec{a}| = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$
$\vec{a'} = |\vec{a'}| \cdot \dfrac{\vec{b}}{|\vec{b}|}$
$\text{SkPr}\vec{a}_{\vec{b}} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$
$\text{VekPr}\vec{a}_{\vec{b}} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \cdot \dfrac{\vec{b}}{|\vec{b}|}$



Zadatak 1.12. Odredite skalarni produkt vektora $\vec{a} = -\vec{i} + 4\vec{j} + 7\vec{k}$ i $\vec{b} = 2\vec{i} - 5\vec{j} + 2\vec{k}$.

Rješenje:

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (-1)\cdot 2 + 4\cdot(-5) + 7\cdot 2\\ &= -2 - 20 + 14\\ &= -8\end{aligned}$$

□

Zadatak 1.13. Zadani su vektori $\vec{a} = m\vec{i} + 3\vec{j} + 4\vec{k}$ i $\vec{b} = 4\vec{i} + m\vec{j} - 7\vec{k}$. Odredite konstantu $m \in \mathbb{R}$ tako da vektori $\vec{a}$ i $\vec{b}$ budu okomiti.

Rješenje:

$$\begin{aligned}\vec{a}\cdot\vec{b} &= m\cdot 4 + 3\cdot m + 4\cdot(-7)\\ &= 4m + 3m - 28\\ &= 7m - 28\end{aligned}$$

Prema uvjetu okomitosti skalarni produkt vektora $\vec{a}$ i $\vec{b}$ iznosi 0.

$$\begin{aligned}7m - 28 &= 0\\ 7m &= 28\\ m &= 4\end{aligned}$$

□

Zadatak 1.14. Nađite skalarnu i vektorsku projekciju vektora $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ na vektor $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$.

Rješenje:

$$\text{SkPr}\vec{a}_{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2+1-2}{\sqrt{4+1+4}} = \frac{1}{3}$$

$$\text{VekPr}\vec{a}_{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\frac{\vec{b}}{|\vec{b}|} = \frac{1}{3}\cdot\frac{2\vec{i}+\vec{j}-2\vec{k}}{3} = \frac{2}{9}\vec{i}+\frac{1}{9}\vec{j}-\frac{2}{9}\vec{k}$$

□

Zadatak 1.15. Zadani su vektori $\vec{a} = \vec{i}+2\vec{j}+3\vec{k}$ i $\vec{b} = 6\vec{i}+4\vec{j}-2\vec{k}$. Odredite:

a) kosinus kuta između vektora $\vec{a}$ i $\vec{b}$

b) ortogonalnu projekciju vektora $\vec{a}$ na vektor $\vec{b}$.

Rješenje:

a) $\varphi = \angle(\vec{a},\vec{b})$

$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$

$|\vec{b}| = \sqrt{36+16+4} = \sqrt{56}$

$$\begin{aligned}
\cos\varphi &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \\
&= \frac{x_1x_2+y_1y_2+z_1z_2}{\sqrt{(x_1^2+y_1^2+z_1^2)\cdot(x_2^2+y_2^2+z_2^2)}} \\
&= \frac{6+8-6}{\sqrt{14\cdot 56}} \\
&= \frac{8}{28} \\
&= \frac{2}{7}
\end{aligned}$$

b)

$$\begin{aligned}
\vec{a'} &= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\frac{\vec{b}}{|\vec{b}|} \\
&= \frac{8}{\sqrt{56}}\cdot\frac{6\vec{i}+4\vec{j}-2\vec{k}}{\sqrt{56}} \\
&= \frac{6}{7}\vec{i}+\frac{4}{7}\vec{j}-\frac{2}{7}\vec{k}
\end{aligned}$$

□

Zadatak 1.16. Koji kut zatvaraju jedinični vektori $\vec{m}$ i $\vec{n}$ ako su vektori $\vec{p} = \vec{m} + 2\vec{n}$ i $\vec{q} = 5\vec{m} - 4\vec{n}$ okomiti.

Rješenje: $|\vec{m}| = |\vec{n}| = 1$

$\vec{p} \perp \vec{q} \;\Rightarrow\; \vec{p} \cdot \vec{q} = 0$

$$\begin{aligned}
0 &= \vec{p} \cdot \vec{q} \\
&= (\vec{m} + 2\vec{n}) \cdot (5\vec{m} - 4\vec{n}) \\
&= 5|\vec{m}|^2 - 4\vec{m} \cdot \vec{n} + 10\vec{m} \cdot \vec{n} - 8|\vec{n}|^2 \\
&= 6\vec{m} \cdot \vec{n} - 3 \\
\vec{m} \cdot \vec{n} &= \frac{1}{2} \\
\cos\varphi &= \frac{\vec{m} \cdot \vec{n}}{|\vec{m}| \cdot |\vec{n}|} \\
&= \frac{1}{2} \\
\varphi &= 60^\circ
\end{aligned}$$

□

Zadatak 1.17. Neka je $|\vec{a}| = 13$, $|\vec{b}| = 19$, $|\vec{a} + \vec{b}| = 24$. Odredite $|\vec{a} - \vec{b}|$.

Rješenje:

$$\begin{aligned}
|\vec{a} + \vec{b}|^2 &= 24^2 \\
&= 576 \\
|\vec{a} + \vec{b}|^2 &= \left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} + \vec{b}\right) \\
&= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \\
2\vec{a} \cdot \vec{b} &= |\vec{a} + \vec{b}|^2 - |\vec{a}|^2 - |\vec{b}|^2 \\
2\vec{a} \cdot \vec{b} &= 576 - 169 - 361 \\
&= 46 \\
\vec{a} \cdot \vec{b} &= 23
\end{aligned}$$

$$\begin{aligned}
|\vec{a} - \vec{b}| &= \sqrt{|\vec{a} - \vec{b}|^2} \\
&= \sqrt{\left(\vec{a} - \vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right)} \\
&= \sqrt{|\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2} \\
&= \sqrt{169 - 46 + 361} \\
&= \sqrt{484} \\
&= 22
\end{aligned}$$

□

Zadatak 1.18. Dokažite da su $\vec{a}$ i $\vec{b}$ ortogonalni (okomiti) ako vrijedi: $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$.

Rješenje:

$$\begin{aligned}
|\vec{a}+\vec{b}| &= |\vec{a}-\vec{b}| /^2 \\
|\vec{a}+\vec{b}|^2 &= |\vec{a}-\vec{b}|^2 \\
\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right) &= \left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right) \\
|\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2 &= |\vec{a}|^2-2\vec{a}\cdot\vec{b}+|\vec{b}|^2 \\
4\vec{a}\cdot\vec{b} &= 0 \\
\vec{a}\cdot\vec{b} &= 0
\end{aligned}$$

Prema uvjetu okomitosti zaključujemo da su vektori $\vec{a}$ i $\vec{b}$ okomiti.

$\square$

Zadatak 1.19. Odredite jedinični vektor $\vec{n_0}$ komplanaran s vektorima $\vec{p}$ i $\vec{q}$ ako je $|\vec{p}| = 2$, $|\vec{q}| = 4$, $\angle(\vec{p},\vec{q}) = \frac{\pi}{3}$, $\vec{n}\cdot\vec{p} = 4$ i $\vec{n}\cdot\vec{q} = -8$.

Rješenje: Kako je $\vec{n}$ komplanaran s vektorima $\vec{p}$ i $\vec{q}$, zaključujemo da su vektori $\vec{n}$, $\vec{p}$ i $\vec{q}$ linearno zavisni pa postoje $\alpha, \beta \in \mathbb{R}$ takvi da se $\vec{n}$ može prikazati kao linearna kombinacija vektora $\vec{p}$ i $\vec{q}$, odnosno da je $\vec{n} = \alpha\vec{p} + \beta\vec{q}$. Odredimo prvo $\vec{p}\cdot\vec{q}$.

$\vec{p}\cdot\vec{q} = |\vec{p}|\cdot|\vec{q}|\cdot\cos\frac{\pi}{3} = 4$

$$\begin{aligned}
\vec{n}\cdot\vec{p} &= 4 \\
(\alpha\vec{p}+\beta\vec{q})\cdot\vec{p} &= 4 \\
\alpha|\vec{p}|^2+\beta\vec{p}\cdot\vec{q} &= 4 \\
4\alpha+4\beta &= 4 \ \ /:4 \\
\alpha+\beta &= 1
\end{aligned}
\qquad\qquad
\begin{aligned}
\vec{n}\cdot\vec{q} &= -8 \\
(\alpha\vec{p}+\beta\vec{q})\cdot\vec{q} &= -8 \\
\alpha\vec{p}\cdot\vec{q}+\beta|\vec{q}|^2 &= -8 \\
4\alpha+16\beta &= -8 \ \ /:4 \\
\alpha+4\beta &= -2
\end{aligned}$$

$$\begin{array}{rl}
\alpha+\beta &= 1 \\
\alpha+4\beta &= -2 \quad /\cdot(-1) \\
\hline
\alpha+\beta &= 1 \\
-\alpha-4\beta &= 2 \\
\hline
-3\beta &= 3 \\
\beta &= -1
\end{array}$$

$$
\begin{aligned}
\alpha+\beta &= 1 \\
\alpha &= 2
\end{aligned}
$$

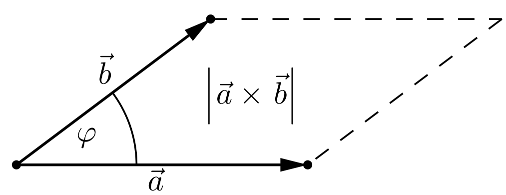
$$
\begin{aligned}
\vec{n} &= 2\vec{p}-\vec{q} \\
|\vec{n}| &= \sqrt{4|\vec{p}|^2-4\vec{p}\cdot\vec{q}+|\vec{q}|^2} \\
&= \sqrt{16-16+16} \\
&= 4
\end{aligned}
\qquad
\begin{aligned}
\vec{n_0} &= \frac{\vec{n}}{|\vec{n}|} \\
&= \frac{1}{4}\cdot(2\vec{p}-\vec{q})
\end{aligned}
$$

$\square$

1.4 Vektorski produkt vektora

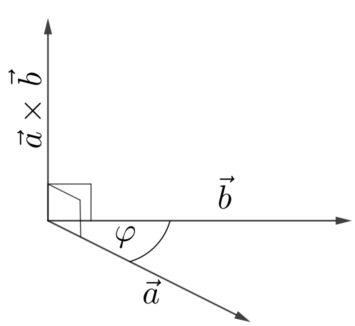
$f : V \times V \to V$...rezultat je vektor
$\left(\vec{a},\vec{b}\right) \longmapsto \vec{a}\times\vec{b}$

Duljina: Za $\vec{a}\times\vec{b}$ vrijedi $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\varphi$



Geometrijski, duljina vektorskog produkta jednaka je površini paralelograma što ga razapinju vektori $\vec{a}$ i $\vec{b}$.

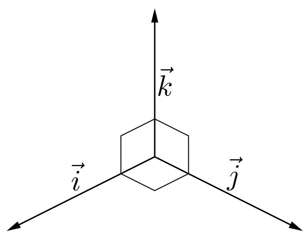
Smjer: Za nekolinearne vektore $\vec{a}$ i $\vec{b}$ vrijedi da su okomiti na svoj vektorski produkt, tj. $\vec{a}\times\vec{b} \perp \vec{a},\vec{b}$.



Uvjet kolinearnosti: Vektorski produkt dva nenul vektora $\vec{a}$ i $\vec{b}$ je jednak $\vec{0}$ ako i samo ako su vektori $\vec{a}$ i $\vec{b}$ kolinearni. ($\vec{a} \times \vec{b} = \vec{0} \iff \vec{a}$ i $\vec{b}$ su kolinearni.)

Orijentacija:

Vektori $\vec{a}$, $\vec{b}$ i $\vec{a} \times \vec{b}$ čine desnu trojku, tj. gledano iz vrha vektora $\vec{a} \times \vec{b}$ rotacija iz $\vec{a}$ u $\vec{b}$ suprotna je gibanju kazaljke na satu.



$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

Svojstva:

1. $\vec{a} \times \vec{a} = \vec{0}$
2. $\lambda\left(\vec{a} \times \vec{b}\right) = \lambda\vec{a} \times \vec{b} = \vec{a} \times \lambda\vec{b}$...homogenost
3. $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$...antikomutativnost
4. $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$...distributivnost

Napomena: asocijativnost ne vrijedi, tj. $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$

Neka su $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ i $\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$. Tada je:

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\
&= \left(y_1z_2 - y_2z_1\right)\vec{i} - \left(x_1z_2 - x_2z_1\right)\vec{j} + \left(x_1y_2 - x_2y_1\right)\vec{k}
\end{aligned}$$

Zadatak 1.20. Odredite vektorski produkt vektora $\vec{a} = 2\vec{i} + 3\vec{j} + 5\vec{k}$ i $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$ i površinu paralelograma određenog tim vektorima.

Rješenje:

$$
\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 5 \\ 1 & 2 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\
&= (3 \cdot 1 - 2 \cdot 5) \cdot \vec{i} - (2 \cdot 1 - 1 \cdot 5) \cdot \vec{j} + (2 \cdot 2 - 1 \cdot 3) \cdot \vec{k} \\
&= -7\vec{i} + 3\vec{j} + \vec{k}
\end{aligned}
$$

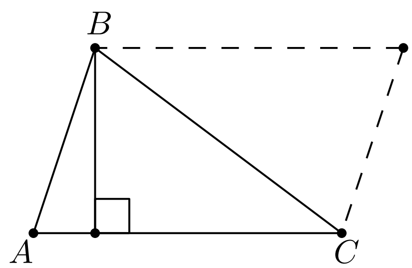
$$P = |\vec{a} \times \vec{b}| = \sqrt{49 + 9 + 1} = \sqrt{59}$$

$\square$

Zadatak 1.21. Točke $A(-1,-1,3)$, $B(-2,-4,3)$ i $C(-1,2,7)$ su vrhovi trokuta. Korištenjem vektorskog računa izračunajte visinu trokuta spuštenog iz vrha B.

Rješenje:

$$
\begin{aligned}
P_\Delta &= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| \\
P_\Delta &= \frac{1}{2} v \cdot |\overrightarrow{AC}|
\end{aligned}
$$



$\overrightarrow{AB} = -\vec{i} - 3\vec{j}$
$\overrightarrow{AC} = 3\vec{j} + 4\vec{k}$

$$
\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & 0 \\ 0 & 3 & 4 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -3 & 0 \\ 3 & 4 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} -1 & 0 \\ 0 & 4 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} -1 & -3 \\ 0 & 3 \end{vmatrix} \\
&= (-3 \cdot 4 - 3 \cdot 0) \cdot \vec{i} - (-1 \cdot 4 - 0 \cdot 0) \cdot \vec{j} + (-1 \cdot 3 - 0 \cdot (-3)) \cdot \vec{k} \\
&= -12\vec{i} + 4\vec{j} - 3\vec{k}
\end{aligned}
$$

$$\begin{aligned} |\overrightarrow{AB} \times \overrightarrow{AC}| &= \sqrt{(-12)^2 + 4^2 + (-3)^2} \\ &= \sqrt{144 + 16 + 9} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$\begin{aligned} |\overrightarrow{AC}| &= \sqrt{0^2 + 3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| &= \frac{1}{2}v \cdot |\overrightarrow{AC}| \\ v &= \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AC}|} \\ &= \frac{13}{5} \end{aligned}$$

□

Zadatak 1.22. Odredite površinu paralelograma određenog vektorima $\vec{a} + 3\vec{b}$ i $3\vec{a} + \vec{b}$, ako je $|\vec{a}| = |\vec{b}| = 1$ i $\angle\left(\vec{a}, \vec{b}\right) = 30°$

Rješenje:

$$\begin{aligned} \left(\vec{a} + 3\vec{b}\right) \times \left(3\vec{a} + \vec{b}\right) &= 3\underbrace{\vec{a} \times \vec{a}}_{\vec{0}} + \vec{a} \times \vec{b} + \underbrace{9\vec{b} \times \vec{a}}_{-9\vec{a}\times\vec{b}} + 3\underbrace{\vec{b} \times \vec{b}}_{\vec{0}} \\ &= \vec{a} \times \vec{b} - 9\vec{a} \times \vec{b} \\ &= -8\vec{a} \times \vec{b} \end{aligned}$$

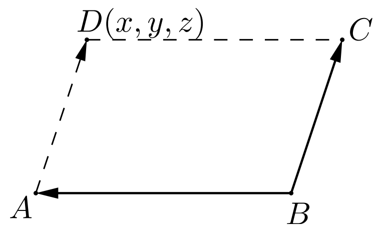
$$\begin{aligned} P &= |-8\vec{a} \times \vec{b}| \\ &= 8|\vec{a} \times \vec{b}| \\ &= 8 \cdot \underbrace{|\vec{a}|}_{1} \cdot \underbrace{|\vec{b}|}_{1} \cdot \sin\angle\left(\vec{a}, \vec{b}\right) \\ &= 8 \cdot \sin 30° \\ &= 8 \cdot \tfrac{1}{2} \\ &= 4 \end{aligned}$$

□

Zadatak 1.23. Neka su $A(5, 1, 3)$, $B(2, 2, 3)$ i $C(2, 1, 5)$ vrhovi paralelograma $ABCD$. Odredite površinu P paralelograma i koordinate vrha D.

Rješenje:

$\overrightarrow{BA} = 3\vec{i} - \vec{j}$
$\overrightarrow{BC} = -\vec{j} + 2\vec{k}$



$$\begin{aligned}
\overrightarrow{BA} \times \overrightarrow{BC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 0 \\ 0 & -1 & 2 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 3 & -1 \\ 3 & -1 \end{vmatrix} \\
&= -2\vec{i} - 6\vec{j} - 3\vec{k}
\end{aligned}$$

$$\begin{aligned}
P &= |\overrightarrow{BA} \times \overrightarrow{BC}| \\
&= \sqrt{(-2)^2 + (-6)^2 + (-3)^2} \\
&= 7
\end{aligned}$$

$\overrightarrow{AD} = \overrightarrow{BC}$

$$(x-5)\,\vec{i} + (y-1)\,\vec{j} + (z-3)\,\vec{k} = -\vec{j} + 2\vec{k}$$

$$\begin{array}{lll} x - 5 = 0 & y - 1 = -1 & z - 3 = 2 \\ x = 5 & y = 0 & z = 5 \end{array}$$

Imamo da je $D(5, 0, 5)$.

$\square$

1.5 Mješoviti produkt vektora

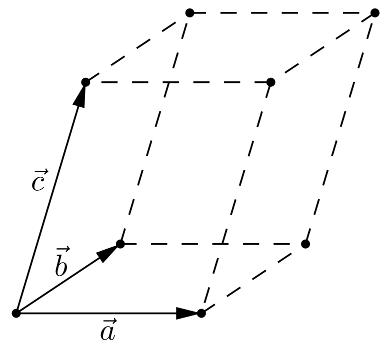
$f : V \times V \times V \to \mathbb{R}$...rezultat je skalar
$$\left(\vec{a}, \vec{b}, \vec{c}\right) \longmapsto \underbrace{\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}}_{Oznaka: \left[\vec{a}, \vec{b}, \vec{c}\right] ili \left(\vec{a}, \vec{b}, \vec{c}\right)}$$

Svojstva:

1. $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \left(\vec{b} \times \vec{c}\right) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b} = \left(\vec{a}, \vec{b}, \vec{c}\right)$

2. $\left(\vec{b}\times\vec{a}\right)\cdot\vec{c}=(\vec{a}\times\vec{c})\cdot\vec{b}=\left(\vec{c}\times\vec{b}\right)\cdot\vec{a}=-\left(\vec{a},\vec{b},\vec{c}\right)$

Geometrijska interpretacija:



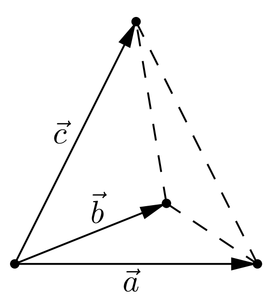
$$V=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$$

Apsolutna vrijednost mješovitog produkta tri nekomplanarna vektora jednaka je obujmu paralelepipeda što ga razapinju ti vektori.

$$\left.\begin{aligned}\vec{a}&=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}\\ \vec{b}&=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}\\ \vec{c}&=x_3\vec{i}+y_3\vec{j}+z_3\vec{k}\end{aligned}\right\}\Rightarrow\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}$$

$$=x_1\cdot\begin{vmatrix}y_2&z_2\\y_3&z_3\end{vmatrix}-y_1\cdot\begin{vmatrix}x_2&z_2\\x_3&z_3\end{vmatrix}+z_1\cdot\begin{vmatrix}x_2&y_2\\x_3&y_3\end{vmatrix}$$

$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=|\vec{a}|\cdot|\vec{b}\times\vec{c}|\cdot\cos\varphi,\varphi=\angle\left(\vec{a},\vec{b}\times\vec{c}\right)$



Obujam tetraedra: $V_T=\dfrac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$

Uvjet komplanarnosti: Za vektore $\vec{a},\vec{b},\vec{c}\neq\vec{0}$ vrijedi da su komplanarni ako i samo ako je njihov mješoviti produkt jednak 0.

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=0\iff\vec{a},\vec{b},\vec{c}\text{ komplanarni.}$$

Zadatak 1.24. Odredi mješoviti produkt vektora

$$\vec{a} = 2\vec{i} - \vec{j} - \vec{k}$$
$$\vec{b} = \vec{i} + 3\vec{j} - \vec{k}$$
$$\vec{c} = \vec{i} + \vec{j} + 4\vec{k}.$$

Rješenje:

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} &= \begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & 4 \end{vmatrix} \\
&= 2 \cdot \begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\
&= 2 \cdot 13 + 1 \cdot 5 - 1 \cdot (-2) \\
&= 26 + 5 + 2 \\
&= 33
\end{aligned}$$

□

Zadatak 1.25. Dokažite da su vektori

$$\vec{a} = 2\vec{i} + 5\vec{j} + 7\vec{k}$$
$$\vec{b} = \vec{i} + \vec{j} - \vec{k}$$
$$\vec{c} = \vec{i} + 2\vec{j} + 2\vec{k}$$

komplanarni.

Rješenje:

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} &= \begin{vmatrix} 2 & 5 & 7 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} \\
&= 2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} - 5 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 7 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\
&= 2 \cdot 4 - 5 \cdot 3 + 7 \cdot 1 \\
&= 8 - 15 + 7 \\
&= 0
\end{aligned}$$

Prema uvjetu komplanarnosti, vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni.

$\square$

Zadatak 1.26. Odredite volumen tetraedra čiji su vrhovi točke $A(2,2,2)$, $B(4,3,3)$, $C(4,5,4)$ i $D(5,5,6)$.

Rješenje: Tetraedar je razapet vektorima $\overrightarrow{AB}$,$\overrightarrow{AC}$ i $\overrightarrow{AD}$

$\overrightarrow{AB} = 2\vec{i} + \vec{j} + \vec{k}$
$\overrightarrow{AC} = 2\vec{i} + 3\vec{j} + 2\vec{k}$
$\overrightarrow{AD} = 3\vec{i} + 3\vec{j} + 4\vec{k}$

$$\begin{aligned}
\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD} &= \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} \\
&= 2 \cdot \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} \\
&= 2 \cdot 6 - 2 - 3 \\
&= 12 - 5 \\
&= 7
\end{aligned}$$

$V_T = \dfrac{1}{6}\left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right| = \dfrac{7}{6}$

$\square$

Zadatak 1.27. Zadani su vektori

$$\begin{aligned}
\vec{a} &= 3\vec{i} - 4\vec{j} + 2\vec{k} \\
\vec{b} &= -\vec{j} + 2\vec{k} \\
\vec{c} &= \vec{i} + \vec{j} + 2\vec{k}.
\end{aligned}$$

Odredite $\alpha \in \mathbb{R}$ tako da obujam tetraedra razapet vektorima $\vec{a}$, $\vec{b}$ i $\alpha\vec{c}$ iznosi 12.

Rješenje: Odredimo mješoviti produkt vektora $\vec{a}$, $\vec{b}$ i $\alpha\vec{c}$.

$\alpha\vec{c} = \alpha\vec{i} + \alpha\vec{j} + 2\alpha\vec{k}.$

$$\begin{aligned}
\left(\vec{a}\times\vec{b}\right)\cdot\alpha\vec{c} &= \begin{vmatrix} 3 & -4 & 2 \\ 0 & -1 & 2 \\ \alpha & \alpha & 2\alpha \end{vmatrix} \\
&= 3\cdot\begin{vmatrix} -1 & 2 \\ \alpha & 2\alpha \end{vmatrix} - (-4)\cdot\begin{vmatrix} 0 & 2 \\ \alpha & 2\alpha \end{vmatrix} + 2\cdot\begin{vmatrix} 0 & -1 \\ \alpha & \alpha \end{vmatrix} \\
&= 3\cdot(-1\cdot 2\alpha - \alpha\cdot 2) + 4\cdot(0\cdot 2\alpha - \alpha\cdot 2) + 2\cdot(0\cdot\alpha - \alpha\cdot(-1)) \\
&= -12\alpha - 8\alpha + 2\alpha \\
&= -18\alpha
\end{aligned}$$

$$\begin{aligned}
V_T &= \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right| \\
\frac{1}{6}\left|18\alpha\right| &= 12 \\
3\left|\alpha\right| &= 12
\end{aligned}$$

Imamo dva rješenja: $\alpha = 4$ ili $\alpha = -4$.

□

Zadatak 1.28. Pokažite da ako za $\vec{a}$, $\vec{b}$ i $\vec{c}$ vrijedi

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0},$$

onda su vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ komplanarni, a vektori $\vec{d}=\vec{a}-\vec{c}$ i $\vec{e}=\vec{b}-\vec{a}$ kolinearni.

Rješenje: Prema uvjetu komplanarnosti dovoljno je pokazati da mješoviti produkt vektora $\vec{a}$, $\vec{b}$ i $\vec{c}$ jednak 0.

$$\begin{aligned}
\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} &= \vec{0} \qquad \Big/\cdot\vec{c} \\
\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}+\underbrace{\left(\vec{b}\times\vec{c}\right)\cdot\vec{c}}_{\vec{0}}+\underbrace{(\vec{c}\times\vec{a})\cdot\vec{c}}_{\vec{0}} &= 0
\end{aligned}$$

Vektor $\vec{b}\times\vec{c}$ je okomit na vektor $\vec{c}$ pa je, prema uvjetu okomitosti, $\left(\vec{c}\times\vec{b}\right)\cdot\vec{c}=0$.

Slično zaključujemo i za vektore $\vec{c}\times\vec{a}$ i $\vec{c}$.

$$(\vec{a}\times\vec{b})\cdot\vec{c} = 0$$

Prema uvjetu kolinearnosti trebali bi pokazati da je vektorski produkt vektora $\vec{d}$ i $\vec{e}$ jednak $\vec{0}$.

$$\begin{aligned}
\vec{d}\times\vec{e} &= (\vec{a}-\vec{c})\times\left(\vec{b}-\vec{a}\right)\\
&= \vec{a}\times\vec{b}-\underbrace{\vec{a}\times\vec{a}}_{\vec{0}}\underbrace{-\vec{c}\times\vec{b}}_{+\vec{b}\times\vec{c}}+\vec{c}\times\vec{a}\\
&= \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\\
&= \vec{0}
\end{aligned}$$

□

Zadatak 1.29. Odredite parametar $t\in\mathbb{R}$ takav da vektori

$$\begin{aligned}
\vec{a} &= 2\vec{i}-3\vec{j}+4\vec{k}\\
\vec{b} &= -\vec{i}-2\vec{j}+2\vec{k}\\
\vec{c} &= -8\vec{i}+t\vec{j}-8\vec{k}.
\end{aligned}$$

budu komplanarni. Prikažite vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rješenje: Prema uvjetu komplanarnosti, mješoviti produkt vektora $\vec{a}$, $\vec{b}$ i $\vec{c}$ mora biti jednak 0.

$$\begin{aligned}
\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} &= \begin{vmatrix} 2 & -3 & 4\\ -1 & -2 & 2\\ -8 & t & -8\end{vmatrix}\\
&= 2\cdot\begin{vmatrix}-2 & 2\\ t & -8\end{vmatrix}-(-3)\cdot\begin{vmatrix}-1 & 2\\ -8 & -8\end{vmatrix}+4\cdot\begin{vmatrix}-1 & -2\\ -8 & t\end{vmatrix}\\
&= 2\left(-2\cdot(-8)-t\cdot 2\right)+3\left(-1\cdot(-8)-(-8)\cdot 2\right)+4\left(-1\cdot t-(-8)\cdot(-2)\right)\\
&= 32-4t+24+48-4t-64\\
&= -8t+40
\end{aligned}$$

$$\begin{aligned}
-8t+40 &= 0\\
8t &= 40\\
t &= 5
\end{aligned}$$

Ostaje prikazati vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$. Odredimo α, $\beta\in\mathbb{R}$ takve da je $\vec{c}=\alpha\vec{a}+\beta\vec{b}$.

$$\begin{aligned}
\vec{c} &= \alpha\vec{a} + \beta\vec{b} \\
-8\vec{i} + 5\vec{j} - 8\vec{k} &= \alpha\left(2\vec{i} - 3\vec{j} + 4\vec{k}\right) + \beta\left(-\vec{i} - 2\vec{j} + 2\vec{k}\right) \\
&= 2\alpha\vec{i} - 3\alpha\vec{j} + 4\alpha\vec{k} - \beta\vec{i} - 2\beta\vec{j} + 2\beta\vec{k} \\
&= (2\alpha - \beta)\vec{i} + (-3\alpha - 2\beta)\vec{j} + (4\alpha + 2\beta)\vec{k}
\end{aligned}$$

$$\begin{aligned}
2\alpha - \beta &= -8 \\
-3\alpha - 2\beta &= 5 \\
4\alpha + 2\beta &= -8
\end{aligned}$$

(2)+(3) $\Rightarrow$ $\alpha = -3$

(1) $\Rightarrow$ $\beta = 2$

$$\left.\begin{array}{l}
\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k} \\
\vec{b} = -\vec{i} - 2\vec{j} + 2\vec{k} \\
\vec{c} = -8\vec{i} + 5\vec{j} - 8\vec{k}
\end{array}\right\} \Longrightarrow \vec{c} = -3\vec{a} + 2\vec{b}$$

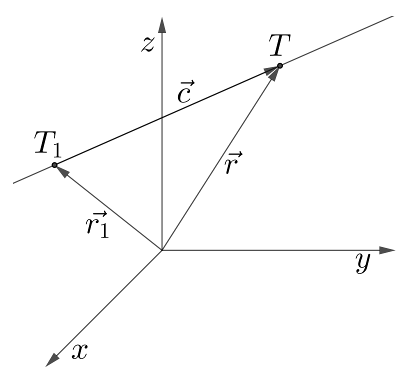
$\square$

Poglavlje 2

Analitička geometrija

2.1 Pravac

Pravac je određen nekom točkom T_1 i vektorom smjera $\vec{c}$ u oznaci $p = (T_1, \vec{c})$



Neka je $\vec{r}$ radijvektor neke proizvoljne točke T pravca p, a $\vec{r_1}$ radijvektor fiksne točke T_1 na pravcu p.

Vektorska jednadžba pravca je: $\vec{r} = \vec{r_1} + t \cdot \vec{c}$.

Ako je $T_1 = (x_1, y_1, z_1)$, $T = (x, y, z)$ i $\vec{c} = (l, m, n)$, vektorska jednadžba je:

$$x\vec{i} + y\vec{j} + z\vec{k} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} + tl\vec{i} + tm\vec{j} + tn\vec{k}$$

Izjednačavanjem koeficijenata dolazimo do parametarskog oblika jednadžbe pravca:

$$p \equiv \begin{cases} x = x_1 + tl \\ y = y_1 + tm \\ z = z_1 + tn \end{cases}$$

Eliminacijom parametra t dobivamo:

$$p \equiv \begin{cases} x = x_1 + tl \\ y = y_1 + tm \\ z = z_1 + tn \end{cases} \Longrightarrow \begin{cases} t = \dfrac{x - x_1}{l} \\ t = \dfrac{y - y_1}{m} \\ t = \dfrac{z - z_1}{n} \end{cases},$$

odnosno kanonski oblik jednadžbe pravca:

$$p \equiv \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n},$$

gdje su l, m i n skalarne komponente vektora smjera, tj. $\vec{c} = (l, m, n)$, a $T_1 = (x_1, y_1, z_1)$ je neka točka pravca p.

Ako je pravac zadan s dvije točke $T_1(x_1, y_1, z_1)$ i $T_2(x_2, y_2, z_2)$, onda je vektor smjera tog pravca

$$\vec{c} = \overrightarrow{T_1T_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Jednadžba pravca koji prolazi točkama T_1 i T_2 je:

$$p \equiv \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Zadatak 2.1. Parametarske jednadžbe pravca p zapiši u kanonskom obliku i odredi vektor smjera $\vec{c}$ ako je: a) $p \equiv \begin{cases} x = 1 + 2t \\ y = -2 + t \\ z = 3 - 2t \end{cases}$ b) $p \equiv \begin{cases} x = 2t \\ y = 1 \\ z = 1 + t \end{cases}$

Rješenje:

a) $$p \equiv \begin{cases} x = 1 + 2t \\ y = -2 + t \\ z = 3 - 2t \end{cases} \Longrightarrow \begin{cases} t = \dfrac{x - 1}{2} \\ t = y + 2 \\ t = \dfrac{z - 3}{-2} \end{cases} \Longrightarrow p \equiv \frac{x - 1}{2} = \frac{y + 2}{1} = \frac{z - 3}{-2}$$

Skalarne komponente vektora smjera vidimo iz nazivnika pa je $\vec{c} = 2\vec{i} + \vec{j} - 2\vec{k}$

b) $$p \equiv \begin{cases} x = 2t \\ y = 1 \\ z = 1 + t \end{cases} \Longrightarrow \begin{cases} t = \dfrac{x}{2} \\ t = z - 1 \end{cases} \Longrightarrow p \equiv \underbrace{\frac{x}{2} = \frac{y - 1}{0} = \frac{z - 1}{1}}_{\text{formalni zapis}}$$

Skalarne komponente vektora smjera vidimo iz nazivnika pa je $\vec{c} = 2\vec{i} + \vec{k}$

Zadatak 2.2. Odredite kanonsku i parametarsku jednadžbu pravca koji prolazi

a) točkom $M(1, 2, -1)$ i ima vektor smjera $\vec{c} = (1, 3, -1)$

b) točkama $M(1, 2, -1)$ i $N(2, 0, 3)$

c) točkama $M(1, 2, -1)$ i $N(1, 1, 2)$

Rješenje:

a) $p \equiv \begin{cases} x = 1 + t \\ y = 2 + 3t \\ z = -1 - t \end{cases}$ i $p \equiv \dfrac{x-1}{1} = \dfrac{y-2}{3} = \dfrac{z+1}{-1}$

b) Vektor smjera $\vec{c} = \overrightarrow{MN} \Longrightarrow \vec{c} = \vec{i} - 2\vec{j} + 4\vec{k}$

$$p \equiv \begin{cases} x = 1 + t \\ y = 2 - 2t \\ z = -1 + 4t \end{cases} \quad \text{i} \quad p \equiv \frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{4}$$

c) Vektor smjera $\vec{c} = \overrightarrow{MN} \Longrightarrow \vec{c} = -\vec{j} + 3\vec{k}$

$$p \equiv \begin{cases} x = 1 \\ y = 2 - t \\ z = -1 + 3t \end{cases} \quad \text{i} \quad p \equiv \frac{x-1}{0} = \frac{y-2}{-1} = \frac{z+1}{3}$$

Zadatak 2.3. Odredite nepoznati parametar λ tako da pravci $p_1 \equiv \dfrac{x+1}{1} = \dfrac{y}{-3} = \dfrac{z-1}{4}$ i $p_2 \equiv \dfrac{x-3}{\lambda} = \dfrac{y+2}{4} = \dfrac{z-4}{-5}$ imaju presječnu točku i odredite je.

Rješenje:

Zapišimo jednadžbe pravaca u parametarskom obliku. Za različite pravce imamo različite parametre.

$$p_1 \equiv \begin{cases} x = -1 + t \\ y = -3t \\ z = 1 + 4t \end{cases} \quad \text{i} \quad p_2 \equiv \begin{cases} x = 3 + \lambda s \\ y = -2 + 4s \\ z = 4 - 5s \end{cases}$$

$$p_1 \cap p_2 \neq \emptyset \Longrightarrow \begin{cases} -1+t = 3+\lambda s \\ \quad -3t = -2+4s \\ 1+4t = 4-5s \end{cases}$$

$$\begin{aligned} -1+t &= 3+\lambda s \\ -3t &= -2+4s \quad / \cdot 4 \\ 1+4t &= 4-5s \quad / \cdot 3 \\ \hline -12t &= -8+16s \\ 3+12t &= 12-15s \\ \hline 3 &= 4+s \\ s &= -1 \end{aligned}$$

$$-3t = -6 \implies t = 2$$

$$\begin{aligned} -1+t &= 3+\lambda s \\ -1+2 &= 3-\lambda \\ \lambda &= 2 \end{aligned}$$

$$\begin{aligned} x &= -1+t && \implies x = 1 \\ y &= -3t && \implies y = -6 \\ z &= 1+4t && \implies z = 9 \end{aligned}$$

Točka presjeka je $S = (1, -6, 9)$

Kut među pravcima $p_1 = (T_1, \vec{c_1})$ i $p_2 = (T_2, \vec{c_2})$ je kut među njihovim vektorima smjera, tj. $\varphi = \angle(\vec{c_1}, \vec{c_2}), \varphi \in \left[0, \frac{\pi}{2}\right]$

Neka je $\vec{c_1} = (l_1, m_1, n_1)$ i $\vec{c_2} = (l_2, m_2, n_2)$. Tada je

$$\cos\varphi = \frac{\vec{c_1} \cdot \vec{c_2}}{|\vec{c_1}| \cdot |\vec{c_2}|} = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}$$

Zadatak 2.4. Odredite kut među pravcima:

a) $p_1 \equiv \dfrac{x-2}{-3} = \dfrac{y-6}{-4} = \dfrac{z-4}{0}$ i $p_2 \equiv \dfrac{x+1}{-3} = \dfrac{y-2}{-4} = \dfrac{z+3}{5}$;

b) $p_1 \equiv \dfrac{x-1}{1} = \dfrac{y}{-2} = \dfrac{z-4}{-7}$ i $p_2 \equiv \dfrac{x+6}{5} = \dfrac{y-2}{1} = \dfrac{z-3}{1}$.

Rješenje:

a) $\vec{c_1} = (-3, -4, 0)$, $\vec{c_2} = (-3, -4, 5)$, $\varphi = \angle(p_1, p_2) = \angle(\vec{c_1}, \vec{c_2})$

$$\begin{aligned}
\cos\varphi &= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}} \\
&= \frac{-3 \cdot (-3) - 4 \cdot (-4) + 0 \cdot 5}{\sqrt{(-3)^2 + (-4)^2 + 0^2} \cdot \sqrt{(-3)^2 + (-4)^2 + 5^2}} \\
&= \frac{25}{\sqrt{25} \cdot \sqrt{50}} \\
&= \frac{25}{25\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

$\varphi = 45°$

b) $\vec{c_1} = (1, -2, -7)$, $\vec{c_2} = (5, 1, 1)$, $\varphi = \angle(p_1, p_2) = \angle(\vec{c_1}, \vec{c_2})$

$$\begin{aligned}
\cos\varphi &= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}} \\
&= \frac{1 \cdot 5 - 2 \cdot 1 - 7 \cdot 1}{\sqrt{1^2 + (-2)^2 + (-7)^2} \cdot \sqrt{5^2 + 1^2 + 1^2}} \\
&= \frac{-4}{\sqrt{54} \cdot \sqrt{27}} \\
&= \frac{-4}{27\sqrt{2}}
\end{aligned}$$

$\varphi = 96°1'$, pa kako je $\varphi > 90°$ zamjenjujemo ga s njegovim suplementarnim kutem.

$\angle(p_1, p_2) = 180° - 96°1' = 83°59'$

Zadatak 2.5. Pravac p prolazi točkama $A(-2, 1, 3)$ i $B(0, -1, 2)$. Odredite kuteve što ga zatvara s koordinatnim osima

Rješenje:

$\vec{c} = \overrightarrow{AB} = 2\vec{i} - 2\vec{j} - \vec{k}$

$|\vec{c}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$\alpha = \angle(\vec{c}, \vec{i})$	$\beta = \angle(\vec{c}, \vec{j})$	$\gamma = \angle(\vec{c}, \vec{k})$
$\cos\alpha = \dfrac{x_{\vec{c}}}{\lvert\vec{c}\rvert} = \dfrac{2}{3}$	$\cos\beta = \dfrac{y_{\vec{c}}}{\lvert\vec{c}\rvert} = -\dfrac{2}{3}$	$\cos\gamma = \dfrac{z_{\vec{c}}}{\lvert\vec{c}\rvert} = -\dfrac{1}{3}$
$\alpha = 48°11'$	$\beta = 131°48'$	$\gamma = 109°28'$
	$\beta > 90° \Rightarrow \begin{array}{l} \beta = 180° - 131°48' \\ \beta = 48°12' \end{array}$	$\gamma > 90° \Rightarrow \begin{array}{l} \gamma = 180° - 109°28' \\ \gamma = 70°32' \end{array}$

Zadatak 2.6. Zadan je pravac $p \equiv \dfrac{x-3}{2} = \dfrac{y+1}{1} = \dfrac{z-1}{4}$.

a) Leži li točka $A(1, 0, -3)$ na pravcu p?

b) Prolazi li pravac p ishodištem $O(0, 0, 0)$?

c) Ako ne prolazi, odredite jednadžbu pravca p_1 za kojeg vrijedi: $p \,||\, p_1$ i $O(0, 0, 0) \in p_1$.

Rješenje:

a) $A(1, 0, -3) \in^? p$

$\dfrac{x-3}{2} = \dfrac{y+1}{1} = \dfrac{z-1}{4}$

$\dfrac{1-3}{2} = \dfrac{1}{1} = \dfrac{-3-1}{4}$

$-1 \neq 1 \neq -1$

$A \notin p$

b) $O(0, 0, 0) \in^? p$

$\dfrac{x-3}{2} = \dfrac{y+1}{1} = \dfrac{z-1}{4}$

$\dfrac{0-3}{2} = \dfrac{0+1}{1} = \dfrac{0-1}{4}$

$-\dfrac{3}{2} \neq 1 \neq -\dfrac{1}{4}$

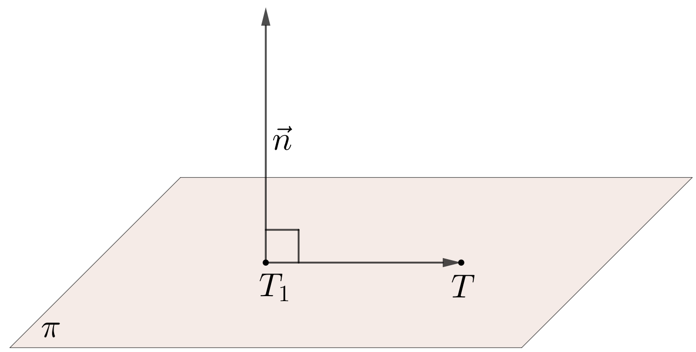
$O \notin p$

c) $\vec{c_1} = \vec{c} = 2\vec{i} + \vec{j} + 4\vec{k}$

$O \in p_1 \Longrightarrow p_1 \equiv \dfrac{x-0}{2} = \dfrac{y-0}{1} = \dfrac{z-0}{4}$, tj. $p_1 \equiv \dfrac{x}{2} = y = \dfrac{z}{4}$

2.2 Ravnina

Ravnina π jedinstveno je određena jednom svojom točkom $T_1 = (x_1, y_1, z_1)$ i vektorom normale $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$. Tako definiranu ravninu označavamo s $\pi = (T_1, \vec{n})$.



Neka je $T = (x, y, z)$ proizvoljna točka ravnine π. Tada je

$$\overrightarrow{T_1T} = (x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k}.$$

Kako $\overrightarrow{T_1T}$ leži u ravnini π, on je okomit na vektor normale $\vec{n}$ pa slijedi:

$$\begin{aligned} \vec{n} \cdot (\overrightarrow{T_1T}) &= 0 \\ \left(A\vec{i} + B\vec{j} + C\vec{k}\right) \cdot \left((x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k}\right) &= 0 \end{aligned}$$

pa je vektorska jednadžba ravnine zadane točkom $T_1 = (x_1, y_1, z_1)$ i vektorom normale $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$:

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

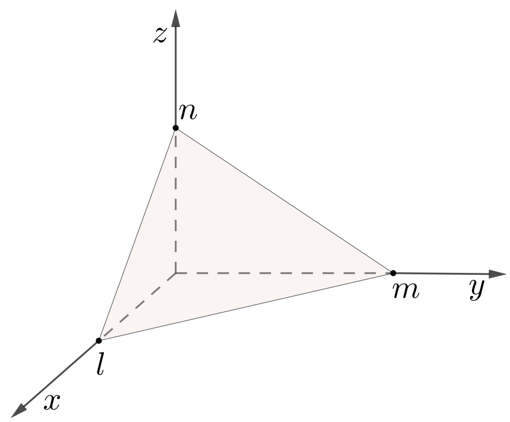
Kanonski oblik jednadžbe ravnine je:

$$Ax + By + Cz + D = 0$$

Ako je $D \neq 0$ dobivamo i segmentni oblik jednadžbe ravnine:

$$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 1,$$

gdje su l, m i n odsječci na koordinatnim osima.



Zadatak 2.7. Napišite jednadžbu ravnine koja prolazi:

a) ishodištem $O(0,0,0)$ i ima vektor normale $\vec{n} = (2,1,0)$.

b) točkom $M(1,0,-1)$ i ima vektor normale $\vec{n} = (0,1,2)$.

c) točkom $T(1,-2,0)$ i okomita je na dužinu $\overline{TS}$, gdje su $T(1,-2,0)$ i $S(0,-1,1)$.

Rješenje:

a) Ravnina π određena je s $\vec{n} = (2,1,0)$ i točkom $O(0,0,0)$:

$$\begin{aligned} A(x-x_1) + B(y-y_1) + C(z-z_1) &= 0 \\ 2(x-0) + 1(y-0) + 0(z-0) &= 0 \\ \pi \equiv 2x + y &= 0 \end{aligned}$$

b) Ravnina π određena je s $\vec{n} = (0,1,2)$ i točkom $M(1,0,-1)$:

$$\begin{aligned} A(x-x_1) + B(y-y_1) + C(z-z_1) &= 0 \\ 0(x-1) + 1(y-0) + 2(z+1) &= 0 \\ \pi \equiv y + 2z + 2 &= 0 \end{aligned}$$

c) $\overline{TS} \perp \pi$ pa je $\vec{n} = \overrightarrow{TS} = -\vec{i} + \vec{j} + \vec{k}$. Ravnina π određena je s $\vec{n} = (-1,1,1)$ i točkom $T(1,-2,0)$

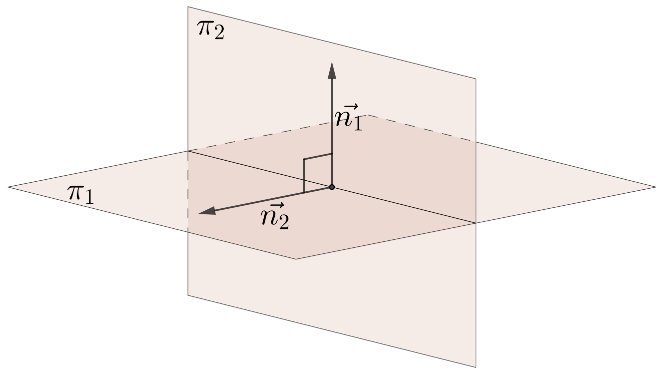
$$\begin{aligned} A(x-x_1) + B(y-y_1) + C(z-z_1) &= 0 \\ -1(x-1) + 1(y+2) + 1(z-0) &= 0 \\ \pi \equiv -x + y + z + 3 &= 0 \end{aligned}$$

Neka su $\pi_1 = (T_1, \vec{n_1})$ i $\pi_2 = (T_2, \vec{n_2})$, gdje su $\vec{n_1} = A_1\vec{i} + B_1\vec{j} + C_1\vec{k}$ i $\vec{n_2} = A_2\vec{i} + B_2\vec{j} + C_2\vec{k}$.

Ravnine su okomite ako su okomiti njihovi vektori normale, tj.

$$\vec{n_1} \perp \vec{n_2} \Longrightarrow \pi_1 \perp \pi_2$$

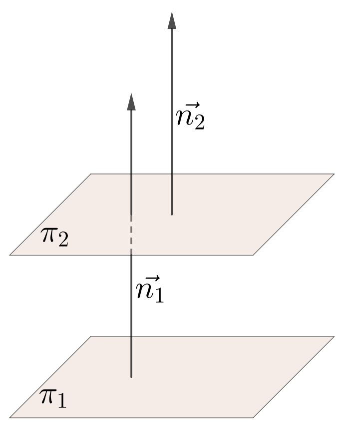
.



Ravnine su usporedne ako su njihovi vektori normale kolinearni, tj.

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

.



Zadatak 2.8. Odredite jednadžbu ravnine koja prolazi

a) točkom $T(1,0,2)$ i okomita je na os x.

b) točkom $T(2,1,2)$ i usporedna je s $xy-$ravninom.

c) točkom $T(2,-1,3)$ i usporedna je s osima y i z.

d) točkom $T(1,3,1)$ i osi x.

e) točkama $T(3,0,2)$, $S(1,2,-1)$ i usporedna je s osi x.

Rješenje:

a) Ravnina π određena je točkom $T(1,0,2)$ i vektorom normale $\vec{n}=\vec{i}$

$1(x-1)=0$, tj. $\pi \equiv x-1=0$

b) Jer je π usporedna s $xy-$ravninom, a $\vec{i},\vec{j}\perp \vec{i}\times\vec{j}$, njezin je vektor normale $\vec{n}=\vec{i}\times\vec{j}=\vec{k}$.

Ravnina π određena je točkom $T(2,1,2)$ i normalom $\vec{n}=\vec{k}$

$\pi \equiv z-2=0$

c) Jer je π usporedna s osima y i z, a $\vec{j},\vec{k}\perp\vec{j}\times\vec{k}$, njezin je vektor normale $\vec{n}=\vec{j}\times\vec{k}=\vec{i}$.

Ravnina π određena je točkom $T(2,-1,3)$ i normalom $\vec{n}=\vec{i}$

$\pi \equiv x-2=0$

d) Jer ravnina π sadrži os x, sadrži i točku $O(0,0,0)$, pa sadrži i vektor $\overrightarrow{OT}=\vec{i}+3\vec{j}+\vec{k}$. $\vec{i},\overrightarrow{OT}\subset\pi$ i $\vec{i},\overrightarrow{OT}\perp\overrightarrow{i}\times\overrightarrow{OT}$, dobivamo da je

$$\begin{aligned}
\vec{n} &= \overrightarrow{i}\times\overrightarrow{OT}\\
&= \vec{i}\times\left(\vec{i}+3\vec{j}+\vec{k}\right)\\
&= \underbrace{\vec{i}\times\vec{i}}_{\vec{0}}+3\vec{i}\times\vec{j}+\vec{i}\times\vec{k}\\
&= 3\vec{k}-\vec{j}\\
&= -\vec{j}+3\vec{k}
\end{aligned}$$

$\pi \equiv -y+3z=0$

e) Jer ravnina π sadrži vektor $\overrightarrow{ST}=2\vec{i}-2\vec{j}+3\vec{k}$ i usporedna je s osi x, dobivamo da je

$$\begin{aligned}
\vec{n} &= \overrightarrow{i}\times\overrightarrow{ST}\\
&= \vec{i}\times\left(2\vec{i}-2\vec{j}+3\vec{k}\right)\\
&= \underbrace{2\vec{i}\times\vec{i}}_{\vec{0}}-2\vec{i}\times\vec{j}+3\vec{i}\times\vec{k}\\
&= -2\vec{k}-3\vec{j}\\
&= -3\vec{j}-2\vec{k}
\end{aligned}$$

$-3(y-2)-2(z+1)=0$

$-3y+6-2z-2=0$

$\pi \equiv 3y+2z-4=0$

Zadatak 2.9. Odredite jednadžbu ravnine π koja je

a) okomita na ravnine $\pi_1 \equiv 2x-y+z-2=0$ i $\pi_2 \equiv x+z+1=0$ i prolazi točkom $T(1,2,-1)$.

b) okomita na ravninu $\pi_1 \equiv 3x-2y+z-3=0$ i prolazi točkama $T(2,1,3)$ i $S(1,0,-1)$.

Rješenje:

a) $\pi \perp \pi_1, \pi_2 \Longrightarrow \vec{n} \perp \vec{n_1}, \vec{n_2}$ i $\vec{n_1}=(2,-1,1)$, $\vec{n_2}=(1,0,1)$ pa imamo:

$$\begin{aligned}
\vec{n} &= \vec{n_1} \times \vec{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \\
&= \vec{i} \cdot (-1-0) - \vec{j} \cdot (2-1) + \vec{k} \cdot (0+1) \\
&= -\vec{i} - \vec{j} + \vec{k} \Longrightarrow \vec{n} = (-1,-1,1)
\end{aligned}$$

$-1(x-1)-1(y-2)+(z+1)=0$

$-x+1-y+2+z+1=0$

$\pi \equiv -x-y+z+4=0$

b) $\pi \perp \pi_1$ i $\overrightarrow{TS} \subset \pi \Longrightarrow \vec{n} \perp \vec{n_1}, \overrightarrow{TS}$ i $\vec{n_1}=(3,-2,1)$, $\overrightarrow{TS}=(-1,-1,-4)$ pa imamo:

$$\begin{aligned}
\vec{n} &= \vec{n_1} \times \overrightarrow{TS} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -1 & -1 & -4 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -2 & 1 \\ -1 & -4 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 3 & 1 \\ -1 & -4 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (8+1) - \vec{j} \cdot (-12+1) + \vec{k} \cdot (-3-2) \\
&= 9\vec{i} + 11\vec{j} - 5\vec{k} \Longrightarrow \vec{n} = (9,11,-5)
\end{aligned}$$

$9(x-1)+11(y-0)-5(z+1)=0$

$9x - 9 + 11y - 5z - 5 = 0$

$\pi \equiv 9x + 11y - 5z - 14 = 0$

Zadatak 2.10. Odredite jednadžbu ravnine određenu točkama $M(1, -1, 2)$,$N(3, 2, 0)$ i $P(1, -2, 1)$.

Rješenje:

$M, N, P \in \pi$ pa slijedi da je $\overrightarrow{MN}, \overrightarrow{MP} \subset \pi$, odnosno $\vec{n} \perp \overrightarrow{MN}, \overrightarrow{MP}$

$\overrightarrow{MN} = 2\vec{i} + 3\vec{j} - 2\vec{k}, \overrightarrow{MP} = -\vec{j} - \vec{k}$

$$\begin{aligned}
\vec{n} &= \overrightarrow{MN} \times \overrightarrow{MP} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -2 \\ 0 & -1 & -1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 2 & -2 \\ 0 & -1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (-3 - 2) - \vec{j} \cdot (-2 - 0) + \vec{k} \cdot (-2 - 0) \\
&= -5\vec{i} + 2\vec{j} - 2\vec{k} \Longrightarrow \vec{n} = (-5, 2, -2)
\end{aligned}$$

$-5(x - 1) + 2(y + 1) - 2(z - 2) = 0$

$-5x + 5 + 2y + 2 - 2z + 4 = 0$

$\pi \equiv -5x + 2y - 2z + 11 = 0$

Neka je $M = (x_0, y_0, z_0)$ točka i $\pi \equiv Ax + By + Cz + D = 0$ ravnina. **Udaljenost točke M od ravnine** π je:

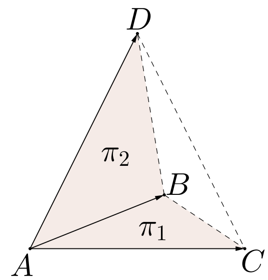
$$d(M, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Kut među ravninama $\pi_1 = (T_1, \vec{n_1})$ i $\pi_2 = (T_2, \vec{n_2})$ je kut što ga zatvaraju njihove normale $\vec{n_1} = (A_1, B_1, C_1)$ i $\vec{n_2} = (A_2, B_2, C_2)$, tj. $\varphi = \angle(\pi_1, \pi_2) = \angle(\vec{n_1}, \vec{n_2}), \varphi \in \left[0, \frac{\pi}{2}\right]$.

$$\cos\varphi = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}| \cdot |\vec{n_2}|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Zadatak 2.11. Točke $A(2, 5, 0)$, $B(1, 6, 2)$, $C(-1, 4, 1)$ i $D(1, 4, 3)$ određuju tetraedar. Odredi kut među stranama ABC i ABD.

Rješenje:



$$\begin{aligned}
\varphi &= \angle\left(\pi_1(A,B,C),\pi_2(A,B,D)\right)\\
&= \angle(\vec{n_1},\vec{n_2})
\end{aligned}$$

$$\begin{aligned}
\vec{n_1} &= \overrightarrow{AB}\times\overrightarrow{AC}\\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & 2\\ -3 & -1 & 1\end{vmatrix}\\
&= \vec{i}\cdot\begin{vmatrix} 1 & 2\\ -1 & 1\end{vmatrix} - \vec{j}\cdot\begin{vmatrix} -1 & 2\\ -3 & 1\end{vmatrix} + \vec{k}\cdot\begin{vmatrix} -1 & 1\\ -3 & -1\end{vmatrix}\\
&= \vec{i}\cdot(1+2)-\vec{j}\cdot(-1+6)+\vec{k}\cdot(1+3)\\
&= 3\vec{i}-5\vec{j}+4\vec{k}\Longrightarrow \vec{n_1}=(3,-5,4)
\end{aligned}$$

$$\begin{aligned}
\vec{n_2} &= \overrightarrow{AB}\times\overrightarrow{AD}\\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & 2\\ -1 & -1 & 3\end{vmatrix}\\
&= \vec{i}\cdot\begin{vmatrix} 1 & 2\\ -1 & 3\end{vmatrix} - \vec{j}\cdot\begin{vmatrix} -1 & 2\\ -1 & 3\end{vmatrix} + \vec{k}\cdot\begin{vmatrix} -1 & 1\\ -1 & -1\end{vmatrix}\\
&= \vec{i}\cdot(3+2)-\vec{j}\cdot(-3+2)+\vec{k}\cdot(1+1)\\
&= 5\vec{i}+\vec{j}+2\vec{k}\Longrightarrow \vec{n_2}=(5,1,2)
\end{aligned}$$

$$\begin{aligned}
\cos\varphi &= \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}\\
&= \frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}}\\
&= \frac{3\cdot5-5\cdot1+4\cdot2}{\sqrt{3^2+5^2+4^2}\cdot\sqrt{5^2+1^2+2^2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{15-5+8}{\sqrt{50}\cdot\sqrt{30}} \\
&= \frac{18}{10\sqrt{15}} \\
\varphi &= 62^\circ 17'
\end{aligned}
$$

Zadatak 2.12. Odredite jednadžbu ravnine π koja sadrži sve točke jednako udaljene od:

a) dviju ravnina $\pi_1 \equiv x-3y-2z+1=0$ i $\pi_2 \equiv 2x-y+3z+3=0$.

b) dviju točaka $T_1(2,-1,3)$ i $T_2(1,2,-1)$.

Rješenje:

a) Za svaku točku $T(x,y,z)\in\pi$ vrijedi $d(T,\pi_1)=d(T,\pi_2)$.

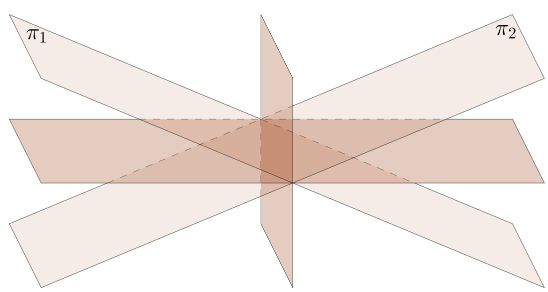
$$\frac{|A_1x+B_1y+C_1z+D_1|}{\sqrt{A_1^2+B_1^2+C_1^2}} = \frac{|A_2x+B_2y+C_2z+D_2|}{\sqrt{A_2^2+B_2^2+C_2^2}}$$

$$\frac{|x-3y-2z+1|}{\sqrt{1^2+3^2+2^2}} = \frac{|2x-y+3z+3|}{\sqrt{2^2+1^2+3^2}} / \cdot \sqrt{14}$$

$$|x-3y-2z+1| = |2x-y+3z+3|$$

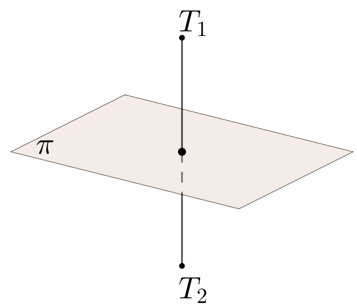
1. slučaj: $x-3y-2z+1=2x-y+3z+3 \Longrightarrow \pi\equiv -x-2y-5z-2=0$

2. slučaj: $x-3y-2z+1=-2x+y-3z-3 \Longrightarrow \pi\equiv 3x-4y+z+4=0$



b) Za svaku točku $T(x,y,z)\in\pi$ vrijedi $d(T,T_1)=d(T,T_2)$.

$$
\begin{aligned}
\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} &= \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2} \\
\sqrt{(x-2)^2+(y+1)^2+(z-3)^2} &= \sqrt{(x-1)^2+(y-2)^2+(z+1)^2} \ /^2 \\
x^2-4x+4+y^2+2y+1+z^2-6z+9 &= x^2-2x+1+y^2-4y+4+z^2+2z+1 \\
-2x+6y-8z+8 &= 0 / : (-2) \\
\pi\equiv x-3y+4z-4 &= 0
\end{aligned}
$$



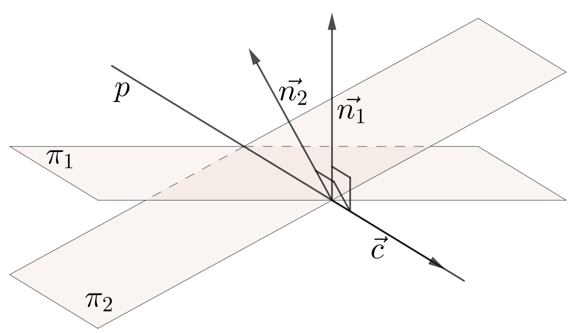
Zadatak 2.13. Pravac p zadan je kao presječnica dviju ravnina

$$p \equiv \begin{cases} x + y - 4z - 5 = 0 \\ 2x - y - 2z - 1 = 0 \end{cases}$$

Odredite njegovu jednadžbu u kanonskom i parametarskom obliku.

Rješenje:

Kako je $p(T, \vec{c}) \subset \pi_1, p \subset \pi_2$, imamo da je $\vec{n_1} = (1, 1, -4) \perp \vec{c}$ i $\vec{n_2} = (2, -1, -2) \perp \vec{c}$, odnosno $\vec{c} = \vec{n_1} \times \vec{n_2}$.



$$\begin{aligned}
\vec{c} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (-2 - 4) - \vec{j} \cdot (-2 + 8) + \vec{k} \cdot (-1 - 2) \\
&= -6\vec{i} - 6\vec{j} - 3\vec{k}
\end{aligned}$$

Možemo pisati i $\vec{c} = -\frac{1}{3}\left(-6\vec{i} - 6\vec{j} - 3\vec{k}\right) = 2\vec{i} + 2\vec{j} + \vec{k}$, jer su kolinearni.

Odredimo sada neku točku $T \in \pi_1 \cap \pi_2$, npr. za $z = -1$

$$\begin{array}{rcl} x + y - 4z - 5 &=& 0 \\ 2x - y - 2z - 1 &=& 0 \\ \hline x + y + 4 - 5 &=& 0 \\ 2x - y + 2 - 1 &=& 0 \\ \hline x + y &=& 1 \\ 2x - y &=& -1 \\ \hline \end{array}$$

$3x = 0 \Longrightarrow x = 0 \Longrightarrow y = 1$

$T(0, 1, -1) \in \pi_1 \cap \pi_2$

$$p(T, \vec{c}) \equiv \frac{x}{2} = \frac{y-1}{2} = \frac{z+1}{1} \quad \text{i} \quad p \equiv \begin{cases} x = 2t \\ y = 1 + 2t \\ z = -1 + t \end{cases}$$

Zadatak 2.14. Odredite jednadžbu ravnine π koja

a) sadrži točku $T(1, 1, 1)$ i okomita je na pravac $p \equiv \dfrac{x-1}{2} = \dfrac{y}{-1} = \dfrac{z-3}{1}$;

b) sadrži točku $T(1, 1, 1)$ i usporedna je s pravcima $p_1 \equiv \dfrac{x-2}{1} = \dfrac{y+1}{0} = \dfrac{z-1}{1}$ i $p_2 \equiv \dfrac{x+1}{2} = \dfrac{y}{1} = \dfrac{z+1}{-1}$.

Rješenje:

a) Kako je $\pi \perp p$, imamo da je $\vec{n} = \vec{c} = (2, -1, 1)$

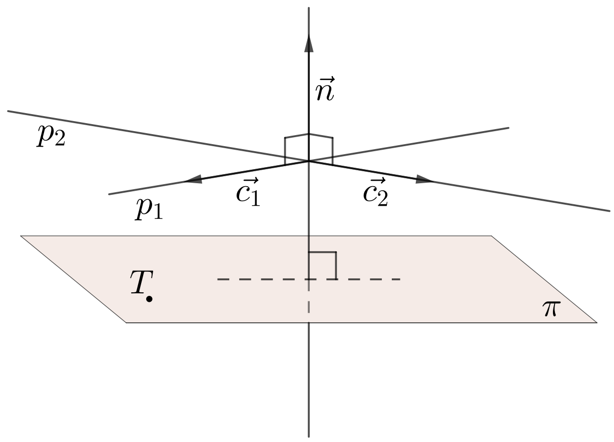
$2(x-1) - 1(y-1) + 1(z-1) = 0$

$\pi \equiv 2x - y + z - 2 = 0$

b) Kako su $p_1(T_1, \vec{c_1}), p_2(T_2, \vec{c_2}) \;||\; \pi$ imamo da je $\vec{n} \perp \vec{c_1}, \vec{c_2}$.

$$\begin{aligned} \vec{n} &= \overrightarrow{c_1} \times \overrightarrow{c_2} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \vec{i}\cdot\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix} - \vec{j}\cdot\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix} + \vec{k}\cdot\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}\\
&= \vec{i}\cdot(0-1) - \vec{j}\cdot(-1-2) + \vec{k}\cdot(1-0)\\
&= -\vec{i}+3\vec{j}+\vec{k} \Longrightarrow \vec{n} = (-1,3,1)
\end{aligned}
$$

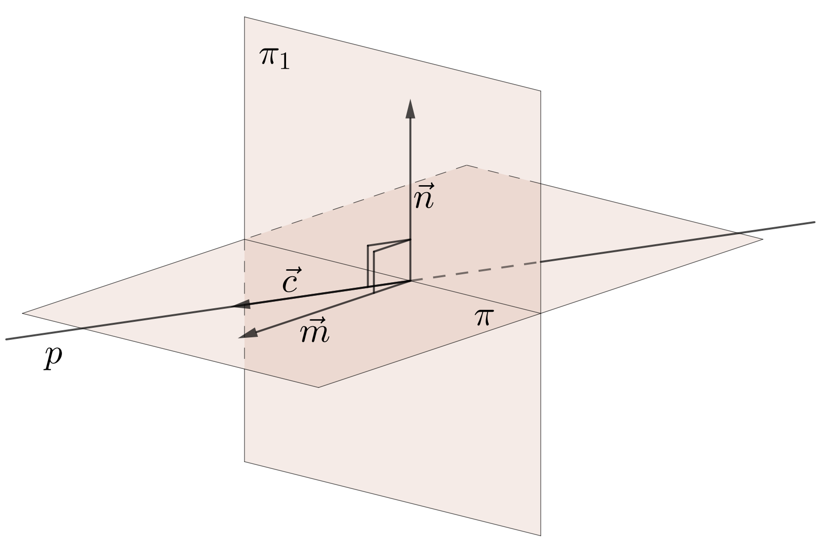


$-1(x-1)+3(y-1)+1(z-1)=0$

$\pi \equiv -x+3y+z-3=0$

Zadatak 2.15. Odredite jednadžbu ravnine π koja sadrži pravac $p \equiv \dfrac{x-1}{1} = \dfrac{y-1}{1} = \dfrac{z-1}{1}$ i okomita je na ravninu $\pi_1 \equiv 2x+3y+z+1=0$.

Rješenje:



Označimo sa $\vec{n}$ normalu ravnine π, a s $\vec{m}=(2,3,1)$ normalu ravnine π_1. Kako je $p \subset \pi$, imamo da je $\vec{c}=(1,1,1)\perp\vec{n}$, a iz $\pi\perp\pi_1$ imamo da je $\vec{n}\perp\vec{m}$

$$\begin{aligned}
\vec{n} &= \overrightarrow{c}\times\overrightarrow{m}\\
&= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & 1\\ 2 & 3 & 1\end{vmatrix}\\
&= \vec{i}\cdot\begin{vmatrix}1 & 1\\ 3 & 1\end{vmatrix}-\vec{j}\cdot\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}+\vec{k}\cdot\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}\\
&= \vec{i}\cdot(1-3)-\vec{j}\cdot(1-2)+\vec{k}\cdot(3-2)\\
&= -2\vec{i}+\vec{j}+\vec{k}\Longrightarrow\vec{n}=(-2,1,1)
\end{aligned}$$

$$-2(x-1)+(y-1)+1(z-1)=0$$

$$\pi\equiv -2x+y+z=0$$

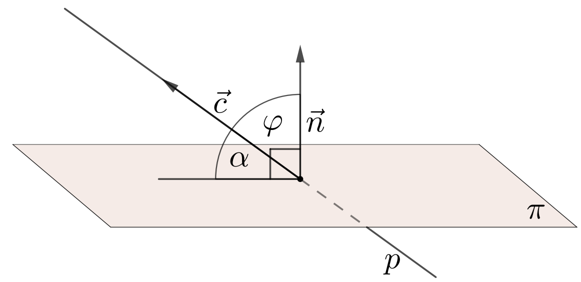
Zadatak 2.16. Pravac p je zadan kao presječnica ravnina $\pi_1\equiv 3x+y-z+1=0$ i $\pi_2\equiv 2x-y+4z-2=0$. Odredite kut što ga zatvaraju pravac p i ravnina $\pi\equiv x-8y+3z-6=0$.

Rješenje: Označimo sa $\vec{n_1}$ vektor normale ravnine π_1, s $\vec{n_2}$ vektor normale ravnine π_2, a s $\vec{c}$ vektor smjera pravca p.

$$\vec{c}\perp\vec{n_1}=(3,1,-1),\ \ \vec{c}\perp\vec{n_2}=(2,-1,4)$$

$$\begin{aligned}
\vec{c} &= \overrightarrow{n_1}\times\overrightarrow{n_2}\\
&= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 1 & -1\\ 2 & -1 & 4\end{vmatrix}\\
&= \vec{i}\cdot\begin{vmatrix}1 & -1\\ -1 & 4\end{vmatrix}-\vec{j}\cdot\begin{vmatrix}3 & -1\\ 2 & 4\end{vmatrix}+\vec{k}\cdot\begin{vmatrix}3 & 1\\ 2 & -1\end{vmatrix}\\
&= \vec{i}\cdot(4-1)-\vec{j}\cdot(12+2)+\vec{k}\cdot(-3-2)\\
&= 3\vec{i}-14\vec{j}-5\vec{k}=(3,-14,-5)
\end{aligned}$$

$$\vec{n}=\vec{i}-8\vec{j}+3\vec{k}=(1,-8,3)$$



Odredimo kut $\varphi = \angle(\vec{n}, \vec{c})$ koji zatvaraju vektor smjera $\vec{c}$ i vektor normale $\vec{n}$.

$$\begin{aligned}
\cos\varphi &= \frac{\vec{n}\cdot\vec{c}}{|\vec{n}|\cdot|\vec{c}|} \\
&= \frac{1\cdot 3 - 8\cdot(-14) + 3\cdot(-5)}{\sqrt{1^2+(-8)^2+3^2}\cdot\sqrt{3^2+(-14)^2+(-5)^2}} \\
&= \frac{3+112-15}{\sqrt{74}\cdot\sqrt{230}} \\
&= \frac{100}{\sqrt{74\cdot 230}} \\
\varphi &= 39°57'
\end{aligned}
\qquad
\begin{aligned}
\alpha &= \angle(\pi, p) \\
&= 90° - \varphi \\
&= 90° - 39°57' \\
&= 50°3'
\end{aligned}$$

Zadatak 2.17. Odredite presjek pravca p i ravnine π ako je:

a) $\pi \equiv \dfrac{x-2}{1} = \dfrac{y}{-2} = \dfrac{z+1}{-1}$ i $\pi \equiv 2x - 3y + z + 4 = 0$

b) $\pi \equiv \dfrac{x-2}{1} = \dfrac{y}{-2} = \dfrac{z+1}{-1}$ i $\pi \equiv 2x - 3y + 8z + 1 = 0$

c) $\pi \equiv \dfrac{x-2}{1} = \dfrac{y}{-2} = \dfrac{z+1}{-1}$ i $\pi \equiv 2x - 3y + 8z + 4 = 0$

Rješenje:

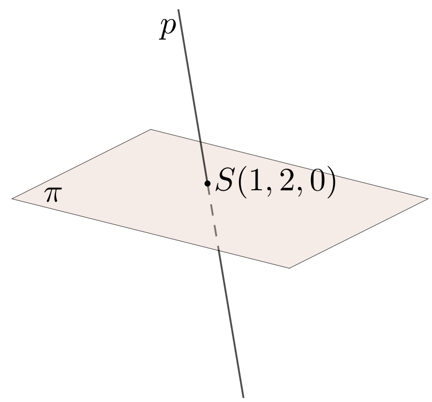
a) Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku: $p \equiv \begin{cases} x = 2 + t \\ y = -2t \\ z = -1 - t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv 2x - 3y + z + 4 = 0$

$$\begin{aligned}
2x - 3y + z + 4 &= 0 \\
2(2+t) - 3(-2t) + (-1-t) + 4 &= 0 \\
4 + 2t + 6t - 1 - t + 4 &= 0
\end{aligned}$$

$$\begin{aligned} 7t &= -7 \\ t &= -1 \end{aligned}$$

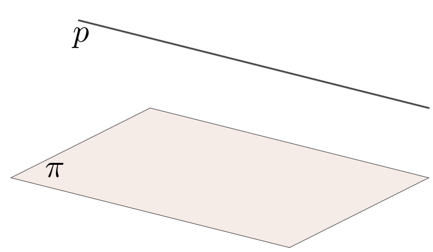
$$p \equiv \begin{cases} x = 2 + (-1) = 1 \\ y = -2(-1) = 2 \\ z = -1 - (-1) = 0 \end{cases} \implies p \cap \pi = S(1,2,0)$$



b) Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku: $p \equiv \begin{cases} x = 2 + t \\ y = -2t \\ z = -1 - t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv 2x - 3y + 8z + 1 = 0$

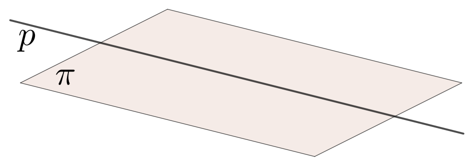
$$\left.\begin{aligned} 2x - 3y + 8z + 1 &= 0 \\ 2(2+t) - 3(-2t) + 8(-1-t) + 1 &= 0 \\ 4 + 2t + 6t - 8 - 8t + 1 &= 0 \\ -3 &\neq 0 \end{aligned}\right. \implies p \cap \pi = \emptyset \implies \pi \,||\, p$$



c) Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku: $p \equiv \begin{cases} x = 2 + t \\ y = -2t \\ z = -1 - t \end{cases}$

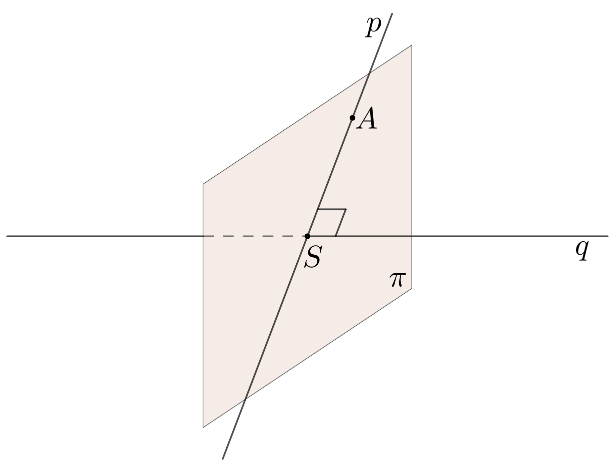
Uvrstimo je u jednadžnu ravnine $\pi \equiv 2x - 3y + 8z + 4 = 0$

$$\left.\begin{aligned} 2x - 3y + 8z + 4 &= 0 \\ 2(2+t) - 3(-2t) + 8(-1-t) + 4 &= 0 \\ 4 + 2t + 6t - 8 - 8t + 4 &= 0 \\ 0 &= 0, \forall t \in \mathbb{R} \end{aligned}\right. \quad \Longrightarrow p \cap \pi = p \text{...pravac leži u ravnini}$$



Zadatak 2.18. Odredite jednadžbu pravca p koji prolazi točkom $A(2, -3, 1)$, okomit je na pravac $q \equiv \dfrac{x-3}{2} = \dfrac{y+3}{-1} = \dfrac{z-5}{3}$ i siječe ga.

Rješenje:



Odredimo jednadžbu ravnine π koja je okomita na q i sadrži točku A, tj. $A \in \pi$ i $\pi \perp q$.

Tada je $\vec{n} = \vec{c_q} = (2, -1, 3)$, pa je

$2(x-2) - (y+3) + 3(z-1) = 0$
$\pi \equiv 2x - y + 3z - 10 = 0$

Odredimo sada točku $S = \pi \cap q$. Parametarski oblik jednadžbe pravca $q \equiv \begin{cases} x = 3 + 2t \\ y = -3 - t \\ z = 5 + 3t \end{cases}$

Uvrstimo je u jednadžbu ravnine $\pi \equiv 2x - y + 3z - 10 = 0$

$$\begin{aligned}
2x-y+3z-10 &= 0\\
2(3+2t)-(-3-t)+3(5+3t)-10 &= 0\\
6+4t+3+t+15+9t-10 &= 0\\
14t+14 &= 0\\
t &= -1
\end{aligned}
\Longrightarrow q \equiv \begin{cases} x = 3+2(-1) = 1\\ y = -3-(-1) = -2\\ z = 5+3(-1) = 2\end{cases}$$

$\pi \cap q = S(1, -2, 2)$

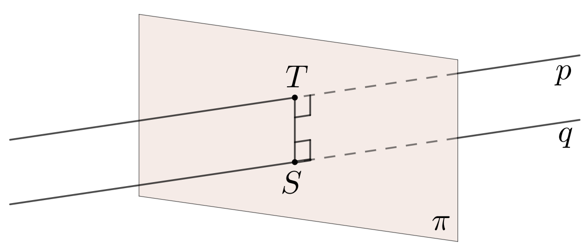
Odredimo sada jednadžbu pravca p koji prolazi točkama A i S.

$\vec{c} = \overrightarrow{AS} = (-1, 1, 1)$, pa je $p \equiv \dfrac{x-2}{-1} = \dfrac{y+3}{1} = \dfrac{z-1}{1}$.

Zadatak 2.19. Odredite udaljenost između pravaca $p \equiv \dfrac{x-1}{3} = \dfrac{y+1}{-2} = \dfrac{z-3}{4}$ i $q \equiv \dfrac{x-6}{3} = \dfrac{y+4}{-2} = \dfrac{z-5}{4}$.

Rješenje:

Odredimo po volji jednu točku $T \in p$, npr. $T(1, -1, 3)$. Odredimo sada jednadžbu ravnine π, za koju vrijedi da je $T \in \pi$ i $\pi \perp p$. Tada je $d(p, q) = d(T, S)$, gdje je $S = \pi \cap q$.



Kako je $\pi \perp p$ imamo da je $\vec{n} = \vec{c} = (3, -2, 4)$

$3(x-1) - 2(y+1) + 4(z-3) = 0$

$\pi \equiv 3x - 2y + 4z - 17 = 0$

Odredimo sada točku $S = \pi \cap q$. Parametarski oblik jednadžbe pravca $q \equiv \begin{cases} x = 6+3t\\ y = -4-2t\\ z = 5+4t\end{cases}$

Uvrstimo je u jednadžbu ravnine $\pi \equiv 3x - 2y + 4z - 17 = 0$

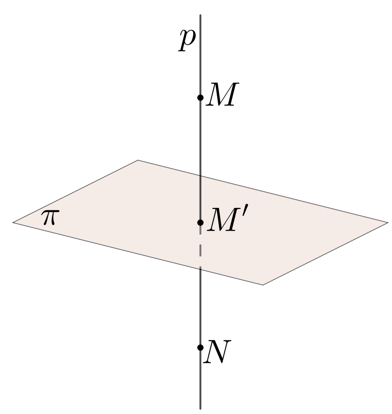
$$
\begin{aligned}
3x - 2y + 4z - 17 &= 0 \\
3(6+3t) - 2(-4-2t) + 4(5+4t) - 17 &= 0 \\
18 + 9t + 8 + 4t + 20 + 16t - 17 &= 0 \\
29t + 29 &= 0 \\
t &= -1
\end{aligned}
\implies q \equiv \begin{cases} x = 6 + 3\cdot(-1) = 3 \\ y = -4 - 2\cdot(-1) = -2 \\ z = 5 + 4\cdot(-1) = 1 \end{cases}
$$

$\pi \cap q = S(3, -2, 1)$

$$
\begin{aligned}
d(p,q) &= d(T,S) \\
&= \sqrt{(x_S - x_T)^2 + (y_S - y_T)^2 + (z_S - z_T)^2} \\
&= \sqrt{(1-3)^2 + (-1+2)^2 + (3-1)^2} \\
&= \sqrt{4+1+4} \\
&= 3
\end{aligned}
$$

Zadatak 2.20. Odredite koordinate točke N koja je simetrična točki $M(1,1,1)$ obzirom na ravninu $\pi \equiv x + y - 2z - 6 = 0$.

Odredimo koordinate točke M', projekcije točke M na ravninu π. Odredimo i jednadžbu pravca p za kojeg vrijedi da je $M \in p$ i $p \perp \pi$. Kako je $p \perp \pi$ imamo da je $\vec{c} = \vec{n} = (1, 1, -2)$ pa je $p \equiv \begin{cases} x = 1 + t \\ y = 1 + t \\ z = 1 - 2t \end{cases}$



$M' = p \cap \pi$

$$\begin{aligned}
x+y-2z-6 &= 0\\
1+t+1+t-2(1-2t)-6 &= 0\\
2+2t-2+4t-6 &= 0 \Longrightarrow p \equiv \begin{cases} x = 1+1=2\\ y=1+1=2\\ z=1-2=-1\end{cases}\\
6t &= 6\\
t &= 1
\end{aligned}$$

$\pi \cap p = M'(2,2,-1)$

Kako je M' polovište dužine $\overline{MN}$ imamo da je $M' = \left(\frac{x_M+x_N}{2}, \frac{y_M+y_N}{2}, \frac{z_M+z_N}{2}\right)$

$$\left.\begin{aligned}
2 &= \frac{1+x_N}{2} \Longrightarrow x_N = 3\\
2 &= \frac{1+y_N}{2} \Longrightarrow y_N = 3\\
-1 &= \frac{1+z_N}{2} \Longrightarrow z_N = -3
\end{aligned}\right\} \Longrightarrow N(3,3,-3)$$

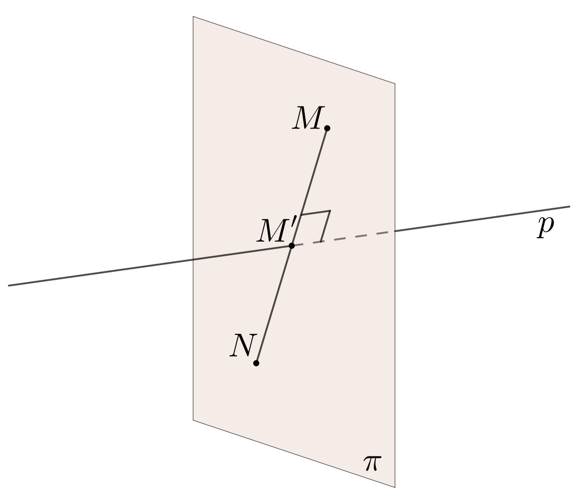
Zadatak 2.21. Odredite točku N simetričnu točki $M(1,0,2)$ obzirom na pravac $p \equiv \frac{x-2}{3} = \frac{y}{5} = \frac{z+1}{1}$.

Rješenje:

Odredimo jednadžbu ravnine π za koju vrijedi da je $\pi \perp p$ i $M \in \pi$. Kako je $\pi \perp p$ imamo da je $\vec{n} = \vec{c} = (3,5,1)$.

$3(x-1)+5(y-0)+1(z-2)=0$

$\pi \equiv 3x+5y+z-5=0$



Odredimo sada koordinate točke $M' = p \cap \pi$.

Parametarski oblik jednadžbe pravca $p \equiv \begin{cases} x = 2 + 3t \\ y = 5t \\ z = -1 + t \end{cases}$

Uvrstimo je u jednadžbu ravnine $\pi \equiv 3x + 5y + z - 5 = 0$

$$\begin{aligned} 3x + 5y + z - 5 &= 0 \\ 3(2 + 3t) + 5(5t) + (-1 + t) - 5 &= 0 \\ 6 + 9t + 25t - 1 + t - 5 &= 0 \\ 35t &= 0 \\ 35t &= 0 \end{aligned} \implies p \equiv \begin{cases} x = 2 + 3 \cdot 0 = 2 \\ y = 5 \cdot 0 = 0 \\ z = -1 + 0 = -1 \end{cases}$$

$p \cap \pi = M'(2, 0, -1)$

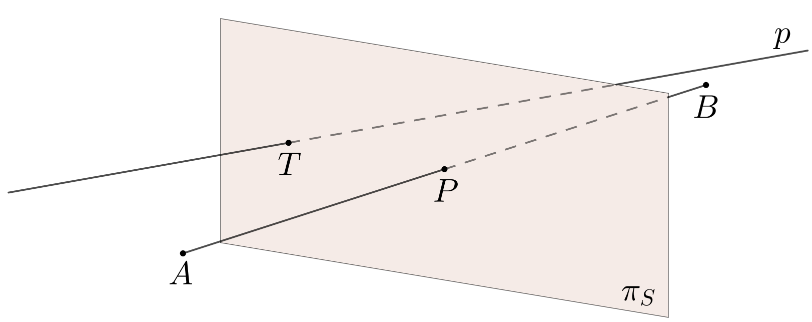
Kako je M' polovište dužine $\overline{MN}$ imamo da je $M' = \left(\frac{x_M + x_N}{2}, \frac{y_M + y_N}{2}, \frac{z_M + z_N}{2} \right)$

$$\left. \begin{aligned} 2 = \frac{1 + x_N}{2} &\implies x_N = 3 \\ 0 = \frac{0 + y_N}{2} &\implies y_N = 0 \\ -1 = \frac{2 + z_N}{2} &\implies z_N = -4 \end{aligned} \right\} \implies N(3, 0, -4)$$

Zadatak 2.22. Na pravcu $p \equiv x = y = z$ odredite točku T koja je jednako udaljena od točaka $A(2, 4, 0)$ i $B(4, 0, 4)$.

Rješenje: Odredimo jednadžbu simetralne ravnine dužine $\overline{AB}$, tj. ravnine $\pi_S(P, \vec{n})$, gdje je P polovište dužine $\overline{AB}$, a $\vec{n} = \overrightarrow{AB} = (2, -4, 4)(\pi_S \perp \overline{AB})$.

$$P = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2} \right) = \left(\frac{2 + 4}{2}, \frac{4 + 0}{2}, \frac{0 + 4}{2} \right) = (3, 2, 2)$$



$2(x-3)-4(y-2)+4(z-2)=0$

$2x-4y+4z-6=0/:2$

$\pi_S \equiv x-2y+2z-3=0$

Parametarski oblik jednadžbe pravca p je $p \equiv \begin{cases} x=t \\ y=t \\ z=t \end{cases}$

Odredimo sada koordinate točke $T = p \cap \pi_S$

$$\begin{aligned} x-2y+2z-3 &= 0 \\ t-2t+2t-3 &= 0 \Longrightarrow p \equiv \begin{cases} x=3 \\ y=3 \\ z=3 \end{cases} \\ t &= 3 \end{aligned}$$

$p \cap \pi_S = T(3,3,3)$

Zadatak 2.23. Odredite udaljenost točke $A(4,-5,4)$ od ravnine određene pravcima zadanima kao presjek ravnina $p_1 \equiv \begin{cases} \pi_1 ... x-y+z-3=0 \\ \pi_2 ... x+y+z-1=0 \end{cases}$ i $p_2 \equiv \begin{cases} \pi_3 ... y=0 \\ \pi_4 ... x+z=0 \end{cases}$

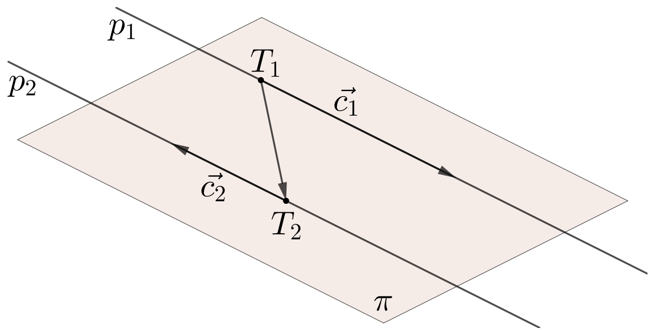
Rješenje: Odredimo vektore smjera pravaca p_1 i p_2.

$$\begin{aligned} \vec{c_1} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \vec{i} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ &= \vec{i} \cdot (-1-1) - \vec{j} \cdot (1-1) + \vec{k} \cdot (1+1) \\ &= -2\vec{i} + 2\vec{k} = (-2,0,2) \end{aligned}$$

$$\begin{aligned} \vec{c_2} &= \overrightarrow{n_3} \times \overrightarrow{n_4} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\ &= \vec{i} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= \vec{i} \cdot (1-0) - \vec{j} \cdot (0-0) + \vec{k} \cdot (0-1) \\ &= \vec{i} - \vec{k} = (1,0,-1) \end{aligned}$$

Uočimo da je $\vec{c_1} = -2\vec{c_2}$ pa izlazi da su kolinearni, tj. $\vec{c_1} \times \vec{c_2} = \vec{0}$

$$\left(\vec{n} = \overrightarrow{c_1} \times \overrightarrow{c_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{vmatrix} = \vec{i} \cdot (0-0) - \vec{j} \cdot (2-2) + \vec{k} \cdot (0-0) = \vec{0} \right)$$



Odredimo točku $T_1 \in p_1$, npr. za $z = 0$ i točku $T_2 \in p_2$, npr. za $z = -1$

$$\begin{array}{rcl} x - y + z - 3 & = & 0 \\ x + y + z - 1 & = & 0 \\ \hline x - y & = & 3 \\ x + y & = & 1 \\ \hline 2x & = & 4 \\ x = 2 \quad \text{i} \quad y & = & -1 \Longrightarrow T_1(2, -1, 0) \end{array} \qquad \begin{array}{rcl} y & = & 0 \\ x + z & = & 0 \\ \hline y & = & 0 \\ x - 1 & = & 0 \\ \hline y & = & 0 \\ x & = & 1 \quad \Longrightarrow T_2(1, 0, -1) \end{array}$$

Sada je normala ravnine π jednaka $\vec{n} = \vec{c_2} \times \overrightarrow{T_1T_2}$

$$\begin{aligned} \vec{n} &= \vec{c_2} \times \overrightarrow{T_1T_2} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{vmatrix} \\ &= \vec{i} \cdot (0 + 1) - \vec{j} \cdot (-1 - 1) + \vec{k} \cdot (1 + 0) \\ &= \vec{i} + 2\vec{j} + \vec{k} = (1, 2, 1) \end{aligned}$$

Jednadžba ravnine π je:

$1(x - 2) + 2(y + 1) + z = 0$

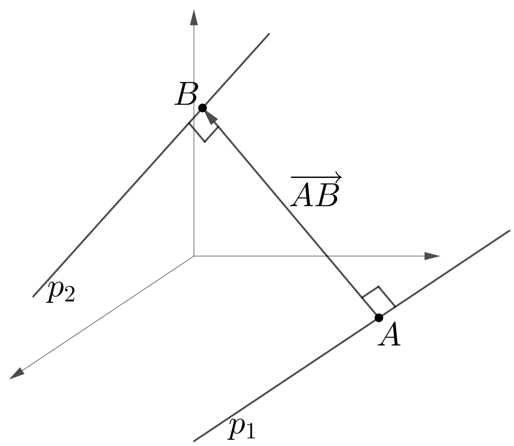
$\pi \equiv x + 2y + z = 0$

$$\begin{aligned} d(A, \pi) &= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|1 \cdot 4 + 2 \cdot (-5) + 1 \cdot 4|}{\sqrt{1^2 + 2^2 + 1^2}} \\ &= \frac{|4 - 10 + 4|}{\sqrt{1 + 4 + 1}} \\ &= \frac{2}{\sqrt{6}} \end{aligned}$$

$$= \frac{2\sqrt{6}}{6}$$
$$= \frac{\sqrt{6}}{3}$$

Zadatak 2.24. Na pravcu $p_1 \equiv \frac{x-6}{3} = \frac{y+2}{-1} = \frac{z-4}{-2}$ odredite točku A koja je najbliža pravcu $p_2 \equiv \begin{cases} x = t \\ y = t \\ z = 2 \end{cases}$.

Rješenje: Među svim vektorima $\overrightarrow{AB}$, gdje je $A \in p_1$, a $B \in p_2$ moramo pronaći onaj za kojeg vrijedi da je $\overrightarrow{AB} \perp p_1$ i $\overrightarrow{AB} \perp p_2$. Tada će $d(A, B)$ biti najmanja.



Kako je $p_1 \equiv \begin{cases} x = 3s + 6 \\ y = -s - 2 \\ z = -2s + 4 \end{cases}$, tada je $A(3s+6, -s-2, -2s+4)$, za neki $s \in \mathbb{R}$, proizvoljna točka pravca p_1, a $B(t, t, 2)$, za neki $t \in \mathbb{R}$, proizvoljna točka pravca p_2.

$\overrightarrow{AB} = (t - 3s - 6)\vec{i} + (t + s + 2)\vec{j} + (2 + 2s - 4)\vec{k}$.

Trebamo pronaći takve t i s da vrijedi: $\overrightarrow{AB} \perp p_1$ i $\overrightarrow{AB} \perp p_2$. No, onda imamo da je $\overrightarrow{AB} \perp \vec{c_1} = (3, -1, -2)$ i $\overrightarrow{AB} \perp \vec{c_2} = (1, 1, 0)$, odnosno $\overrightarrow{AB} \cdot \vec{c_1} = 0$ i $\overrightarrow{AB} \cdot \vec{c_2} = 0$

$$\begin{array}{rcl} \overrightarrow{AB} \cdot \vec{c_1} & = & 0 \\ \overrightarrow{AB} \cdot \vec{c_2} & = & 0 \\ \hline 3(t - 3s - 6) - (t + s + 2) - 2(2 + 2s - 4) & = & 0 \end{array}$$

$$
\begin{aligned}
t-3s-6+t+s+2 &= 0 \\
\hline
3t-9s-18-t-s-2+4-4s &= 0 \\
2t-2s &= 4 \\
\hline
t-7s &= 8 \\
t-s &= 2 \\
\hline
t-7s-(t-s) &= 8-2 \\
-6s &= 6 \\
s &= -1 \\
t &= 1
\end{aligned}
$$

$A(3s+6,-s-2,-2s+4) \Longrightarrow A\,(3,-1,6)$

$B(t,t,2) \Longrightarrow B\,(1,1,2)$.

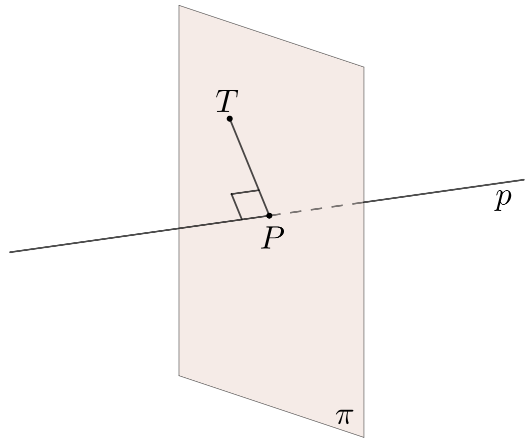
Zadatak 2.25. Odredite ortgonalnu projekciju točke $T(0,5,-1)$ na pravac $p \equiv \begin{cases} x+y=0 \\ x-z+3=0 \end{cases}.$

Rješenje: Odredimo vektor smjera $\vec{c}$ pravca p.

$$
\begin{aligned}
\vec{c} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\
&= \vec{i} \cdot (-1-0) - \vec{j} \cdot (-1-0) + \vec{k} \cdot (0-1) \\
&= -\vec{i} + \vec{j} - \vec{k} = (-1,1,-1)
\end{aligned}
$$

Za proizvoljnu točku $T_1 \in p$, npr. za $x=0$ imamo $T_1(0,0,3)$ i parametarski oblik jednadžbe pravca $p \equiv \begin{cases} x=-t \\ y=t \\ z=3-t \end{cases}$

Odredimo jednadžbu ravnine π tako da je $\pi \perp p$ i $T \in \pi$. Tada je $\vec{n} = \vec{c} = (-1,1,-1)$.



$-1(x-0)+1(y-5)-1(z+1)=0$

$\pi \equiv -x+y-z-6=0$

Sada je ortogonalna projekcija točke T, točka $P=\pi\cap p$.

Odredimo sada koordinate točke $P=p\cap\pi$

$$\begin{aligned} -x+y-z-6 &= 0 \\ -(-t)+t-(3-t)-6 &= 0 \\ t+t-3+t-6 &= 0 \\ 3t &= 9 \\ t &= 3 \end{aligned} \Longrightarrow p \equiv \begin{cases} x=-3 \\ y=3 \\ z=0 \end{cases}$$

$\pi\cap p = P(-3,3,0)$

Zadatak 2.26. Odredite $A\in\mathbb{R}$ za koji je ravnina $\pi\equiv Ax+3y-5z-31=0$ usporedna s pravcem $p\equiv \dfrac{x-1}{4}=\dfrac{y+1}{3}=z$. Odredite otrogonalnu projekciju pravca p na ravninu π.

Rješenje:

$\pi\equiv Ax+3y-5z-31=0$ pa je $\vec{n}=A\vec{i}+3\vec{j}-5\vec{k}=(A,3,-5)$. $p\equiv\dfrac{x-1}{4}=\dfrac{y+1}{3}=\dfrac{z}{1}$ pa je $\vec{c}=4\vec{i}+3\vec{j}+1\vec{k}=(4,3,1)$

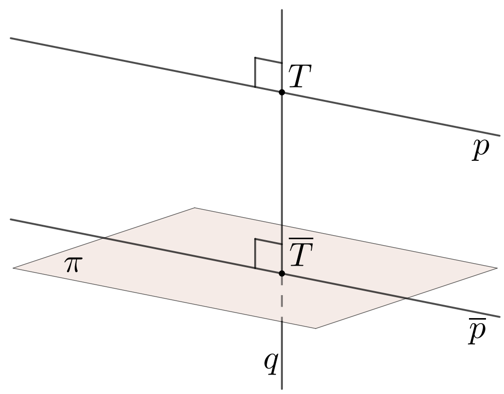
$\pi\,||\,p\Longrightarrow\vec{n}\perp\vec{c}\Longleftarrow\vec{n}\cdot\vec{c}=0$

$$\begin{aligned} \vec{n}\cdot\vec{c} &= 0 \\ 4A+3\cdot 3-5\cdot 1 &= 0 \\ 4A+9-5 &= 0 \\ 4A &= -4 \end{aligned}$$

$$A = -1 \implies \vec{n} = (-1, 3, -5)$$

$\pi \equiv -x + 3y - 5z - 31 = 0$

Odaberimo proizvoljno neku točku $T \in p$, npr. za $x = 1$ imamo $T(1, -1, 0)$. Odredimo jednadžbu pravca q tako da je $q \perp \pi$ i $T \in q$.



$\vec{c_q} \,||\, \vec{n} \implies \vec{c_q} = \vec{n} = (-1, 3, -5)$, pa je $q \equiv \begin{cases} x = 1 - t \\ y = -1 + 3t \\ z = -5t \end{cases}$

Odredimo koordinate točke $\overline{T} = \pi \cap q$.

$$\begin{aligned} -x + 3y - 5z - 31 &= 0 \\ -(1-t) + 3(-1+3t) - 5(-5t) - 31 &= 0 \\ -1 + t - 3 + 9t + 25t - 31 &= 0 \implies q \equiv \begin{cases} x = 1 - 1 = 0 \\ y = -1 + 3 = 2 \\ z = -5 \end{cases} \\ 35t - 35 &= 0 \\ t &= 1 \end{aligned}$$

$\pi \cap p = \overline{T}(0, 2, -5)$

Ortogonalna projekcija pravca p je $\overline{p} = (\overline{T}, \vec{c})$

$\overline{p} \equiv \dfrac{x}{4} = \dfrac{y-2}{3} = \dfrac{z+5}{1}$

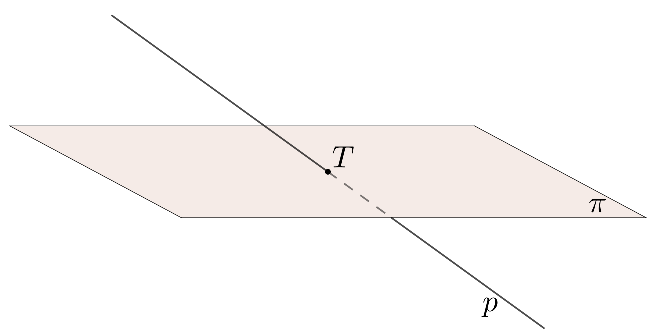
Zadatak 2.27. Odredite međusobni položaj pravca $p \equiv \dfrac{x-2}{-2} = \dfrac{y-3}{-3} = \dfrac{z-6}{-2}$ i ravnine $\pi \equiv x + 2y - z + 4 = 0$. Odredite otrogonalnu projekciju pravca p na ravninu π.

Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku: $p \equiv \begin{cases} x = 2 - 2t \\ y = 3 - 3t \\ z = 6 - 2t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv x + 2y - z + 4 = 0$

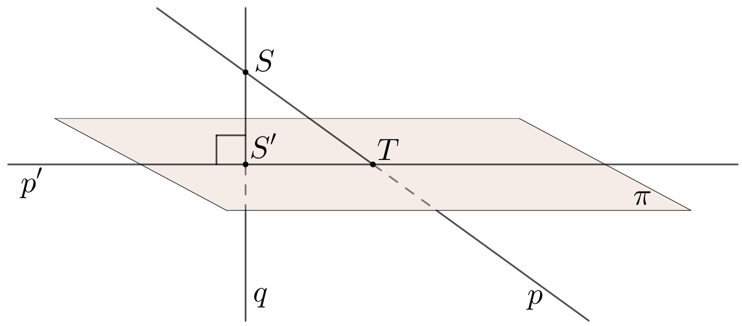
$$\begin{aligned} x + 2y - z + 4 = 0 &= 0 \\ 2 - 2t + 2(3 - 3t) - (6 - 2t) + 4 &= 0 \\ 2 - 2t + 6 - 6t - 6 + 2t + 4 &= 0 \\ -6t &= -6 \\ t &= 1 \end{aligned}$$

$p \equiv \begin{cases} x = 2 - 2 \cdot 1 = 0 \\ y = 3 - 3 \cdot 1 = 0 \\ z = 6 - 2 \cdot 1 = 4 \end{cases}$ Pravac p probada ravninu π u točki $T(0, 0, 4)$.



Kako bi odredili ortogonalnu projekciju p' pravca p na ravninu π, dovoljno je odrediti ortogonalnu projekciju dvije po volji odabrane točke s pravca p na ravninu π. Pravac koji prolazi tim točkama bit će tražena ortogonalna projekcija pravca p. Uočimo da ortogonalnu projekciju jedne točke pravca p već imamo, to je točka probodišta T, jer kako već leži u ravnini, ona je sama sebi ortogonalna projekcija. Dakle, dovoljno je odabrati još samo jednu točku, npr. za $t = 0$, imamo točku $S(2, 3, 6)$. Odredimo i jednadžbu pravca q koji je okomit na ravninu π i prolazi točkom S. Dakle, $q \perp \pi$ pa je $\vec{n} = \vec{c_q} = (1, 2, -1)$.

Odredimo koordinate probodišta pravca q i ravnine π. To će biti ortogonalna projekcija točke S na ravninu π.



Jednadžba pravca q u parametarskom obliku: $q \equiv \begin{cases} x = 2 + t \\ y = 3 + 2t \\ z = 6 - t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv x + 2y - z + 4 = 0$

$$\begin{aligned} x + 2y - z + 4 = 0 &= 0 \\ 2 + t + 2(3 + 2t) - (6 - t) + 4 &= 0 \\ 2 + t + 6 + 4t - 6 + t + 4 &= 0 \\ 6t &= -6 \\ t &= -1 \end{aligned}$$

$q \equiv \begin{cases} x = 2 + (-1) = 1 \\ y = 3 + 2 \cdot (-1) = 1 \\ z = 6 - (-1) = 7 \end{cases}$ Pravac q probada ravninu π u točki $S'(1, 1, 7)$.

Jednadžbu pravca p' dobivamo kroz dvije točke, T i S':

$$p' \equiv \frac{x}{1} = \frac{y}{1} = \frac{z - 4}{3}$$

Poglavlje 3

Matrice i linearni sustavi

3.1 Osnovno o matricama

Definicija 3.1. Svako preslikavanje sa skupa $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ u polje $\mathbb{R}$ ili $\mathbb{C}$ nazivamo **matrica**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ & \ddots & \vdots & & \\ & & a_{ij} & & \\ & & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{m(n-1)} & a_{mn} \end{bmatrix} \qquad a_{ij} \text{ je element u } i\text{-tom retku i } j\text{-tom stupcu}$$

Matrica A je tipa $m \times n \, (A \in \mathbb{R}^{m \times n})$ pa ima m redaka i n stupaca. S a_{ij} označavamo opći član matrice A.

Kvadratna matrica je matrica tipa $n \times n$ (ima jednak broj redaka i stupaca).
Glavna dijagonala kvadratne matrice sadrži elemente $a_{11}, a_{22}, ..., a_{nn}$.

Gornje trokutasta kvadratna matrica je matrica kojoj su svi elementi ispod glavne dijagonale 0, tj. $a_{ij} = 0, i > j$. Primjer: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Donje trokutasta kvadratna matrica je matrica kojoj su svi elementi iznad glavne dijagonale 0, tj. $a_{ij} = 0, i < j$. Primjer: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$.

Dijagonalna kvadratna matrica je matrica kojoj su svi elementi izvan glavne dijagonale 0, tj. $a_{ij} = 0, i \neq j$. Primjer: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

Posebno, ako je $a_{ij} = 0$ za $i \neq j$ i $a_{ij} = 1$ za $i = j$ imamo **jediničnu matricu** $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Nul matrica je matrica kojoj su svi elementi 0, tj. $a_{ij} = 0, \forall i, j$.

Transponirana matrica matrice A je matrica A^T za koju vrijedi $a_{ij} = b_{ji}, \forall i, j$, gdje su a_{ij} elementi matrice A, a b_{ji} elementi matrice A^T.

Primjer: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Za kvadratnu matricu A vrijedi:

- A je **simetrična** ako je $A^T = A \quad (a_{ij} = a_{ji}, \forall i, j)$
- A je **antisimetrična** ako je $A^T = -A \quad (a_{ij} = -a_{ji}, \forall i, j)$

3.2 Operacije s matricama

1. **Množenje matrice skalarom $\lambda \in \mathbb{R}$**

   $A = (a_{ij}), \lambda \in \mathbb{R} \Longrightarrow \lambda \cdot A = (\lambda \cdot a_{ij})$

   $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Longrightarrow \lambda \cdot A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

2. **Zbrajanje matrica istog tipa**

$A, B \in \mathbb{R}^{m \times n}$ $A = (a_{ij}), B = (b_{ij}) \Longrightarrow A + B = (a_{ij} + b_{ij})$

$$
\begin{aligned}
A + B &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}
\end{aligned}
$$

3. **Množenje matrica**

Za $A \in \mathbb{R}^{m \times n}$ i $B \in \mathbb{R}^{n \times p}$ definiramo $A \cdot B = C = (c_{ik}) \in \mathbb{R}^{m \times p}$

$$c_{ik} = \sum_{j=1}^{n} a_{ij} \cdot b_{jk}$$

Primjer: $A = \begin{bmatrix} 1 & 5 \\ 3 & 6 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \in \mathbb{R}^{4 \times 2}, B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$

$$
\begin{aligned}
A \cdot B &= \begin{bmatrix} 1 & 5 \\ 3 & 6 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 2 + 5 \cdot 0 & 1 \cdot 1 + 5 \cdot (-1) & 1 \cdot 0 + 5 \cdot 1 \\ 3 \cdot 2 + 6 \cdot 0 & 3 \cdot 1 + 6 \cdot (-1) & 3 \cdot 0 + 6 \cdot 1 \\ 1 \cdot 2 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot (-1) & 1 \cdot 0 + 0 \cdot 1 \\ 2 \cdot 2 + 3 \cdot 0 & 2 \cdot 1 + 3 \cdot (-1) & 2 \cdot 0 + 3 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -4 & 5 \\ 6 & -3 & 6 \\ 2 & 1 & 0 \\ 4 & -1 & 3 \end{bmatrix} \in \mathbb{R}^{4 \times 3}
\end{aligned}
$$

Zadatak 3.1. Zadane su matrice

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}, B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix} \in \mathbb{R}^{2\times 3}.$$

Izračunajte $2A + B^T$.

Rješenje:

$$\begin{aligned} 2 \cdot A + B^T &= 2 \cdot \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}^T \\ &= \begin{bmatrix} 6 & 2 \\ -2 & 4 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 1 \\ -2 & 5 \\ 1 & 5 \end{bmatrix} \end{aligned}$$

Zadatak 3.2. Zadane su kvadratne matrice

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Izračunajte $A \cdot B$ i $B \cdot A$.

Rješenje:

$$A \cdot B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 7 & 2 \\ 6 & 8 & 4 \\ -1 & 11 & 4 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 11 & 9 \\ 0 & -6 & 0 \\ 6 & 6 & 8 \end{bmatrix}$$

$A \cdot B \neq B \cdot A \Longrightarrow$ množenje matrica nije komutativno

Zadatak 3.3. Zadane su matrice

$$A = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{1\times 3}.$$

Izračunajte $A^T \cdot B$ i $A \cdot B^T$.

Rješenje:

$A^T = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^{3\times 1}$, $B^T = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \in \mathbb{R}^{3\times 1}$

$A^T \cdot B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -3 & 3 \\ 2 & -1 & 1 \\ 4 & -2 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

$A \cdot B^T = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix} \in \mathbb{R}^{1\times 1}$

Zadatak 3.4. Odredite $f(A)$, ako je

$$f(x) = 2x^2 + 3x - 4 \text{ i } A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Rješenje:

$f(A) = 2A^2 + 3A - 4I$, gdje je $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ jedinična matrica i vrijedi $A \cdot I = I \cdot A = A$

$$\begin{aligned}
f(A) &= 2A^2+3A-4I\\
&= 2\cdot\begin{bmatrix}2 & -1\\ 0 & 4\end{bmatrix}\cdot\begin{bmatrix}2 & -1\\ 0 & 4\end{bmatrix}+3\begin{bmatrix}2 & -1\\ 0 & 4\end{bmatrix}-4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
&= 2\cdot\begin{bmatrix}4 & -6\\ 0 & 16\end{bmatrix}+\begin{bmatrix}6 & -3\\ 0 & 12\end{bmatrix}-\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\\
&= \begin{bmatrix}8 & -12\\ 0 & 32\end{bmatrix}+\begin{bmatrix}6 & -3\\ 0 & 12\end{bmatrix}-\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\\
&= \begin{bmatrix}10 & -15\\ 0 & 40\end{bmatrix}
\end{aligned}$$

Zadatak 3.5. Za matricu $A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}\in\mathbb{R}^{2\times 2}$ odredite matricu B tako da vrijedi $A\cdot B=0$.

Rješenje:

$B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\mathbb{R}^{2\times 2}$

$$\begin{aligned}
\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} &= \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}\cdot\begin{bmatrix}a & b\\ c & d\end{bmatrix}\\
&= \begin{bmatrix}a+2c & b+2d\\ 3a+6c & 3b+6d\end{bmatrix}
\end{aligned}$$

$$\left.\begin{aligned}a+2c&=0\\ b+2d&=0\\ 3a+6c&=0\\ 3b+6d&=0\end{aligned}\right\}\implies\begin{aligned}a&=-2c\\ b&=-2d\end{aligned}\implies B=\begin{bmatrix}-2c & -2d\\ c & d\end{bmatrix},\forall c,d\in\mathbb{R}$$

3.3 Determinanta kvadratne matrice

Definicija 3.2. Determinanta kvadratne matrice je preslikavanje det : $\mathbb{R}^{n\times n} \to \mathbb{R}$ koje matrici pridružuje realni broj na sljedeći način:

1. $n = 2$:

   Za $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ definiramo $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

2. $n \geq 3$: Laplaceov razvoj po bilo kojem retku ili stupcu

   Za matricu $A \in \mathbb{R}^{n\times n}$ definiramo $A_{ij}, i, j \in \{1, 2, ..., n\}$ kao matricu dobivenu od A nakon izbacivanja i-tog retka i j-tog stupca.

   Razvoj po i-tom retku:

   $$\det A = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

   Razvoj po j-tom stupcu:

   $$\det A = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

Zadatak 3.6. Izračunajte sljedeće determinante:

a) Determinantu računamo razvojem po prvom retku:

$$\begin{aligned}\begin{vmatrix} 2 & 1 & -1 \\ 3 & 1 & -2 \\ 1 & 0 & 1 \end{vmatrix} &= 2 \cdot \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \\ &= 2(1-0) - 1(3+2) - 1(0-1) = 2 - 5 + 1 = -2\end{aligned}$$

Determinantu možemo računati razvojem po bilo kojem retku ili stupcu, npr. trećem retku:

$$\begin{aligned}\begin{vmatrix} 2 & 1 & -1 \\ 3 & 1 & -2 \\ 1 & 0 & 1 \end{vmatrix} &= (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} + (-1)^{3+3} \cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \\ &= 1(-2+1) + 1(2-3) = -1 - 1 = -2\end{aligned}$$

b)

$$\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix} = (-1)^{3+1}\cdot 2\cdot \begin{vmatrix} 1 & -1 & 2 \\ 1 & 3 & -4 \\ -5 & 3 & -3 \end{vmatrix} + (-1)^{3+3}\cdot 1\cdot \begin{vmatrix} 3 & 1 & 2 \\ -5 & 1 & -4 \\ 1 & -5 & -3 \end{vmatrix} + $$

$$+(-1)^{3+4}\cdot(-1)\cdot \begin{vmatrix} 3 & 1 & -1 \\ -5 & 1 & 3 \\ 1 & -5 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & 3 & -4 \\ -5 & 3 & -3 \end{vmatrix} = 1(-9+12)+1(-3-20)+2(3+15) = 3-23+36 = 16$$

$$\begin{vmatrix} 3 & 1 & 2 \\ -5 & 1 & -4 \\ 1 & -5 & -3 \end{vmatrix} = 3(-3-20)-1(15+4)+2(25-1) = -69-19+48 = -40$$

$$\begin{vmatrix} 3 & 1 & -1 \\ -5 & 1 & 3 \\ 1 & -5 & 3 \end{vmatrix} = 3(3+15)-1(-15-3)-1(25-1) = 54+18-24 = 48$$

$$\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix} = 2\cdot 16 + 1\cdot(-40) + 1\cdot 48 = 40$$

c)

$$\begin{aligned} \begin{vmatrix} 2 & 8 & 0 \\ 0 & 0 & -6 \\ 1 & -4 & 2 \end{vmatrix} &= (-1)^{2+3}\cdot(-6)\cdot \begin{vmatrix} 2 & 8 \\ 1 & -4 \end{vmatrix} \\ &= (-1)\cdot(-6)\cdot(2\cdot(-4)-1\cdot 8) \\ &= 6\cdot(-16) \\ &= -96 \end{aligned}$$

d)

$$\begin{vmatrix} 5 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 5 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \end{vmatrix}$$

$$\begin{aligned}
&= 5 \cdot \left((-1)^{2+3} \cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} + (-1)^{3+3} \cdot 3 \cdot \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} \right) \\
&= 5 \cdot (-1 \cdot (1 \cdot 2 - 3 \cdot 2) + 3 \cdot (1 \cdot 1 - 3 \cdot (-1))) \\
&= 5 \cdot (-1 \cdot (-4) + 3 \cdot 4) \\
&= 5 \cdot 16 \\
&= 80
\end{aligned}$$

e)

$$\begin{vmatrix} 3 & 1 & 2 & 4 \\ 0 & 0 & -1 & 6 \\ 2 & 1 & 3 & 1 \\ 2 & -2 & 3 & 1 \end{vmatrix} = (-1)^{2+3} \cdot (-1) \cdot \begin{vmatrix} 3 & 1 & 4 \\ 2 & 1 & 1 \\ 2 & -2 & 1 \end{vmatrix} + (-1)^{2+4} \cdot 6 \cdot \begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix}$$

$$= (*)$$

$$\begin{aligned}
\begin{vmatrix} 3 & 1 & 4 \\ 2 & 1 & 1 \\ 2 & -2 & 1 \end{vmatrix} &= (-1)^{1+1} \cdot 3 \cdot \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + \\
&\quad + (-1)^{1+3} \cdot 4 \cdot \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} \\
&= 3 \cdot (1 \cdot 1 - (-2) \cdot 1) - (2 \cdot 1 - 2 \cdot 1) + 4 \cdot (2 \cdot (-2) - 2 \cdot 1) \\
&= 3 \cdot 3 - 0 + 4 \cdot (-6) \\
&= -15
\end{aligned}$$

$$\begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 3 \cdot \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} +$$

$$
\begin{aligned}
&+(-1)^{1+3}\cdot 2\cdot\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix}\\
&= 3\cdot(1\cdot 3-(-2)\cdot 3)-(2\cdot 3-2\cdot 3)+2\cdot(2\cdot(-2)-2\cdot 1)\\
&= 3\cdot 9-0+2\cdot(-6)\\
&= 15
\end{aligned}
$$

$$
\begin{aligned}
(*) &= -15+6\cdot 15\\
&= 75
\end{aligned}
$$

f)

$$
\begin{aligned}
\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix} &= (-1)^{4+1}\cdot d\cdot\begin{vmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{vmatrix}\\
&= -d\cdot(-1)^{3+1}\cdot c\cdot\begin{vmatrix} 2 & a \\ b & 0 \end{vmatrix}\\
&= -dc\cdot(2\cdot 0-a\cdot b)\\
&= -dc\cdot(-ab)\\
&= abcd
\end{aligned}
$$

Svojstva determinante

1. Zamjenom dva retka (ili stupca) determinanta mijenja predznak.
2. Ako su svi elementi nekog retka ili stupca 0, determinanta je 0.
3. Ako su dva retka ili stupca matrice jednaka ili proporcionalna, determinanta je 0.
4. Determinanta trokutaste matrice jednaka je umnošku elemenata na dijagonali.
5. **Binet-Cauchy**: $\det(A\cdot B)=\det A\cdot\det B$
6. Dodavanje jednog retka (stupca), pomnoženog nekim skalarom, drugom retku (stupcu), ne mijenja determinantu.
7. Množenje retka (stupca) skalarom λ mijenja determinantu za faktor λ. Posebno, $\det(\lambda A)=\lambda^n\det A$, gdje je n broj redaka (stupaca) matrice A.

8. $\det A = \det A^T$

3.4 Rang matrice

Rang matrice je broj linearno nezavisnih redaka (stupaca) matrice

Oznaka: Za $A \in \mathbb{R}^{m\times n}$ rang matrice označavamo s $r(A)$, pri čemu je $r(A) \leq \min\{m, n\}$.

Reducirani oblik matrice dobiva se elementarnim transformacijama.

Elementarne transformacije su:

1. zamjena dvaju redaka (stupaca) matrice
2. množenje retka (stupca) skalarom različitim od 0
3. dodavanje retka (stupca) nekom drugom retku (stupcu)

Za $A \in \mathbb{R}^{n\times n}$ vrijedi: $r(A) = n \Longleftrightarrow \det A \neq 0$.

Zadatak 3.7. Svedite na reducirani oblik matricu

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

i odredite joj rang.

Rješenje:

$A \in \mathbb{R}^{3\times 4} \Longrightarrow r(A) \leq 3$

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} \\ \mathrm{II}-2\mathrm{I} \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -4 & -1 & -3 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} \\ /:(-4) \\ /:2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Longrightarrow r(A) = 3$

Zadatak 3.8. Odredite rang matrice

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

Rješenje:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ \mathrm{II}-\mathrm{I} \\ \\ \\ \\ \end{array} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ /:(-4) \\ \\ \mathrm{IV}-\mathrm{III} \\ \mathrm{V}-\mathrm{III} \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -3 & -1 \end{bmatrix} \begin{array}{l} \mathrm{I}-2\mathrm{II} \\ \\ \\ /:(-1) \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 & -1 \end{bmatrix} \begin{array}{l} \\ \\ \mathrm{III}-3\mathrm{IV} \\ \mathrm{V}+3\mathrm{IV} \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \mathrm{III}-\frac{1}{2}\mathrm{V} \\ \mathrm{IV}-\frac{1}{2}\mathrm{V} \\ /:2 \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow r(A) = 5$

Zadatak 3.9. U ovisnosti o parametru λ odredite rang matrice

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & \lambda \\ 2 & 10 & 0 \\ -6 & -2 & -2\lambda \end{bmatrix}$$

Rješenje: $r(A) \leq 4$

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & \lambda \\ 2 & 10 & 0 \\ -6 & -2 & -2\lambda \end{bmatrix} \begin{matrix} \\ \mathrm{II}-3\mathrm{I} \\ \mathrm{III}-2\mathrm{I} \\ \mathrm{IV}+6\mathrm{I} \end{matrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & \lambda \\ 0 & 14 & 0 \\ 0 & -14 & -2\lambda \end{bmatrix} \begin{matrix} \\ \\ \mathrm{III}-2\mathrm{II} \\ \mathrm{IV}+2\mathrm{II} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & \lambda \\ 0 & 0 & -2\lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \mathrm{II}+\frac{1}{2}\mathrm{III} \\ /\cdot(-\frac{1}{2}) \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

$$r(A) = \begin{cases} 2, \lambda = 0 \\ 3, \lambda \neq 0 \end{cases}$$

3.5 Inverzna matrica

Definicija 3.3. Za matricu $A \in \mathbb{R}^{n\times n}$ kažemo da je **regularna** ako postoji A^{-1} za koju vrijedi da je $A \cdot A^{-1} = A^{-1} \cdot A = I$. Za A^{-1} kažemo da je **inverzna matrica** matrice A.

Vrijedi sljedeće: $A \in \mathbb{R}^{n\times n}$ je regularna $\Longleftrightarrow r(A) = n \Longleftrightarrow \det(A) \neq 0$.

Zadatak 3.10. Odredite inverz matrice $A = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix}$.

Rješenje:

$$
\left[\begin{array}{rrr|rrr} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \mathrm{II}\leftrightarrow\mathrm{I} \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ -3 & 1 & -1 & 1 & 0 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \mathrm{II}+3\mathrm{I} \\ \mathrm{III}+2\mathrm{I} \end{array}
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 3 & 0 \\ 0 & 2 & 5 & 0 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{III}-2\mathrm{II} \end{array} \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] \begin{array}{l} \mathrm{I}-\mathrm{III} \\ \mathrm{II}-2\mathrm{III} \\ \\ \end{array}
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 5 & -1 \\ 0 & 1 & 0 & 5 & 11 & -2 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]
$$

$$
A^{-1} = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 11 & -2 \\ -2 & -4 & 1 \end{bmatrix}
$$

Zadatak 3.11. Za matricu $A \in \mathbb{R}^{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ pokažite da je $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Rješenje:

$$
\begin{aligned}
A \cdot A^{-1} &= \frac{1}{\det A} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&= \frac{1}{\det A} \cdot \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \mathrm{I}
\end{aligned}
$$

Zadatak 3.12. Odredite inverz matrice $A = \begin{bmatrix} 3 & 8 \\ 1 & 2 \end{bmatrix}$.

Rješenje: 1. način:

$$\left[\begin{array}{cc:cc} 3 & 8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \begin{array}{c} \text{II}\leftrightarrow\text{I} \\ \end{array} \sim \left[\begin{array}{cc:cc} 1 & 2 & 0 & 1 \\ 3 & 8 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \text{II}-3\text{I} \end{array}$$

$$\sim \left[\begin{array}{cc:cc} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & -3 \end{array}\right] \begin{array}{c} \\ /:2 \end{array} \sim \left[\begin{array}{cc:cc} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} \end{array}\right] \begin{array}{c} \text{I}-2\text{II} \\ \end{array}$$

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & -1 & 4 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 4 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

2.način: $\det A = -2$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 2 & -8 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Zadatak 3.13. Odredite inverz matrice $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

Rješenje:

$$\left[\begin{array}{cccc:cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \text{I}\leftrightarrow\text{II} \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ \text{III}-\text{I} \\ \text{IV}-\text{I} \end{array}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{III}-\mathrm{II} \\ \mathrm{IV}-\mathrm{II} \end{array}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \mathrm{IV}-\frac{1}{2}\mathrm{III} \end{array}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ /:(-2) \\ \cdot\frac{-2}{3} \end{array}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{array}\right] \begin{array}{l} \mathrm{I}-\mathrm{IV} \\ \mathrm{II}-\mathrm{IV} \\ \mathrm{III}-\frac{1}{2}\mathrm{IV} \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{array}\right] \begin{array}{l} \mathrm{I}-\mathrm{III} \\ \mathrm{II}-\mathrm{III} \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Zadatak 3.14. Riješite matričnu jednadžbu $A \cdot X = B$ ako je $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ i $B = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$.

Rješenje:

$$\begin{aligned} A^{-1}\cdot / \qquad A \cdot X &= B \\ \underbrace{A^{-1} \cdot A}_{I} \cdot X &= A^{-1} \cdot B \\ X &= A^{-1} \cdot B \end{aligned}$$

$\det A = -5 \Longrightarrow A^{-1} = -\frac{1}{5} \cdot \begin{bmatrix} 4 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix}$

$X = A^{-1} \cdot B = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{7}{5} \\ \frac{4}{5} & \frac{6}{5} \end{bmatrix}$

Zadatak 3.15. Riješite matričnu jednadžbu:

i) $X \cdot A = B$, $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

ii) $B \cdot X = I - A$, $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$

iii) $A \cdot X \cdot B = I$, $A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

Zadatak 3.16. Zadana je matrica $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ i $B = A^2 - \lambda A$. Odredite $\lambda \in \mathbb{R}$ tako da matrica B bude singularna (nije regularna).

Rješenje:

$$
\begin{aligned}
B &= A^2 - \lambda A \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & \lambda & 0 \\ 0 & \lambda & 0 \\ \lambda & 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & \lambda & 0 \\ 0 & \lambda & 0 \\ \lambda & 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} 1-\lambda & 2-\lambda & 0 \\ 0 & 1-\lambda & 0 \\ 2-\lambda & 1 & 1-\lambda \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \det B \\
&= \begin{vmatrix} 1-\lambda & 2-\lambda & 0 \\ 0 & 1-\lambda & 0 \\ 2-\lambda & 1 & 1-\lambda \end{vmatrix} \\
&= (-1)^{3+3}(1-\lambda) \begin{vmatrix} 1-\lambda & 2-\lambda \\ 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)(1-\lambda)^2 \\
&= (1-\lambda)^3
\end{aligned}
$$

$\lambda = 1$

Zadatak 3.17. Zadana je matrica $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 5 \\ 2 & \lambda & 1 \end{bmatrix}$. Odredite $\lambda \in \mathbb{R}$ tako da matrica A bude regularna.

3.6 Sustavi linearnih jednadžbi

Definicija 3.4. Linearna jednadžba s varijablama $x_1, x_2, ..., x_n$ je jednadžba oblika $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$, gdje su $a_1, a_2, ..., a_n$ koeficijenti, a d je konstanta, $a_i, d \in \mathbb{R}, \forall i$.

Sustav linearnih m jednadžbi s n nepoznanica je

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_1 \\ \vdots & & & & \ddots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

Sustav možemo zapisati u matričnom obliku $A \cdot X = B$, gdje je

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ i } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Matricu A nazivamo matricom sustava, a matricu $A_p = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$ proširenom matricom sustava. Ako je $r(A) \neq r(A_p)$ sustav nema rješenje. Ako je $r = r(A) = r(A_p)$ sustav ima jedinstveno rješenje. Ako je $r > r(A) = r(A_p)$, onda sustav ima beskonačno mnogo rješenja koja zapisujemo pomoću $r - r(A)$ parametara.

Homogeni sustavi ($b = 0$) uvijek imaju barem trivijalno rješenje.

Zadatak 3.18. Riješite sustav: $\begin{cases} x_1 - x_3 + 2x_4 & = & -3 \\ 2x_2 + x_3 + x_4 & = & 1 \\ x_1 + x_2 - x_3 & = & -1 \\ 2x_1 + x_2 + x_3 + x_4 & = & 3 \end{cases}$

Rješenje:

Neka je $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $b = \begin{bmatrix} -3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \in \mathbb{R}^{4\times 1}$ i $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$.

Općenito, $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^{n\times 1}$ i $b \in \mathbb{R}^{m\times 1}$ i $A \cdot x = b$.

Elementarnim transformacijama dovodimo matricu $\left[\begin{array}{c:c} A & b \end{array}\right]$ do reduciranog oblika.

$$\left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & -3 \\ 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 2 & 1 & 1 & 1 & 3 \end{array}\right] \begin{array}{l} \\ \\ \text{III}-\text{I} \\ \text{IV}-2\text{I} \end{array} \sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & -3 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -3 & 9 \end{array}\right] \begin{array}{l} \\ \text{II}\leftrightarrow\text{III} \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 0 & -2 & 2 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & -3 & 9 \end{array}\right] \begin{array}{l} \\ \\ \text{III}-2\text{II} \\ \text{IV}-\text{II} \end{array} \sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 1 & 5 & -3 \\ 0 & 0 & 3 & -1 & 7 \end{array}\right] \begin{array}{l} \\ \\ \\ \text{IV}-3\text{III} \end{array}$$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 1 & 5 & -3 \\ 0 & 0 & 0 & -16 & 16 \end{array}\right]$$

$$\begin{array}{rcr} x_1 - x_3 + 2x_4 & = & -3 \\ x_2 - 2x_4 & = & 2 \\ x_3 + 5x_4 & = & -3 \\ -16x_4 & = & 16 \end{array} \Longrightarrow x = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

Zadatak 3.19. Riješite sustav: $\left\{ \begin{array}{rcl} x_1 - x_3 + 2x_4 & = & 0 \\ 2x_2 + x_3 + x_4 & = & 0 \\ x_1 + x_2 - x_3 & = & 0 \\ 2x_1 + x_2 + x_3 + x_4 & = & 0 \end{array} \right.$

Rješenje:

Neka je $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{4\times 1}$ i $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$.

Općenito, $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^{n\times 1}$ i $b \in \mathbb{R}^{m\times 1}$ i $A \cdot x = b$.

Elementarnim transformacijama dovodimo matricu $\left[\begin{array}{c:c} A & b \end{array}\right]$ do reduciranog oblika.

$$\left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{III}-\mathrm{I} \\ \mathrm{IV}-2\mathrm{I} \end{array} \sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ \mathrm{II}\leftrightarrow\mathrm{III} \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{III}-2\mathrm{II} \\ \mathrm{IV}-\mathrm{II} \end{array} \sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \mathrm{IV}-3\mathrm{III} \end{array}$$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & -16 & 0 \end{array}\right]$$

$$\left.\begin{array}{rcl} x_1 - x_3 + 2x_4 & = & 0 \\ x_2 - 2x_4 & = & 0 \\ x_3 + 5x_4 & = & 0 \\ -16x_4 & = & 0 \end{array}\right\} \Longrightarrow x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Zadatak 3.20. Riješite sustav: $\left\{\begin{array}{rcl} 3x_1 + 2x_2 + 4x_3 & = & 0 \\ 2x_1 + x_2 + x_3 & = & 0 \\ x_1 + x_2 + 3x_3 & = & 0 \end{array}\right.$

Rješenje: Elementarnim transformacijama dovodimo matricu $\left[\begin{array}{c:c} A & b \end{array}\right]$ do reduciranog oblika.

$$\left[\begin{array}{ccc:c} 3 & 2 & 4 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \begin{array}{l} \mathrm{I}\leftrightarrow\mathrm{III} \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 3 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \mathrm{II}-2\mathrm{I} \\ \mathrm{III}-3\mathrm{I} \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 3 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{III}-\mathrm{II} \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 3 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 + 3x_3 = 0$

$-x_2 - 5x_3 = 0$

$(2) \implies x_2 = -5x_3$

$(1) \implies x_1 = 2x_3$

Kako imamo dvije linearno nezavisne jednadžbe, a tri nepoznanice, rješenje sustava će biti parametarsko. Uvedimo parametar t i neka je $x_3 = t$. Zapišimo rješenje u matričnom obliku.

$x_1 = 2t$

$x_2 = -5t$

$x_3 = t$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -5t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

Skup rješenja sustava je $\{(2t, -5t, t) : t \in \mathbb{R}\}$. U ovom slučaju kažemo da sustav ima jednoparametarski skup rješenja.

Zadatak 3.21. Riješite sustav: $\begin{cases} x + y &= 0 \\ 2x - y + 3z &= 3 \\ x - 2y - z &= 0 \end{cases}$

Zadatak 3.22. Riješite sustav: $\begin{cases} x + y + z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{cases}$

Rješenje:

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \begin{array}{l} \\ \text{II}-2\text{I} \\ \text{III}-3\text{I} \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 3 & -8 & -27 \end{array}\right] \begin{array}{l} \\ \\ \text{III}-\frac{3}{2}\text{II} \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right] \implies$$

$$\left.\begin{array}{rcr} x + y + z &=& 9 \\ 2y - 5z &=& -17 \\ -\frac{1}{2}z &=& -\frac{3}{2} \end{array}\right\} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$$

Zadatak 3.23. Riješite sustav: $\left\{\begin{array}{rcl} 2x-2z &=& 6 \\ y+z &=& 1 \\ 2x+y-z &=& 7 \\ 3y+3z &=& 0 \end{array}\right.$

Rješenje:

$$\left[\begin{array}{ccc:c} 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -1 & 7 \\ 0 & 3 & 3 & 0 \end{array}\right] / : (2) \sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -1 & 7 \\ 0 & 3 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{III}-2\text{I} \\ \\ \end{array} \sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{III}-\text{II} \\ \text{IV}-3\text{II} \end{array} \sim$$

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right] \Longrightarrow \left.\begin{array}{rcr} x-z &=& 3 \\ y+z &=& 1 \\ 0 &=& 0 \\ 0 &=& -3 \end{array}\right\} \Longrightarrow \text{nema rješenja (zbog } 0=-3)$$

Zadatak 3.24. Riješite sustav: $\left\{\begin{array}{rcr} x+3y &=& 1 \\ 2x+y &=& -3 \\ 2x+2y &=& -2 \end{array}\right.$

Rješenje:

$$\left[\begin{array}{cc:c} 1 & 3 & 1 \\ 2 & 1 & -3 \\ 2 & 2 & -2 \end{array}\right] \begin{array}{l} \\ \text{II}-2\text{I} \\ \text{III}-2\text{I} \end{array} \sim \left[\begin{array}{cc:c} 1 & 3 & 1 \\ 0 & -5 & -5 \\ 0 & -4 & -4 \end{array}\right] \begin{array}{l} \\ / : (-5) \\ / : (-4) \end{array} \sim \left[\begin{array}{cc:c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{III}-\text{II} \end{array} \sim \left[\begin{array}{cc:c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{rcl} x+3y &=& 1 \\ y &=& 1 \end{array} \Longrightarrow x=-2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Zadatak 3.25. Riješite sustav: $\left\{\begin{array}{rcl} 2x+z &=& 3 \\ x-y-z &=& 1 \\ 3x-y &=& 4 \end{array}\right.$

Rješenje:

$$\left[\begin{array}{rrr|r} 2 & 0 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 3 & -1 & 0 & 4 \end{array}\right] \begin{array}{l} \mathrm{I}\leftrightarrow\mathrm{II} \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r} 1 & -1 & -1 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & -1 & 0 & 4 \end{array}\right] \begin{array}{l} \\ \mathrm{II}-2\mathrm{I} \\ \mathrm{III}-3\mathrm{I} \end{array} \sim$$

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{III}-\mathrm{II} \end{array} \sim \left[\begin{array}{rrr|r} 1 & -1 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x-y-z &= 1 \\ 2y+3z &= 1 \Longrightarrow z=t \\ 0 &= 0 \end{aligned}$$

$2y+3z=1 \Longrightarrow y=\frac{1}{2}-\frac{3}{2}z=\frac{1}{2}-\frac{3}{2}t$

$x-y-z=1 \Longrightarrow x=1+y+z=1+\frac{1}{2}-\frac{3}{2}t+t=\frac{3}{2}-\frac{1}{2}t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2}-\frac{1}{2}t \\ \frac{1}{2}-\frac{3}{2}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + t\cdot \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}, t\in\mathbb{R}$$

Teorem 3.1. Sustav linearnih jednadžbi je rješiv ako i samo ako matrica tog sustava i njegova proširena matrica imaju isti rang, tj. $r(A|b)=r(A)$.

Zadatak 3.26. Gaussovom metodom riješite sustav: $\left\{\begin{aligned} x+2y+3z &= 5 \\ 2x-y-z &= 1 \\ x+3y+4z &= 6 \end{aligned}\right.$

Rješenje:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 2 & -1 & -1 & 1 \\ 1 & 3 & 4 & 6 \end{array}\right] \begin{array}{l} \\ \mathrm{II}-2\mathrm{I} \\ \mathrm{III}-\mathrm{I} \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & -5 & -7 & -9 \\ 0 & 1 & 1 & 1 \end{array}\right] \mathrm{II}\leftrightarrow\mathrm{III} \sim$$

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -7 & -9 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{III}+5\mathrm{II} \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right] \begin{array}{l} \\ \\ /:(-2) \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$r(A)=r(A|b)=3$ pa postoji rješenje.

$$\begin{aligned} x+2y+3z &= 5 \\ y+z &= 1 \\ z &= 2 \end{aligned}$$

$y+z=1 \Longrightarrow y=-1$

$x+2y+3z=5 \Longrightarrow x=5-3z-2y=5-6+2=1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Zadatak 3.27. Gaussovom metodom riješite sustav: $\begin{cases} 3x-y+3z = 4 \\ 6x-2y+6z = 1 \\ 5x+4y = 2 \end{cases}$

Rješenje:

$$\left[\begin{array}{ccc:c} 3 & -1 & 3 & 4 \\ 6 & -2 & 6 & 1 \\ 5 & 4 & 0 & 2 \end{array}\right] \begin{array}{l} \\ \mathrm{II}-2\mathrm{I} \\ \mathrm{III}-\frac{5}{3}\mathrm{I} \end{array} \sim \left[\begin{array}{ccc:c} 3 & -1 & 3 & 4 \\ 0 & 0 & 0 & -7 \\ 0 & \frac{17}{3} & -5 & -\frac{14}{3} \end{array}\right] \mathrm{II}\leftrightarrow\mathrm{III} \sim \left[\begin{array}{ccc:c} 3 & -1 & 3 & 4 \\ 0 & \frac{17}{3} & -5 & -\frac{14}{3} \\ 0 & 0 & 0 & -7 \end{array}\right]$$

$\Rightarrow r(A) = 2 \neq r(A|b) = 3$

Sustav nema rješenje.

Zadatak 3.28. Gaussovom metodom riješite sustav:

$$\begin{cases} x_1+2x_2+2x_3+3x_4+x_5 = 3 \\ 2x_1-x_3-x_4+5x_5 = 2 \\ x_1+2x_2+6x_3-x_4+5x_5 = 3 \\ x_1-2x_2+5x_3-12x_4+12x_5 = -1 \end{cases}$$

Rješenje:

$$\left[\begin{array}{ccccc:c} 1 & 2 & 2 & 3 & 1 & 3 \\ 2 & 0 & -1 & -1 & 5 & 2 \\ 1 & 2 & 6 & -1 & 5 & 3 \\ 1 & -2 & 5 & -12 & 12 & -1 \end{array}\right] \begin{array}{l} \\ \mathrm{II}-2\mathrm{I} \\ \mathrm{III}-\mathrm{I} \\ \mathrm{IV}-\mathrm{I} \end{array} \sim \left[\begin{array}{ccccc:c} 1 & 2 & 2 & 3 & 1 & 3 \\ 0 & -4 & -5 & -7 & 3 & -4 \\ 0 & 0 & 4 & -4 & 4 & 0 \\ 0 & -4 & 3 & -15 & 11 & -4 \end{array}\right] \begin{array}{l} \\ \\ /:4 \\ \mathrm{IV}-\mathrm{II} \end{array} \sim$$

$$\left[\begin{array}{rrrrr:r} 1 & 2 & 2 & 3 & 1 & 3 \\ 0 & -4 & -5 & -7 & 3 & -4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 8 & -8 & 8 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \mathrm{IV}-8\mathrm{III} \end{array} \sim \left[\begin{array}{rrrrr:r} 1 & 2 & 2 & 3 & 1 & 3 \\ 0 & -4 & -5 & -7 & 3 & -4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 + x_5 &= 3 \\ -4x_2 - 5x_3 - 7x_4 + 3x_5 &= -4 \\ x_3 - x_4 + x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

$x_4 = t_1,\ x_5 = t_2,\ t_1, t_2 \in \mathbb{R}$

$$\begin{aligned} x_3 - x_4 + x_5 &= 0 \\ x_3 &= x_4 - x_5 \\ x_3 &= t_1 - t_2 \end{aligned}$$

$$\begin{aligned} -4x_2 - 5x_3 - 7x_4 + 3x_5 &= -4 \\ -4x_2 &= -4 + 5x_3 + 7x_4 - 3x_5 \\ -4x_2 &= -4 + 5t_1 - 5t_2 + 7t_1 - 3t_2 \\ -4x_2 &= -4 + 12t_1 - 8t_2 / : (-4) \\ x_2 &= 1 - 3t_1 + 2t_2 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 + x_5 &= 3 \\ x_1 &= 3 - 2x_2 - 2x_3 - 3x_4 - x_5 \\ x_1 &= 3 - 2(1 - 3t_1 + 2t_2) - 2(t_1 - t_2) - 3t_1 - t_2 \\ x_1 &= 1 + 6t_1 - 4t_2 - 2t_1 + 2t_2 - 3t_1 - t_2 \\ x_1 &= 1 + t_1 - 3t_2 \end{aligned}$$

$$x = \begin{bmatrix} 1 + t_1 - 3t_2 \\ 1 - 3t_1 + 2t_2 \\ t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, t_1, t_2 \in \mathbb{R}$$

Dvoparametarski skup rješenja.

3.7 Svojstveni vektori i svojstvene vrijednosti

Definicija 3.5. **Svojstvena vrijednost** matrice $A \in \mathbb{R}^{n\times n}$ je skalar $\lambda \in \mathbb{R}$ za koji postoji vektor $v \neq \vec{0}$ tako da je $A \cdot v = \lambda \cdot v$.

Definicija 3.6. **Svojstveni vektor** matrice A je vektor v koji pripada svojstvenoj vrijednosti λ.

Definicija 3.7. Polinom $k(\lambda) = \det(\lambda I - A)$ nazivamo **karakteristični polinom** matrice A.

Teorem 3.2. Nultočke karakterističnog polinoma su svojstvene vrijednosti matrice.

Zadatak 3.29. Odredite svojstvene vrijednosti i svojstvene vektore matrice $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

Rješenje:

$$\begin{aligned} \lambda I - A &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} k(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 4 \end{vmatrix} \\ &= (\lambda - 1)(\lambda - 4) + 2 \\ &= \lambda^2 - 4\lambda - \lambda + 4 + 2 \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

$$\begin{aligned} \lambda^2 - 5\lambda + 6 &= 0 \\ \lambda_1, \lambda_2 &= \tfrac{5 \pm \sqrt{5^2 - 4\cdot 1\cdot 6}}{2} \\ \lambda_1, \lambda_2 &= \tfrac{5 \pm 1}{2} \\ \lambda_1 = 2 \quad &, \quad \lambda_2 = 3 \end{aligned}$$

Sada tražimo pripadne svojstvene vektore:

$\lambda = 2$

$$\begin{aligned} Av &= \lambda v \\ \lambda I v - Av &= 0 \\ (\lambda I - A)v &= 0 \end{aligned}$$

$$\begin{aligned} \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \right) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ -2x_1 - 2x_2 &= 0 \end{aligned} \Longrightarrow x_1 = -x_2, x_2 = t \in \mathbb{R} \Longrightarrow v = \begin{bmatrix} -t \\ t \end{bmatrix}, t \in \mathbb{R} \setminus \{0\}$$

$\lambda = 3$

$(\lambda I - A)v = 0$

$$\begin{aligned} \left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \right) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ -2x_1 - x_2 &= 0 \end{aligned} \Longrightarrow x_2 = -2x_1, x_1 = t \in \mathbb{R} \Longrightarrow v = \begin{bmatrix} t \\ -2t \end{bmatrix}, t \in \mathbb{R} \setminus \{0\}$$

Zadatak 3.30. Odredite svojstvene vrijednosti i svojstvene vektore matrice $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$.

$$\begin{aligned} \lambda I - A &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{bmatrix} \end{aligned}$$

$$\begin{aligned} k(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{vmatrix} \\ &= (\lambda - 3)\lambda + 2 \\ &= \lambda^2 - 3\lambda + 2 \end{aligned} \qquad \begin{aligned} \lambda^2 - 3\lambda + 2 &= 0 \\ \lambda_1, \lambda_2 &= \tfrac{3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 2}}{2} \\ \lambda_1, \lambda_2 &= \tfrac{3 \pm 1}{2} \\ \lambda_1 = 1 \quad &, \quad \lambda_2 = 2 \end{aligned}$$

Sada tražimo pripadne svojstvene vektore:

$\lambda = 1$

$(\lambda I - A)v = 0$

$$
\begin{aligned}
\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}\right)\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
\begin{bmatrix}-2 & -2\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
-2x_1 - 2x_2 &= 0\\
x_1 + x_2 &= 0
\end{aligned}
\Longrightarrow x_2 = -x_1, x_1 = t \in \mathbb{R} \Longrightarrow v = \begin{bmatrix}t\\ -t\end{bmatrix}, t \in \mathbb{R}\setminus\{0\}
$$

$\lambda = 2$

$(\lambda I - A)v = 0$

$$
\begin{aligned}
\left(\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} - \begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}\right)\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
\begin{bmatrix}-1 & -2\\ 1 & 2\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
-x_1 - 2x_2 &= 0\\
x_1 + 2x_2 &= 0
\end{aligned}
\Longrightarrow x_1 = -2x_2, x_2 = t \in \mathbb{R} \Longrightarrow v = \begin{bmatrix}-2t\\ t\end{bmatrix}, t \in \mathbb{R}\setminus\{0\}
$$

Zadatak 3.31. Odredite svojstvene vrijednosti i svojstvene vektore matrice $A = \begin{bmatrix}2 & 0 & 3\\ 2 & 2 & 2\\ 3 & 0 & 2\end{bmatrix}$.

$$
\begin{aligned}
\lambda I - A &= \begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} - \begin{bmatrix}2 & 0 & 3\\ 2 & 2 & 2\\ 3 & 0 & 2\end{bmatrix}\\
&= \begin{bmatrix}\lambda-2 & 0 & -3\\ -2 & \lambda-2 & -2\\ -3 & 0 & \lambda-2\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
k(\lambda) &= \det(\lambda I - A) \\
&= \begin{vmatrix} \lambda-2 & 0 & -3 \\ -2 & \lambda-2 & -2 \\ -3 & 0 & \lambda-2 \end{vmatrix} \\
&= (-1)^{2+2}(\lambda-2)\begin{vmatrix} \lambda-2 & -3 \\ -3 & \lambda-2 \end{vmatrix} \\
&= (\lambda-2)\left[(\lambda-2)^2-9\right] \\
&= (\lambda-2)\left(\lambda^2-4\lambda-5\right) \\
&= (\lambda-2)(\lambda-5)(\lambda+1)
\end{aligned}
$$

$\lambda_1 = 2,\ \lambda_2 = 5,\ \lambda_3 = -1$

Sada tražimo pripadne svojstvene vektore:

$\lambda_1 = 2$

$(\lambda_1 I - A)v = 0$

$$
\begin{bmatrix} 0 & 0 & -3 \\ -2 & 0 & -2 \\ -3 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \left.\begin{aligned} -3x_3 &= 0 \\ -2x_1 - 2x_3 &= 0 \\ -3x_1 &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} x_1 &= 0 \\ x_2 &= t \\ x_3 &= 0 \end{aligned}
$$

$$
v = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}, t \in \mathbb{R} \setminus \{0\}
$$

$\lambda_2 = 5$

$(\lambda_2 I - A)v = 0$

$$
\begin{bmatrix} 3 & 0 & -3 \\ -2 & 3 & -2 \\ -3 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \left.\begin{aligned} 3x_1 - 3x_3 &= 0 \\ -2x_1 + 3x_2 - 2x_3 &= 0 \\ -3x_1 + 3x_3 = 0 &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} x_3 &= t \\ x_1 &= x_3 \\ x_2 &= \frac{4}{3}t \end{aligned}
$$

$$
v = \begin{bmatrix} t \\ \frac{4}{3}t \\ t \end{bmatrix}, t \in \mathbb{R} \setminus \{0\}
$$

$\lambda_3 = -1$

$(\lambda_3 I - A)v = 0$

$$\begin{bmatrix} -3 & 0 & -3 \\ -2 & -3 & -2 \\ -3 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \left.\begin{aligned} -3x_1 - 3x_3 = 0 \\ -2x_1 - 3x_2 - 2x_3 = 0 \\ -3x_1 - 3x_3 = 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} x_1 &= -x_3 \\ x_2 &= 0 \\ x_3 &= t \end{aligned}$$

$$v = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}, t \in \mathbb{R} \setminus \{0\}$$

Zadatak 3.32. Odredite svojstvene vrijednosti matrice $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

$$\begin{aligned} \lambda I - A &= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda - 1 & 1 & -1 \\ -1 & \lambda - 1 & 1 \\ -1 & 1 & \lambda - 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} k(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 1 & 1 & -1 \\ -1 & \lambda - 1 & 1 \\ -1 & 1 & \lambda - 1 \end{vmatrix} \\ &= (-1)^{1+1}(\lambda - 1) \begin{vmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{vmatrix} + (-1)^{1+2} \begin{vmatrix} -1 & 1 \\ -1 & \lambda - 1 \end{vmatrix} + (-1)^{1+3}(-1) \begin{vmatrix} -1 & \lambda - 1 \\ -1 & 1 \end{vmatrix} \\ &= (\lambda - 1)\left[(\lambda - 1)^2 - 1\right] - \left[-(\lambda - 1) + 1\right] - (-1 + \lambda - 1) \\ &= (\lambda - 1)\left(\lambda^2 - 2\lambda + 1 - 1\right) - (-\lambda + 2) - \lambda + 2 \\ &= (\lambda - 1)\left(\lambda^2 - 2\lambda\right) + \lambda - 2 - \lambda + 2 \\ &= (\lambda - 1)\lambda\,(\lambda - 2) \end{aligned}$$

$\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 2$

Zadatak 3.33. Odredite barem jedan svojstveni vektor matrice A koji pripada svojstvenoj vrijednosti λ ako je:

a) $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & -4 \\ 1 & -2 & 4 \end{bmatrix}$ i $\lambda = -1$

b) $A = \begin{bmatrix} 5 & 4 & -4 \\ 1 & 3 & 3 \\ 2 & 0 & -9 \end{bmatrix}$ i $\lambda = 1$

c) $A = \begin{bmatrix} 4 & 4 & 6 \\ 3 & 3 & -1 \\ 1 & -3 & -5 \end{bmatrix}$ i $\lambda = 2$

Rješenje:

$\lambda = -1$

$$\begin{aligned} (\lambda I - A)v &= 0 \\ (-I - A)v &= 0 \\ (I + A)v &= 0 \end{aligned}$$

$$\begin{aligned} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & -4 \\ 1 & -2 & 4 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & -4 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\left[\begin{array}{ccc:c} 2 & 3 & 1 & 0 \\ 1 & 5 & -4 & 0 \\ 1 & -2 & 5 & 0 \end{array} \right] \begin{array}{l} \\ \mathrm{II} \leftrightarrow \mathrm{I} \\ \\ \end{array} \sim \left[\begin{array}{ccc:c} 1 & 5 & -4 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & -2 & 5 & 0 \end{array} \right] \begin{array}{l} \\ \mathrm{II} - 2\mathrm{I} \\ \mathrm{III} - \mathrm{I} \end{array} \sim$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -4 & 0 \\ 0 & -7 & 9 & 0 \\ 0 & -7 & 9 & 0 \end{array} \right] \begin{array}{l} \\ \\ \mathrm{III} - \mathrm{II} \end{array} \sim \left[\begin{array}{ccc:c} 1 & 5 & -4 & 0 \\ 0 & -7 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} x_1 + 5x_2 - 4x_3 &= 0 \\ -7x_2 + 9x_3 &= 0 \end{aligned}$$

$x_3 = t,\ t \in \mathbb{R}$

$$\begin{aligned} -7x_2 + 9x_3 &= 0 \\ x_2 &= \tfrac{9}{7}x_3 \\ x_2 &= \tfrac{9}{7}t \end{aligned}$$

$$\begin{aligned} x_1 + 5x_2 - 4x_3 &= 0 \\ x_1 &= -5x_2 + 4x_3 \\ x_1 &= -5\tfrac{9}{7}t + 4t \\ x_1 &= -\tfrac{17}{7}t \end{aligned}$$

$$x = \begin{bmatrix} -\frac{17}{7}t \\ \frac{9}{7}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{17}{7} \\ \frac{9}{7} \\ 1 \end{bmatrix}, t \in \mathbb{R} \setminus \{0\}, \text{ npr. za } t = 7 \text{ imamo } v_1 = \begin{bmatrix} -17 \\ 9 \\ 7 \end{bmatrix}$$

Zadatak 3.34. Odredite svojstvene vrijednosti i svojstvene vektore matrice $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -2 & 2 & 0 \end{bmatrix}$.

$$\begin{aligned} \lambda I - A &= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -2 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda - 1 & -1 & 1 \\ 1 & \lambda - 3 & 1 \\ 2 & -2 & \lambda \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
k(\lambda) &= \det(\lambda I - A) \\
&= \begin{vmatrix} \lambda-1 & -1 & 1 \\ 1 & \lambda-3 & 1 \\ 2 & -2 & \lambda \end{vmatrix} \\
&= (\lambda-1)\begin{vmatrix} \lambda-3 & 1 \\ -2 & \lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & \lambda \end{vmatrix} + \begin{vmatrix} 1 & \lambda-3 \\ 2 & -2 \end{vmatrix} \\
&= (\lambda-1)\left[\lambda(\lambda-3)+2\right] + \lambda - 2 - 2 - 2\,(\lambda-3) \\
&= (\lambda-1)\left[\lambda^2-3\lambda+2\right] + \lambda - 2 - 2 - 2\lambda + 6 \\
&= \lambda^3 - 3\lambda^2 + 2\lambda - \lambda^2 + 3\lambda - 2 - \lambda + 2 \\
&= \lambda^3 - 4\lambda^2 + 4\lambda \\
&= \lambda\,(\lambda^2 - 4\lambda + 4) \\
&= \lambda\,(\lambda-2)^2
\end{aligned}$$

$\lambda_1 = 0$, $\lambda_{2,3} = 2 \leftarrow$ dvostruka svojstvena vrijednost

Sada tražimo pripadne svojstvene vektore:

$\lambda_1 = 0$

$(\lambda_1 I - A)v = 0$

$Av = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -2 & 2 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \text{II+I} \\ \text{III+2I} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{III}-\text{II} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 4x_2 - 2x_3 &= 0 \end{aligned} \Longrightarrow \begin{aligned} x_1 &= \frac{1}{2}t \\ x_2 &= \frac{1}{2}t \\ x_3 &= t \end{aligned}$$

$v = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{bmatrix}$, $t \in \mathbb{R} \setminus \{0\}$, npr. za $t = 2$, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\lambda_{2,3} = 2$

$(\lambda_{2,3} I - A)v = 0$

$(2I-A)v=0$

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \text{II}-\text{I} \\ \text{III}-2\text{I} \end{array} \sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - x_2 + x_3 = 0 \Longrightarrow \begin{array}{l} x_1 = t_1 - t_2 \\ x_2 = t_1 \\ x_3 = t_2 \end{array}$$

$$v = \begin{bmatrix} t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, t_1, t_2 \in \mathbb{R}, (t_1, t_2) \neq (0,0)$$

Npr. za $t_1 = 0, t_2 = 1$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ i za $t_1 = 1, t_2 = 0$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Teorem 3.3. Svaka matrica poništava svoj karakteristični polinom, tj. $k(A) = 0$.

Zadatak 3.35. Koji od vektora $x = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}$ i $y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ je svojstveni vektor matrice $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ za svojstvenu vrijednost $\lambda = 1$? Pokažite da matrica poništava svoj karakteristični polinom.

Rješenje:

$Ax \stackrel{?}{=} \lambda x$, za $\lambda = 1$

$$Ax = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}$$

x je svojstveni vektor za $\lambda = 1$

$$Ay = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \neq y$$

y nije svojstveni vektor za $\lambda = 1$

$$\begin{aligned} k(\lambda) &= \det(\lambda I - A) \\ &= \begin{bmatrix} \lambda - 1 & 2 & -1 \\ 0 & \lambda - 2 & -3 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \\ &= (\lambda - 1) \begin{vmatrix} \lambda - 2 & -3 \\ 0 & \lambda + 1 \end{vmatrix} \\ &= (\lambda - 1)(\lambda - 2)(\lambda + 1) \end{aligned}$$

$$\begin{aligned} k(A) &= (A - I)(A - 2I)(A + I) \\ &= \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -9 \\ 0 & 0 & -6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Zadatak 3.36. Pronađite svojstvene vrijednosti matrice A i provjerite je li vektor v svojstveni vektor koji odgovara svojstvenoj vrijednosti λ ako je:

a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\lambda = 1$

b) $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$, $\lambda = -1$

Zadatak 3.37. Gaussovom metodom riješite sustav:

$$\begin{cases} x + y + z + w = 2 \\ -2x - 4y + z + w = 1 \\ x - y - z + w = 5 \end{cases}$$

Poglavlje 4

Nizovi i redovi

4.1 Nizovi

Definicija 4.1. Svaka funkcija $f : \mathbb{N} \to A$, gdje je A neki skup, naziva se **niz**.

- $f : \mathbb{N} \to \mathbb{N}$
- $f : \mathbb{N} \to \mathbb{R}$
- $f : \mathbb{N} \to \mathbb{Q}$
- $a : \mathbb{N} \to \mathbb{R}$

Općenito, niz $a : \mathbb{N} \to \mathbb{R}$ označavat ćemo s $(a_n)_{n\in\mathbb{N}}$.

Definicija 4.2. Za niz realnih brojeva $(a_n)_{n\in\mathbb{N}}$ kažemo da je **ograničen** ako postoji $M \in \mathbb{R}$, $M > 0$ takav da je $|a_n| \leq M, \forall n \in \mathbb{N}$.

Zadatak 4.1. Ispišite prvih nekoliko članova niza i odredite je li ograničen, ako je

a) $a_n = \dfrac{n^2}{2n+1}$

b) $b_n = (-1)^n$

c) $c_n = \dfrac{n}{n+1}$

d) $d_n = (-1)^n \dfrac{1}{n}$

e) $e_n = \frac{1}{n}$

f) $f_n = \frac{3n+1}{n^2+1}$

g) $g_n = 3$

Rješenje:

a) $\frac{1}{3}, \frac{4}{5}, \frac{9}{7}, \frac{16}{9}, \ldots$ Niz nije ograničen.

b) $-1, 1, -1, 1, \ldots$

Imamo npr. $M = 2$ jer je $|b_n| \leq 2, \forall n \in \mathbb{N}$.

c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

$|c_n| = \frac{n}{n+1} \leq 1, \forall n \in \mathbb{N}$.

d) $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$

$|d_n| = \frac{1}{n} \leq 1, \forall n \in \mathbb{N}$.

e) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$|e_n| = \frac{1}{n} \leq 1, \forall n \in \mathbb{N}$.

f) $2, \frac{7}{5}, 1, \frac{13}{17}, \frac{16}{26}, \ldots$

$|f_n| = \frac{3n+1}{n^2+1} < \frac{3n^2+3}{n^2+1} = 3, \forall n \in \mathbb{N}$.

g) $3, 3, 3, 3, \ldots$

$|g_n| = 3 \leq 4, \forall n \in \mathbb{N}$.

Definicija 4.3. Za niz $(a_n)_{n\in\mathbb{N}}$ kažemo da je

- **padajući** ako je $a_{n+1} \leq a_n, \forall n \in \mathbb{N}$
- **strogo padajući** ako je $a_{n+1} < a_n, \forall n \in \mathbb{N}$
- **rastući** ako je $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$
- **strogo rastući** ako je $a_n < a_{n+1}, \forall n \in \mathbb{N}$

- **monoton** ako je rastući ili padajući
- **strogo monoton** ako je strogo rastući ili strogo padajući

Zadatak 4.2. Provjerite je li niz (strogo) rastući ili (strogo) padajući:

a) $a_n = \dfrac{n^2}{2n+1}$

b) $b_n = \dfrac{n}{n+1}$

c) $c_n = \dfrac{1}{n}$

d) $d_n = \dfrac{3n+1}{n^2+1}$

Rješenje:

a) $a_n = \dfrac{n^2}{2n+1}$

$$\begin{aligned} a_n &\overset{?}{<} a_{n+1}, \forall n \in \mathbb{N} \\ \frac{n^2}{2n+1} &< \frac{(n+1)^2}{2(n+1)+1} \\ \frac{n^2}{2n+1} &< \frac{n^2+2n+1}{2n+3} \qquad / \cdot (2n+1)(2n+3) > 0 \\ n^2(2n+3) &< (2n+1)(n^2+2n+1) \\ 2n^3+3n^2 &< 2n^3+4n^2+2n+n^2+2n+1 \\ 0 &< 2n^2+4n+1, \forall n \in \mathbb{N} \end{aligned}$$

b) $b_n = \dfrac{n}{n+1}$

$$\begin{aligned} b_n &\overset{?}{<} b_{n+1}, \forall n \in \mathbb{N} \\ \frac{n}{n+1} &< \frac{n+1}{n+2} \qquad / \cdot (n+1)(n+2) > 0 \\ n(n+2) &< (n+1)(n+1) \\ n^2+2n &< n^2+2n+1 \\ 0 &< 1, \forall n \in \mathbb{N} \end{aligned}$$

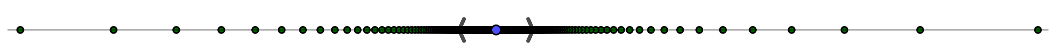
c) $c_n = \dfrac{1}{n}$

$$\begin{aligned} c_n &\overset{?}{>} c_{n+1}, \forall n \in \mathbb{N} \\ \frac{1}{n} &> \frac{1}{n+1} \quad / \cdot n(n+1) > 0 \\ n+1 &> n \\ 1 &> 0, \forall n \in \mathbb{N} \end{aligned}$$

d) $d_n = \dfrac{3n+1}{n^2+1}$

$$\begin{aligned} d_n &\overset{?}{>} d_{n+1}, \forall n \in \mathbb{N} \\ \frac{3n+1}{n^2+1} &> \frac{3(n+1)+1}{(n+1)^2+1} \\ \frac{3n+1}{n^2+1} &> \frac{3n+4}{n^2+2n+2} \quad / \cdot (n^2+1)(n^2+2n+2) > 0 \\ (3n+1)(n^2+2n+2) &> (3n+4)(n^2+1) \\ 3n^3+6n^2+6n+n^2+2n+2 &> 3n^3+3n+4n^2+4 \\ 3n^2+5n-2 &> 0, \forall n \in \mathbb{N} \end{aligned}$$

Definicija 4.4. Neka je $(a_n)_{n\in\mathbb{N}}$ niz realnih brojeva. Ako postoji $L \in \mathbb{R}$ tako da vrijedi: $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow |a_n - L| < \varepsilon$, kažemo da je L **limes niza** i označavamo sa $L = \lim\limits_{n\to+\infty} a_n$. Za niz koji ima limes kažemo da je **konvergentan**. Inače kažemo da je **divergentan**.



$L-\varepsilon \; L \; L+\varepsilon$

$|a_n - L| < \varepsilon \Rightarrow -\varepsilon < a_n - L < \varepsilon \Rightarrow L - \varepsilon < a_n < L + \varepsilon$

Limes niza je broj, takav da svaka okolina tog broja, kolikogod mala ona bila, sadrži beskonačno mnogo članova, a istovremeno je izvan te okoline najviše konačno mnogo članova niza.

Svojstva konvergentnih nizova: Neka su $(a_n)_{n\in\mathbb{N}}$ i $(b_n)_{n\in\mathbb{N}}$ dva konvergentna niza te $A = \lim\limits_{n\to+\infty} a_n$ i $B = \lim\limits_{n\to+\infty} b_n$. Tada vrijedi:

i) Niz $(a_n + b_n)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty} (a_n + b_n) = A + B$

ii) Za svaki $c \in \mathbb{R}$, niz $(c \cdot a_n)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty} (c \cdot a_n) = c \cdot A$

iii) Niz $(a_n \cdot b_n)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}(a_n \cdot b_n) = A \cdot B$

iv) Ako je $b_n \neq 0, \forall n \in \mathbb{N}$ i $B \neq 0$, niz $\left(\dfrac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{A}{B}$

v) Niz $\left(a_n^{b_n}\right)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}\left(a_n^{b_n}\right) = A^B$

- npr. za $b_n = \dfrac{1}{2}, \forall n \in \mathbb{N} \Longrightarrow \lim\limits_{n\to+\infty}\sqrt{a_n} = \sqrt{A}$

Primjeri nekih limesa:

i) $a_n = A, \forall n \in \mathbb{N}$ je konstantni niz i $\lim\limits_{n\to+\infty} a_n = A$

ii) $a_n = \dfrac{1}{n}$ i $\lim\limits_{n\to+\infty}\dfrac{1}{n} = 0$ (Za bilo koji $\varepsilon > 0$ odaberemo $n_0 \in \mathbb{N}$ takav da je $\dfrac{1}{n_0} < \varepsilon$ pa za $n > n_0$ imamo $|a_n - 0| = \dfrac{1}{n} \leq \dfrac{1}{n_0} < \varepsilon$)

iii) $a_n = q^n$, za neki $q \in \mathbb{R}$

- $|q| < 1$ imamo da je $\lim\limits_{n\to+\infty} q^n = 0$
- $q = 1$ imamo da je $\lim\limits_{n\to+\infty} q^n = 1$
- $|q| > 1$ niz divergira
- $q = -1$ imamo niz $-1, 1, -1, 1, ..$ pa divergira

iv) $a_n = \sqrt[n]{n}$ i $\lim\limits_{n\to+\infty}\sqrt[n]{n} = 1$

v) $a_n = \sqrt[n]{a}$, za neki $a > 0$ i $\lim\limits_{n\to+\infty}\sqrt[n]{a} = 1$

vi) $a_n = \left(1 + \dfrac{1}{n}\right)^n$ i $\lim\limits_{n\to+\infty}\left(1 + \dfrac{1}{n}\right)^n = e$

Teorem 4.1. Svaki ograničen i monoton niz je konvergentan.

Teorem 4.2. Neka su $(a_n)_{n\in\mathbb{N}}$ i $(b_n)_{n\in\mathbb{N}}$ konvergentni i neka je $a_n \leq b_n, \forall n \in \mathbb{N}$. Tada je $\lim\limits_{n\to+\infty} a_n \leq \lim\limits_{n\to+\infty} b_n$. Svaki ograničen i monoton niz je konvergentan.

Zadatak 4.3. Izračunajte limese sljedećih nizova:

a) $a_n = \dfrac{n}{n+1}$

b) $b_n = \dfrac{n+1}{n^2+2}$

c) $c_n = \dfrac{n^2+n-1}{n^2+2}$

d) $d_n = \dfrac{n^3+n^2-1}{2n^2+3}$

e) $e_n = \left[\left(1+\dfrac{3}{n}\right)\left(2-\dfrac{4}{n}\right)^2\left(\dfrac{5}{n^2}-1\right)\right]$

f) $f_n = \left(\sqrt{n+1}-\sqrt{n}\right)$

g) $g_n = \left(\dfrac{1+2+\cdots+n}{n+2}-\dfrac{n}{2}\right)$

h) $h_n = \left(\dfrac{n+1}{n-1}\right)^n$

i) $i_n = \dfrac{\sin n}{n}$

Rješenje:

$$\begin{aligned}\lim_{n\to+\infty} a_n &= \lim_{n\to+\infty}\frac{n}{n+1}\\ &= \lim_{n\to+\infty}\frac{n}{n+1}:\frac{n}{n}\\ &= \lim_{n\to+\infty}\frac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}}\\ &= \frac{1}{1+0}\\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{n\to+\infty} b_n &= \lim_{n\to+\infty}\frac{n+1}{n^2+2}\\ &= \lim_{n\to+\infty}\frac{n+1}{n^2+2}:\frac{n^2}{n^2}\\ &= \lim_{n\to+\infty}\frac{\overbrace{\frac{1}{n}}^{\to 0}+\overbrace{\frac{1}{n^2}}^{\to 0}}{1+\underbrace{\frac{2}{n^2}}_{\to 0}}\\ &= \frac{0+0}{1+0}\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{n\to+\infty} c_n &= \lim_{n\to+\infty}\frac{n^2+n-1}{n^2+2}\\ &= \lim_{n\to+\infty}\frac{n^2+n-1}{n^2+2}:\frac{n^2}{n^2}\\ &= \lim_{n\to+\infty}\frac{1+\frac{1}{n}-\frac{1}{n^2}}{1+\underbrace{\frac{2}{n^2}}_{\to 0}}\\ &= \frac{1+0-0}{1+0}\\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{n\to+\infty} d_n &= \lim_{n\to+\infty}\frac{n^3+n^2-1}{2n^2+3}\\ &= \lim_{n\to+\infty}\frac{n^3+n^2-1}{2n^2+3}:\frac{n^3}{n^3}\\ &= \lim_{n\to+\infty}\frac{1+\overbrace{\frac{1}{n}}^{\to 0}-\overbrace{\frac{1}{n^3}}^{\to 0}}{\underbrace{\frac{2}{n}}_{\to 0}+\underbrace{\frac{3}{n^3}}_{\to 0}}\\ &= \frac{1}{0}\\ &= +\infty\end{aligned}$$

$$\begin{aligned}\lim_{n\to+\infty} e_n &= \lim_{n\to+\infty}\left[\left(1+\frac{3}{n}\right)\left(2-\frac{4}{n}\right)^2\left(\frac{5}{n^2}-1\right)\right]\\ &= (1+0)\,(2-0)^2\,(0-1)\\ &= -4\end{aligned}$$

$$\begin{aligned}
\lim_{n\to+\infty} f_n &= \lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = (\infty-\infty)\\
&= \lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\\
&= \lim_{n\to+\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\\
&= \lim_{n\to+\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} : \frac{\sqrt{n}}{\sqrt{n}}\\
&= \lim_{n\to+\infty}\frac{\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}+1}\\
&= \frac{0}{\sqrt{1+0}+1}\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to+\infty} g_n &= \lim_{n\to+\infty}\left(\frac{1+2+\cdots+n}{n+2}-\frac{n}{2}\right) \qquad \left(1+2+\cdots+n=\frac{n(n+1)}{2}\right)\\
&= \lim_{n\to+\infty}\left(\frac{n(n+1)}{2(n+2)}-\frac{n}{2}\right) =\\
&= \lim_{n\to+\infty}\frac{n(n+1)-n(n+2)}{2(n+2)}\\
&= \lim_{n\to+\infty}\frac{n^2+n-n^2-2n}{2n+4}\\
&= \lim_{n\to+\infty}\frac{-n}{2n+4} : \frac{n}{n}\\
&= \lim_{n\to+\infty}\frac{-1}{2+\frac{4}{n}}\\
&= \frac{-1}{2+0}\\
&= \frac{-1}{2}
\end{aligned}$$

$$
\begin{aligned}
\lim_{n\to+\infty} h_n &= \lim_{n\to+\infty}\left(\frac{n+1}{n-1}\right)^n = (1^\infty)\\
&= \lim_{n\to+\infty}\left(\frac{n-1}{n-1}+\frac{2}{n-1}\right)^n\\
&= \lim_{n\to+\infty}\left(1+\frac{2}{n-1}\right)^n\\
&= \lim_{n\to+\infty}\left(1+\frac{1}{\frac{n-1}{2}}\right)^n\\
&= \lim_{n\to+\infty}\left(\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}}\right)^{\frac{2n}{n-1}}\\
&= \left(\lim_{n\to+\infty}\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}}\right)^{\lim\limits_{n\to+\infty}\frac{2n}{n-1}} \qquad n\to+\infty \implies \left(\tfrac{n-1}{2}\right)\to+\infty\\
&= e^{\lim\limits_{n\to+\infty}\frac{2}{1-\frac{1}{n}}}\\
&= e^2
\end{aligned}
$$

$$
\lim_{n\to+\infty} i_n = \lim_{n\to+\infty}\frac{\sin n}{n}
$$

$$
\begin{aligned}
-1 \le & \quad \sin n \quad \le 1 \quad / :n\\
\frac{-1}{n} \le & \quad \frac{\sin n}{n} \quad \le \frac{1}{n} \quad / \lim_{n\to+\infty}\\
\underbrace{\lim_{n\to+\infty}\frac{-1}{n}}_{0} \le & \lim_{n\to+\infty}\frac{\sin n}{n} \le \underbrace{\lim_{n\to+\infty}\frac{1}{n}}_{0}\\
& \lim_{n\to+\infty}\frac{\sin n}{n} = 0
\end{aligned}
$$

Napomena: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$

Zadatak 4.4. Za niz a_n odredite limes L te $n_0 \in \mathbb{N}$ takav da vrijedi: $n \ge n_0 \Rightarrow |a_n - L| < \varepsilon$ ako je

a) $a_n = \dfrac{(-1)^n}{n} + 1$ i $\varepsilon = 0.02$,

b) $a_n = \dfrac{3n^2 - n}{n^2}$ i $\varepsilon = 0.001$.

Rješenje:

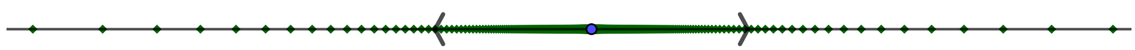
a)

$$\begin{aligned}
\lim_{n\to+\infty} a_n &= \lim_{n\to+\infty}\left(\frac{(-1)^n}{n}+1\right)\\
&= \lim_{n\to+\infty}\frac{(-1)^n}{n}+1\\
&= 0+1\\
&= 1
\end{aligned}$$

Odredimo sada n_0 takav da je $|a_n - 1| < 0.02$.

$$\begin{aligned}
\left|\frac{(-1)^n}{n}+1-1\right| &< 0.02\\
\frac{1}{n} &< 0.02 \quad / \cdot n\\
1 &< 0.02\cdot n \quad / : 0.02\\
50 &< n\\
n_0 &= 51
\end{aligned}$$

Svi članovi niza, od 51-og pa nadalje nalaze se u okolini $\langle 1-0.02, 1+0.02\rangle$



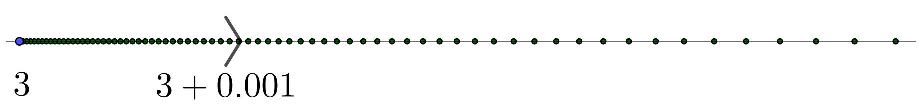
b)

$$\begin{aligned}
\lim_{n\to+\infty} a_n &= \lim_{n\to+\infty}\frac{3n^2-n}{n^2}\\
&= \lim_{n\to+\infty}\frac{3n^2-n}{n^2} : \frac{n^2}{n^2}\\
&= \lim_{n\to+\infty}\frac{3-\frac{1}{n}}{1}\\
&= \frac{3-0}{1}\\
&= 3
\end{aligned}$$

Odredimo sada n_0 takav da je $|a_n - 3| < 0.001$.

$$\begin{aligned}
\left|\frac{3n^2-n}{n^2}-3\right| &< 0.001\\
\left|\frac{3n^2-n-3n^2}{n^2}\right| &< 0.001\\
\left|\frac{-1}{n}\right| &< 0.001\\
\frac{1}{n} &< 0.001 \quad / \cdot n\\
1 &< 0.001\cdot n \quad / : 0.001\\
1000 &< n\\
n_0 &= 1001
\end{aligned}$$

Svi članovi niza, od 1001-og pa nadalje nalaze se u okolini $\langle 3-0.001, 3+0.001\rangle$



4.2 Redovi

Definicija 4.5. Neka je $(a_n)_{n\in\mathbb{N}}$ niz realnih brojeva. $S_n = a_1 + a_2 + \cdots + a_n = \sum\limits_{k=1}^{n} a_k$ kažemo da je $n-$**ta parcijalna suma** niza $(a_n)_{n\in\mathbb{N}}$.

Definicija 4.6. Uređeni par (a_n, S_n), gdje je a_n opći član niza, a S_n niz pripadajućih parcijalnih suma od a_n naziva se **red**.

Zadatak 4.5. Odredite opći član reda:

a) $\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots$

b) $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\cdots$

Rješenje:

a) $a_n = \frac{1}{2n+2}$

b) $a_n = \frac{n+2}{(n+1)^2}$

Zadatak 4.6. Odredite 3. i 4. parcijalnu sumu redova:

a) $\sum\limits_{n=1}^{\infty} \frac{n}{2^{n-1}}$

b) $\sum\limits_{n=1}^{\infty} \frac{1}{n}$

Rješenje:

$$\begin{aligned} S_3 &= \sum_{n=1}^{3} \frac{n}{2^{n-1}} \\ &= \frac{1}{2^{1-1}} + \frac{2}{2^{2-1}} + \frac{3}{2^{3-1}} \\ &= \frac{1}{1} + \frac{2}{2} + \frac{3}{4} \\ &= \frac{11}{4} \end{aligned}$$

$$\begin{aligned} S_4 &= \sum_{n=1}^{4} \frac{n}{2^{n-1}} \\ &= S_3 + \frac{4}{2^{4-1}} \\ &= \frac{11}{4} + \frac{4}{8} \\ &= \frac{13}{4} \end{aligned}$$

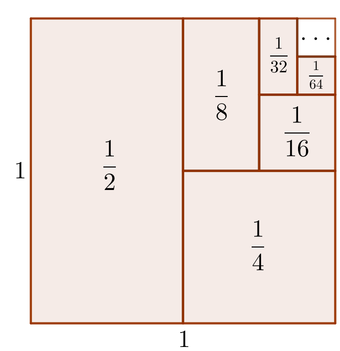
$$\begin{aligned} S_3 &= \sum_{n=1}^{3} \frac{1}{n} \\ S_3 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \\ S_3 &= \frac{11}{6} \end{aligned}$$

$$\begin{aligned} S_4 &= \sum_{n=1}^{4} \frac{1}{n} \\ &= S_3 + \frac{1}{4} \\ &= \frac{11}{6} + \frac{1}{4} \\ &= \frac{25}{12} \end{aligned}$$

Definicija 4.7. Za red (a_n, S_n) kažemo da je **konvergentan** ako je S_n konvergentan. Ako postoji limes $s = \lim\limits_{n \to \infty} S_n$, kažemo da je s **suma** reda (a_n, S_n) i pišemo

$$s = \sum_{n=1}^{\infty} a_n.$$

Primjer: $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ konvergira i $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1$.



Kriteriji konvergencije:

i) **Kriterij uspoređivanja**: Neka su $\sum\limits_{n=1}^{\infty} a_n$ i $\sum\limits_{n=1}^{\infty} b_n$ redovi i neka postoji $K > 0$ takav da je $0 \le a_n < K \cdot b_n, \forall n \in \mathbb{N}$. Tada vrijedi:

- Ako red $\sum\limits_{n=1}^{\infty} b_n$ konvergira, konvergira i red $\sum\limits_{n=1}^{\infty} a_n$. $\left(\sum\limits_{n=1}^{\infty} a_n \le K \sum\limits_{n=1}^{\infty} b_n\right)$
- Ako red $\sum\limits_{n=1}^{\infty} a_n$ divergira, divergira i red $\sum\limits_{n=1}^{\infty} b_n$.
- Ako postoji $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} \neq 0$, onda oba reda $\sum\limits_{n=1}^{\infty} a_n$ i $\sum\limits_{n=1}^{\infty} b_n$ ili konvergiraju ili divergiraju.

ii) **D'Alembertov kriterij**: Neka je $\sum\limits_{n=1}^{\infty} a_n$ red s pozitivnim članovima i postoji $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = q$. Ako je:

- $q < 1$, onda red $\sum\limits_{n=1}^{\infty} a_n$ konvergira.
- $q > 1$, onda red $\sum\limits_{n=1}^{\infty} a_n$ divergira.
- $q = 1$, onda ne možemo donijeti zaključak.

iii) **Cauchyjev kriterij**: Neka je $\sum\limits_{n=1}^{\infty} a_n$ red s pozitivnim članovima i postoji $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = q$. Ako je:

- $q < 1$, onda red $\sum\limits_{n=1}^{\infty} a_n$ konvergira.
- $q > 1$, onda red $\sum\limits_{n=1}^{\infty} a_n$ divergira.
- $q = 1$, onda ne možemo donijeti zaključak.

iv) **Leibnitzov kriterij**: Neka je $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ alternirajući red. Ako vrijedi:

- $a_1 \ge a_2 \ge a_3 \ge \cdots$, tj. niz (a_n) je padajući i
- $\lim\limits_{n\to\infty} a_n = 0$

onda red $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ konvergira.

Napomena:

i) Ako red $\sum\limits_{n=1}^{\infty} a_n$ konvergira, onda je $\lim\limits_{n\to\infty} a_n = 0$. Obrat ne vrijedi.

ii) Ako red $\sum\limits_{n=1}^{\infty} |a_n|$ konvergira, onda konvergira i red $\sum\limits_{n=1}^{\infty} a_n$. Kažemo da apsolutna konvergencija povlači običnu konvergenciju.

Primjeri redova:

i) Geometrijski red $\sum\limits_{n=1}^{\infty} q^n$

- konvergira, ako je $|q| < 1$,
- divergira, ako je $|q| \geq 1$.

ii) Harmonijski red $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ divergira. Red $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ konvergira ako je $k > 1$. Npr. red $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ konvergira.

Zadatak 4.7. Ispitajte konvergenciju redova:

a) $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}$

b) $\sum\limits_{n=1}^{\infty} \frac{n+1}{n \cdot 2^n}$

c) $\sum\limits_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

d) $\sum\limits_{n=1}^{\infty} \frac{\sin n}{1+n^2}$

e) $\sum\limits_{n=1}^{\infty} \frac{n^2}{n^3+2n^2+5}$

f) $\sum\limits_{n=1}^{\infty} \frac{2n-1}{2^n}$

g) $\sum\limits_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

h) $\sum\limits_{n=1}^{\infty} \frac{n!}{3^n}$

i) $\sum\limits_{n=1}^{\infty} n \cdot \left(\frac{4}{3}\right)^{n+1}$

j) $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} \cdot \left(1+\frac{1}{n}\right)^{n^2}$

k) $\sum\limits_{n=1}^{\infty} \frac{2^n}{n^{10}}$

l) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Rješenje:

a) $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$, pa red divergira jer opći član ne teži prema 0.

b) $\sum\limits_{n=1}^{\infty} \frac{n+1}{n \cdot 2^n}$

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\frac{n+2}{(n+1)\cdot 2^{n+1}}}{\frac{n+1}{n\cdot 2^n}} \\
&= \lim_{n\to\infty}\frac{n(n+2)}{2(n+1)^2} \\
&= \lim_{n\to\infty}\frac{n^2+2n}{2n^2+4n+2} : \frac{n^2}{n^2} \\
&= \lim_{n\to\infty}\frac{1+\frac{2}{n}}{2+\frac{4}{n}+\frac{2}{n^2}} \\
&= \tfrac{1}{2} < 1
\end{aligned}$$

Prema D'Alembertovom kriteriju, red konvergira.

c) $\sum\limits_{n=1}^{\infty}\frac{1}{(2n+1)^2}$

$0 \leq \frac{1}{(2n+1)^2} \leq \frac{1}{n^2}$ Kako $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ konvergira, to konvergira i $\sum\limits_{n=1}^{\infty}\frac{1}{(2n+1)^2}$ prema kriteriju uspoređivanja.

d) $\sum\limits_{n=1}^{\infty}\frac{\sin n}{1+n^2}$

$0 \leq \frac{|\sin n|}{1+n^2} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$

$\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ konvergira, pa konvergira i $\sum\limits_{n=1}^{\infty}\frac{|\sin n|}{1+n^2}$, pa naposlijetku konvergira i $\sum\limits_{n=1}^{\infty}\frac{\sin n}{1+n^2}$ jer apsolutna konvergencija povlači i konvergenciju reda.

e) $\sum\limits_{n=1}^{\infty}a_n = \sum\limits_{n=1}^{\infty}\frac{n^2}{n^3+2n^2+5}$

Znamo da $\sum\limits_{n=1}^{\infty}b_n = \sum\limits_{n=1}^{\infty}\frac{1}{n}$ divergira i vrijedi:

$$
\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{n^2}{n^3+2n^2+5}}{\frac{1}{n}}\\
&= \lim_{n\to\infty}\frac{n^3}{n^3+2n^2+5}\\
&= \lim_{n\to\infty}\frac{n^3}{n^3+2n^2+5} : \frac{n^3}{n^3}\ ,\\
&= \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{5}{n^3}}\\
&= \frac{1}{1+0+0}\\
&= 1\neq 0
\end{aligned}
$$

pa prema 2. kriteriju uspoređivanja, red $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{n^3+2n^2+5}$ divergira.

f) $\sum\limits_{n=1}^{\infty}\dfrac{2n-1}{2^n}$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\frac{2(n+1)-1}{2^{n+1}}}{\frac{2n-1}{2^n}}\\
&= \lim_{n\to\infty}\frac{2n+1}{2(2n-1)}\\
&= \lim_{n\to\infty}\frac{2n+1}{4n-2} : \frac{n}{n}\\
&= \lim_{n\to\infty}\frac{2+\frac{1}{n}}{4-\frac{2}{n}}\\
&= \frac{1}{2} < 1
\end{aligned}
$$

Prema D'Alembertovom kriteriju, red konvergira.

g) $\sum\limits_{n=1}^{\infty}\left(\dfrac{n}{2n+1}\right)^n$

$$
\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} a_n^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\left(\frac{n}{2n+1}\right)^n\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n}{2n+1} : \frac{n}{n} \\
&= \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} \\
&= \frac{1}{2} < 1
\end{aligned}
$$

Prema Cauchyjevom kriteriju, red konvergira.

h) $\sum\limits_{n=1}^{\infty} \dfrac{n!}{3^n}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} \\
&= \lim_{n\to\infty} \frac{\frac{n!\cdot(n+1)}{3\cdot 3^n}}{\frac{n!}{3^n}} \\
&= \lim_{n\to\infty} \frac{n+1}{3} \\
&= \infty
\end{aligned}
$$

Prema D'Alembertovom kriteriju, red divergira.

i) $\sum\limits_{n=1}^{\infty} n \cdot \left(\dfrac{4}{3}\right)^{n+1}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)\cdot\left(\frac{4}{3}\right)^{n+2}}{n\cdot\left(\frac{4}{3}\right)^{n+1}} \\
&= \frac{4}{3} \lim_{n\to\infty} \frac{n+1}{n} \\
&= \frac{4}{3} > 1
\end{aligned}
$$

Prema D'Alembertovom kriteriju, red divergira.

j) $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n} \cdot \left(1+\dfrac{1}{n}\right)^{n^2}$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{a_n} &= \lim_{n\to\infty}\sqrt[n]{\frac{1}{2^n}\cdot\left(1+\frac{1}{n}\right)^{n^2}}\\ &= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\\ &= \frac{1}{2}e>1\end{aligned}$$

Prema Cauchyjevom kriteriju, red divergira.

k) $\sum\limits_{n=1}^{\infty}\frac{2^n}{n^{10}}$

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)^{10}}}{\frac{2^n}{n^{10}}}\\ &= 2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{10}\\ &= 2\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^{10}\\ &= 2\cdot 1^{10}\\ &= 2>1\end{aligned}$$

Prema D'Alembertovom kriteriju, red divergira.

l) $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n=\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

$a_1\geq a_2\geq a_3\cdots$ i $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1}{n}=0$

Prema Leibnitzovom kriteriju, red konvergira.

Zadatak 4.8. Odredite opći član i ispitajte konvergenciju reda:

a) $\frac{3\cdot 1}{\sqrt[3]{1}}+\frac{3\cdot 2}{\sqrt[3]{2}}+\frac{3\cdot 4}{\sqrt[3]{3}}+\frac{3\cdot 8}{\sqrt[3]{4}}+\cdots$

b) $\frac{2}{2\cdot 3}+\frac{4}{4\cdot 9}+\frac{6}{8\cdot 27}+\frac{8}{16\cdot 81}+\cdots$

c) $\frac{1}{3!}+\frac{\sqrt{3}}{5!}+\frac{\sqrt{5}}{7!}+\frac{\sqrt{7}}{9!}+\cdots$

Rješenje: a) $a_n=\frac{3\cdot 2^{n-1}}{\sqrt[3]{n}}$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\dfrac{3\cdot 2^n}{\sqrt[3]{n+1}}}{\dfrac{3\cdot 2^{n-1}}{\sqrt[3]{n}}}\\
&= \lim_{n\to\infty} 2\cdot\sqrt[3]{\frac{n}{n+1}}\\
&= 2\cdot\sqrt[3]{\lim_{n\to\infty}\frac{n}{n+1}:\frac{n}{n}}\\
&= 2\cdot\sqrt[3]{\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}}\\
&= 2\cdot\sqrt[3]{\frac{1}{1+0}}\\
&= 2\cdot\sqrt[3]{1}\\
&= 2\cdot 1\\
&= 2>1
\end{aligned}
$$

Prema D'Alembertovom kriteriju, red divergira.

b) $a_n=\dfrac{2n}{2^n\cdot 3^n}$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\dfrac{2(n+1)}{2^{n+1}\cdot 3^{n+1}}}{\dfrac{2n}{2^n\cdot 3^n}}\\
&= \lim_{n\to\infty}\frac{n+1}{6n}\\
&= \frac{1}{6}\underbrace{\lim_{n\to\infty}\frac{n+1}{n}}_{=1}\\
&= \frac{1}{6}<1
\end{aligned}
$$

Prema D'Alembertovom kriteriju, red konvergira.

c) $a_n=\dfrac{\sqrt{2n-1}}{(2n+1)!}$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\dfrac{\sqrt{2(n+1)-1}}{(2(n+1)+1)!}}{\dfrac{\sqrt{2n-1}}{(2n+1)!}} \\
&= \lim_{n\to\infty}\frac{\dfrac{\sqrt{2n+1}}{(2n+3)!}}{\dfrac{\sqrt{2n-1}}{(2n+1)!}} \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{2n+1}}{\sqrt{2n-1}}\cdot\frac{1}{(2n+2)(2n+3)}\right) \\
&= \lim_{n\to\infty}\sqrt{\frac{2n+1}{2n-1}}\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)} \\
&= \sqrt{\lim_{n\to\infty}\frac{2n+1}{2n-1}:\frac{n}{n}}\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)} \\
&= \sqrt{\lim_{n\to\infty}\frac{2+\frac{1}{n}}{2-\frac{1}{n}}}\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)} \\
&= \sqrt{\frac{2+0}{2-0}}\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)} \\
&= 1\cdot 0 \\
&= 0<1
\end{aligned}
$$

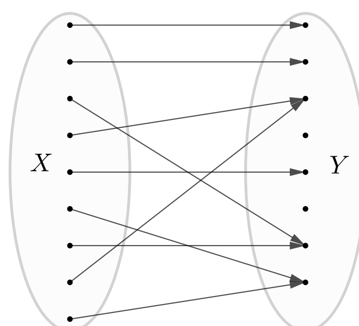
Prema D'Alembertovom kriteriju, red konvergira.

Poglavlje 5

Funkcije

5.1 Osnovni pojmovi

Definicija 5.1. Neka su X i Y skupovi, f pravilo preslikavanja elemenata skupa X u Y. Ako $\forall x \in X, \exists! y \in y$ takav da je $f(x) = y$ onda uređenu trojku (X, f, Y) nazivamo **funkcija** i pišemo $f : X \to Y$.



Definicija 5.2. Neka je (X, f, Y) funkcija. Skup X nazivamo domena, a skup Y kodomena funkcije f.

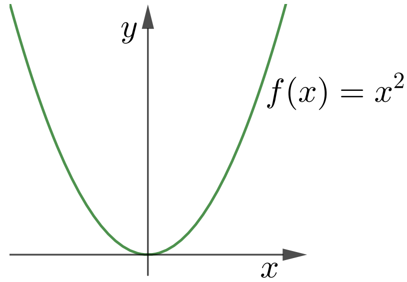
Definicija 5.3. Skup $\mathrm{Im} f = \{y \in Y : \exists x \in X, f(x) = y\}$ nazivamo **slika funkcije**.

Definicija 5.4. Skup $\Gamma_f = \{(x, f(x)), x \in X\} \subset X \times Y$ nazivamo **graf funkcije**.

Primjer: $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$\text{Im} f = \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$



Definicija 5.5. Za dvije funkcije kažemo da su jednake ako imaju istu domenu, istu kodomenu i isto pravilo.

Primjer:

$$\left.\begin{array}{l} f_1 : \mathbb{R} \to \mathbb{R}, \quad f_1(x) = x^2 \\ f_2 : \mathbb{R} \to \mathbb{R}^+, f_2(x) = x^2 \end{array}\right\} f_1 \neq f_2$$

Definicija 5.6. Neka je $f : X \to Y$ funkcija. Za funkciju $f_1 : A \subseteq X \to Y$ kažemo da je restrikcija funkcije f na A ako je $f_1(x) = f(x), \forall x \in A$ i označavamo $f_1 = f|_A$. Za funkciju $f_2 : A \supseteq X \to Y$ kažemo da je proširenje funkcije f na A ako je $f_2(x) = f(x), \forall x \in X$.

Definicija 5.7. Za funkciju $f : X \to Y$ kažemo da je **injekcija** ako $\forall x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ ili ekvivalentno, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Definicija 5.8. Za funkciju $f : X \to Y$ kažemo da je **surjekcija** ako $\forall y \in Y, \exists x \in X, f(x) = y$, odnosno, $\text{Im} f = Y$.

Definicija 5.9. Za funkciju $f : X \to Y$ kažemo da je **bijekcija** ako je i injekcija i surjekcija.

5.2 Realna funkcija realne varijable

$f : D \to \mathbb{R}, D \subseteq \mathbb{R}$

Prirodna domena funkcije (u oznaci $D(f)$ ili D_f) je skup svih realnih brojeva za koje je funkcija dobro definirana.

Zadatak 5.1. Odredite prirodnu domenu funkcije:

a) $f(x) = \dfrac{x}{(x-2)(x-5)}$

b) $f(x) = \sqrt{x^2 - 5x + 6}$

c) $f(x) = \ln(2x - 4)$

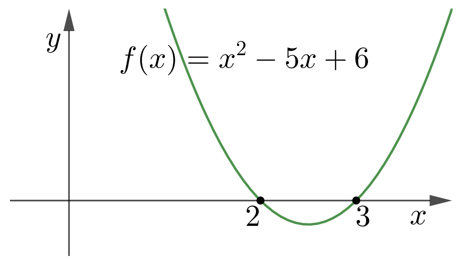
d) $f(x) = \dfrac{x}{\sqrt{-x^2 + 3x - 2}}$

Rješenje: a) $f(x) = \dfrac{x}{(x-2)(x-5)}$

$x \neq 2$ i $x \neq 5 \Longrightarrow D_f = \langle -\infty, 2\rangle \bigcup \langle 2, 5\rangle \bigcup \langle 5, +\infty\rangle = \mathbb{R} \setminus \{2, 5\}$

b) $f(x) = \sqrt{x^2 - 5x + 6}$

$$x^2 - 5x + 6 \geq 0$$



$$x_{1,2} = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6 \cdot 1}}{2 \cdot 1}$$

$$x_{1,2} = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$x_1 = 2 \quad , \quad x_2 = 3$$

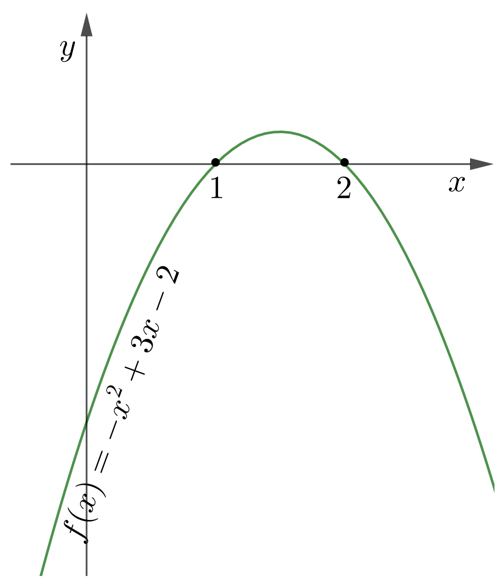
$D_f = \langle -\infty, 2] \bigcup [3, +\infty\rangle$

c) $f(x) = \ln(2x - 4)$

$2x - 4 > 0 \Longrightarrow x > 2$

$D_f = \langle 2, +\infty\rangle$

d) $f(x) = \dfrac{x}{\sqrt{-x^2 + 3x - 2}}$



$$-x^2 + 3x - 2 > 0$$

$$x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot (-2) \cdot (-1)}}{2 \cdot (-1)}$$

$$x_{1,2} = \frac{-3 \pm \sqrt{1}}{-2}$$

$$x_1 = 1 \quad , \quad x_2 = 2$$

$D_f = \langle 1, 2\rangle$

Definicija 5.10. Neka je $f : D \subseteq \mathbb{R} \to \mathbb{R}$ funkcija. Kažemo da je f:

- **(strogo) rastuća** ako $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) (f(x_1) < f(x_2))$

- **(strogo) padajuća** ako $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)(f(x_1) > f(x_2))$
- **parna** ako je $f(-x) = f(x), \forall x \in D$
- **neparna** ako je $f(-x) = -f(x), \forall x \in D$
- **periodična** ako $\exists a \in \mathbb{R}$ takav da je $f(x+a) = f(x), \forall x \in D$
- **omeđena** ako $\exists M \in \mathbb{R}, M > 0$ takav da je $|f(x)| \leq M, \forall x \in D$, tj. $-M \leq f(x) \leq M, \forall x \in D$

Definicija 5.11. Neka su $f : X \subseteq \mathbb{R} \to \mathbb{R}$ i $g : Y \subseteq \mathrm{Im} f \to \mathbb{R}$ funkcije. **Kompozicija funkcija** f i g je funkcija $h : X \to \mathbb{R}$, definirana pravilom $h(x) = g(f(x))$. Kompoziciju funkcija f i g označavamo: $h = g \circ f$, odnosno $(g \circ f)(x) = g(f(x))$.

Zadatak 5.2. Odredite $f \circ g$ i $g \circ f$ ako je:

a) $f(x) = 3x$ i $g(x) = \sqrt{x-1}$

b) $f(x) = \dfrac{x+2}{x-1}$ i $g(x) = \dfrac{x}{x-1}$

c) $f(x) = \sqrt{x}$ i $g(x) = \dfrac{1}{1+x^2}$

Rješenje: a)

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) & \qquad (g \circ f)(x) &= g(f(x)) \\ &= f(\sqrt{x-1}) & &= g(3x) \\ &= 3\sqrt{x-1} & &= \sqrt{3x-1} \end{aligned}$$

b)

$$
\begin{aligned}
(f\circ g)(x) &= f(g(x)) \\
&= f\left(\frac{x}{x-1}\right) \\
&= \frac{\frac{x}{x-1}+2}{\frac{x}{x-1}-1} \\
&= \frac{\frac{x+2x-2}{x-1}}{\frac{x-x+1}{x-1}} \\
&= \frac{3x-2}{1} \\
&= 3x-2
\end{aligned}
\qquad
\begin{aligned}
(g\circ f)(x) &= g(f(x)) \\
&= g\left(\frac{x+2}{x-1}\right) \\
&= \frac{\frac{x+2}{x-1}}{\frac{x+2}{x-1}-1} \\
&= \frac{\frac{x+2}{x-1}}{\frac{x+2-x+1}{x-1}} \\
&= \frac{x+2}{3}
\end{aligned}
$$

c)

$$
\begin{aligned}
(f\circ g)(x) &= f(g(x)) \\
&= f\left(\frac{1}{1+x^2}\right) \\
&= \sqrt{\frac{1}{1+x^2}}
\end{aligned}
\qquad
\begin{aligned}
(g\circ f)(x) &= g(f(x)) \\
&= g\left(\sqrt{x}\right) \\
&= \frac{1}{1+\left(\sqrt{x}\right)^2} \\
&= \frac{1}{1+x}
\end{aligned}
$$

Napomena: $\sqrt{x^2}=|x|, \left(\sqrt{x}\right)^2=x$

5.2.1 Elementarne funkcije

Polinomi

Definicija 5.12. Neka su $a_0, a_1, a_2, \cdots, a_n \in \mathbb{R}$ i $a_n \neq 0$. Funkciju $f:\mathbb{R}\to\mathbb{R}$ oblika $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ nazivamo **polinomom** n-tog stupnja.

Svaki polinom $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ može se zapisati u obliku $f(x)=a_n(x-x_1)(x-x_2)\cdots(x-x_n)$ gdje su $x_1,x_2,...,x_n\in\mathbb{C}$ **nultočke** polinoma.

Ako je $\alpha+\beta i$ nultočka polinoma f, onda je i $\alpha-\beta i$ također nultočka tog polinoma, pa imamo:

$$
\begin{aligned}
(x-(\alpha+\beta i))\,(x-(\alpha-\beta i)) &= (x-\alpha-\beta i)\,(x-\alpha+\beta i) \\
&= (x-\alpha)^2-(\beta i)^2 \qquad , \\
&= x^2-2x\alpha+\alpha^2+\beta^2
\end{aligned}
$$

odnosno, polinom možemo rastaviti na linearne i kvadratne članove s realnim koeficijentima.

Racionalna funkcija

Definicija 5.13. Neka su $P_m(x)$ i $Q_n(x)$ polinomi m-tog, odnosno n-tog stupnja. Funkciju oblika $f(x) = \dfrac{P_m(x)}{Q_n(x)}$ nazivamo **racionalna funkcija**.

Ako je $m < n$ kažemo da je racionalna funkcija **prava**. Nepravu racionalnu funkciju $f(x) = \dfrac{P_m(x)}{Q_n(x)}, m \geq n$ možemo zapisati kao $f(x) = R_{m-n}(x) + \dfrac{S_t(x)}{Q_n(x)}, t < n$

Rastav funkcije $f(x) = \dfrac{P_m(x)}{Q_n(x)}$ na parcijalne razlomke:

$Q_n(x)$ faktoriziramo na linearne i kvadratne faktore:

- Ako se u faktorizaciji pojavi $(x - x_0)^s$, onda u rastavu na parcijalne razlomke imamo
$$\frac{A_1}{x - x_0} + \frac{A_2}{(x - x_0)^2} + \cdots + \frac{A_s}{(x - x_0)^s}$$
- Ako se u faktorizaciji pojavi $(x^2 + ax + b)^t$, onda u rastavu na parcijalne razlomke imamo
$$\frac{A_1 x + B_1}{x^2 + ax + b} + \frac{A_2 x + B_2}{(x^2 + ax + b)^2} + \cdots + \frac{A_t x + B_t}{(x^2 + ax + b)^t}$$

Zadatak 5.3. Rastavite na parcijalne razlomke racionalnu funkciju

a) $Q(x) = \dfrac{1}{x^4 - 16}$

b) $Q(x) = \dfrac{2x^2 - 3x + 5}{(x+2)(x-1)(x-3)}$

c) $Q(x) = \dfrac{1}{(x-2)^2(x^2+1)}$

.

Rješenje: a)

$$\frac{1}{x^4 - 16} = \frac{1}{(x^2 - 4)(x^2 + 4)} = \frac{1}{(x-2)(x+2)(x^2+4)}$$

$$\frac{1}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx + D}{x^2 + 4} \quad / \cdot (x-2)(x+2)(x^2+4)$$

$$\begin{aligned}
1 &= A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x+2)(x-2)\\
1 &= A(x^3+4x+2x^2+8)+B(x^3+4x-2x^2-8)+(Cx+D)(x^2-4)\\
1 &= Ax^3+4Ax+2Ax^2+8A+Bx^3+4Bx-2Bx^2-8B+Cx^3-4Cx+Dx^2-4D\\
1 &= x^3(A+B+C)+x^2(2A-2B+D)+x(4A+4B-4C)+(8A-8B-4D)
\end{aligned}$$

Dva su polinoma jednaka ako su koeficijenti uz odgovarajuće potencije jednaki.

$$\begin{aligned}
A+B+C &= 0\\
2A-2B+D &= 0\\
4A+4B-4C &= 0\\
8A-8B-4D &= 1
\end{aligned}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0\\ 2 & -2 & 0 & 1 & 0\\ 4 & 4 & -4 & 0 & 0\\ 8 & -8 & 0 & -4 & 1 \end{array}\right] /:4 \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0\\ 2 & -2 & 0 & 1 & 0\\ 1 & 1 & -1 & 0 & 0\\ 8 & -8 & 0 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \mathrm{II}-2\mathrm{I}\\ \mathrm{III}-\mathrm{I}\\ \mathrm{IV}-8\mathrm{I} \end{array} \sim$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0\\ 0 & -4 & -2 & 1 & 0\\ 0 & 0 & -2 & 0 & 0\\ 0 & -16 & -8 & -4 & 1 \end{array}\right] \begin{array}{l} \mathrm{II}-\frac{1}{4}\mathrm{IV}\\ :/(-2) \end{array} \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & -\frac{1}{4}\\ 0 & 0 & 1 & 0 & 0\\ 0 & -16 & -8 & -4 & 1 \end{array}\right] \mathrm{IV}\leftrightarrow\mathrm{II} \sim$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0\\ 0 & -16 & -8 & -4 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & -\frac{1}{4} \end{array}\right] \begin{array}{l} \mathrm{II}+8\mathrm{III}\\ \mathrm{II}+2\mathrm{IV} \end{array} \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0\\ 0 & -16 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & -\frac{1}{4} \end{array}\right]$$

$D=-\dfrac{1}{8}$, $C=0$, $B=-\dfrac{1}{32}$

$A+B+C=0$

$A=\dfrac{1}{32}$

$$\frac{1}{(x-2)(x+2)(x^2+4)} = \frac{\frac{1}{32}}{x-2}-\frac{\frac{1}{32}}{x+2}-\frac{\frac{1}{8}}{x^2+4}$$

b)

$$\frac{2x^2-3x+5}{(x+2)(x-1)(x-3)} = \frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{x-3} \quad /\cdot(x+2)(x-1)(x-3)$$

$$\begin{aligned}
2x^2-3x+5 &= A(x-1)(x-3)+B(x+2)(x-3)+C(x+2)(x-1)\\
2x^2-3x+5 &= A(x^2-4x+3)+B(x^2-x-6)+C(x^2+x-2)\\
2x^2-3x+5 &= Ax^2-4Ax+3A+Bx^2-Bx-6B+Cx^2+Cx-2C\\
2x^2-3x+5 &= x^2(A+B+C)+x(-4A-B+C)+(3A-6B-2C)
\end{aligned}$$

$$\begin{aligned}
A+B+C &= 2\\
-4A-B+C &= -3\\
3A-6B-2C &= 5
\end{aligned}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 2\\ -4 & -1 & 1 & -3\\ 3 & -6 & -2 & 5\end{array}\right]\begin{array}{l}\\ \text{II}+4\text{I}\\ \text{III}-3\text{I}\end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2\\ 0 & 3 & 5 & 5\\ 0 & -9 & -5 & -1\end{array}\right]\begin{array}{l}\\ \\ \text{III}+3\text{II}\end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2\\ 0 & 3 & 5 & 5\\ 0 & 0 & 10 & 14\end{array}\right]$$

$$C=\frac{7}{5}$$

$$\begin{aligned}
3B+5C &= 5\\
3B &= 5-5\cdot\frac{7}{5}\\
3B &= -2\\
B &= -\frac{2}{3}
\end{aligned}
\qquad
\begin{aligned}
A+B+C &= 2\\
A &= 2+\frac{2}{3}-\frac{7}{5}\\
A &= \frac{19}{15}
\end{aligned}$$

c)

$$\frac{1}{(x-2)^2(x^2+1)} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+1} \quad / \cdot (x-2)^2(x^2+1)$$

$$\begin{aligned}
1 &= A(x-2)(x^2+1)+B(x^2+1)+(Cx+D)(x-2)^2\\
1 &= A(x^3-2x^2+x-2)+Bx^2+B+(Cx+D)(x^2-4x+4)\\
1 &= Ax^3-2Ax^2+Ax-2A+Bx^2+B+Cx^3+Dx^2-4Cx^2-4Dx+4Cx+4D\\
1 &= x^3(A+C)+x^2(-2A+B-4C+D)+x(A+4C-4D)+(-2A+B+4D)
\end{aligned}$$

$$\begin{aligned}
A+C &= 0\\
-2A+B-4C+D &= 0\\
A+4C-4D &= 0\\
-2A+B+4D &= 1
\end{aligned}$$

$$\left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0\\ -2 & 1 & -4 & 1 & 0\\ 1 & 0 & 4 & -4 & 0\\ -2 & 1 & 0 & 4 & 1\end{array}\right]\begin{array}{l}\\ \text{II}+2\text{I}\\ \text{III}-\text{I}\\ \text{IV}+2\text{I}\end{array} \sim \left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 1 & 0\\ 0 & 0 & 3 & -4 & 0\\ 0 & 1 & 2 & 4 & 1\end{array}\right]\begin{array}{l}\\ \\ \\ \text{IV}-\text{II}\end{array} \sim$$

$$\left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & 3 & 1 \end{array}\right] \begin{array}{c} \\ \\ \\ \mathrm{IV}-\frac{4}{3}\mathrm{III} \end{array} \sim \left[\begin{array}{cccc:c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & \frac{25}{3} & 1 \end{array}\right]$$

$$\begin{aligned} D &= \frac{3}{25} \\ 3C-4D &= 0 \\ 3C &= 4\cdot\frac{3}{25} \\ C &= \frac{4}{25} \end{aligned} \qquad \begin{aligned} B-2C+D &= 0 \\ B &= 2C-D \\ B &= 2\cdot\frac{4}{25}-\frac{3}{25} \\ B &= \frac{1}{5} \end{aligned} \qquad \begin{aligned} A+C &= 0 \\ A &= -C \\ A &= -\frac{4}{25} \end{aligned}$$

Eksponencijalna funkcija

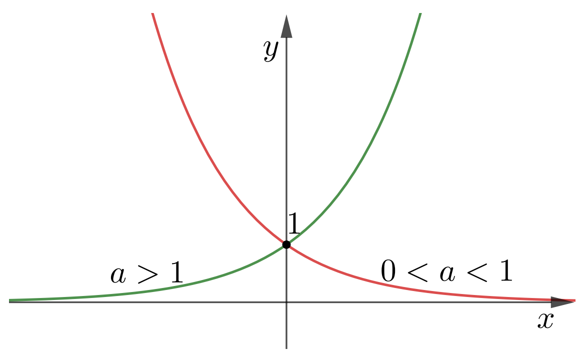
Definicija 5.14. Neka je $a \in \mathbb{R}, a \neq 1, a > 0$. Funkcija $f : \mathbb{R} \to \mathbb{R}^+$, definirana sa $f(x) = a^x$ naziva se **eksponencijalna** funkcija.

$a^0 = 1, \forall a \in \mathbb{R}$

$a^{x_1+x_2} = a^{x_1} \cdot a^{x_2}$

$a^{x_1-x_2} = \frac{a^{x_1}}{a^{x_2}}$

$a^{x_1 \cdot x_2} = (a^{x_1})^{x_2}$



Logaritamska funkcija

Definicija 5.15. Neka je $a \in \mathbb{R}, a \neq 1, a > 0$. Funkcija $f : \mathbb{R}^+ \to \mathbb{R}$, definirana sa $f(x) = \log_a x$ naziva se **logaritamska** funkcija.

$\log_a x = b \Longleftrightarrow a^b = x$

$\log_a(x_1 \cdot x_2) = \log_a x_1 + \log_a x_2$

$\log_a \left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2$

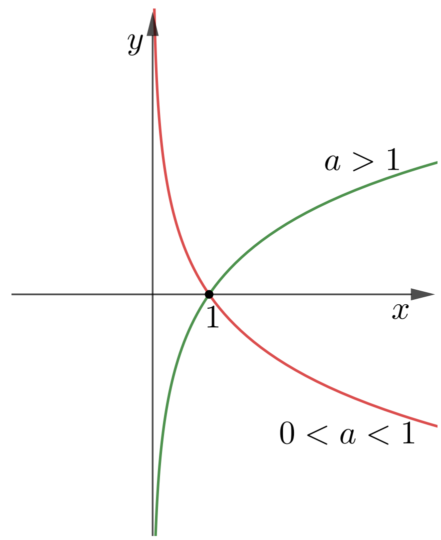
$\log_a x = \frac{\log_b x}{\log_b a}$

$\log_a b = \frac{1}{\log_b a}$

$a^{\log_a x} = \log_a a^x = x$

$\log_a x^n = n \cdot \log_a x, \forall n > 0$

$\log_e x = \ln x, e \approx 2.71$



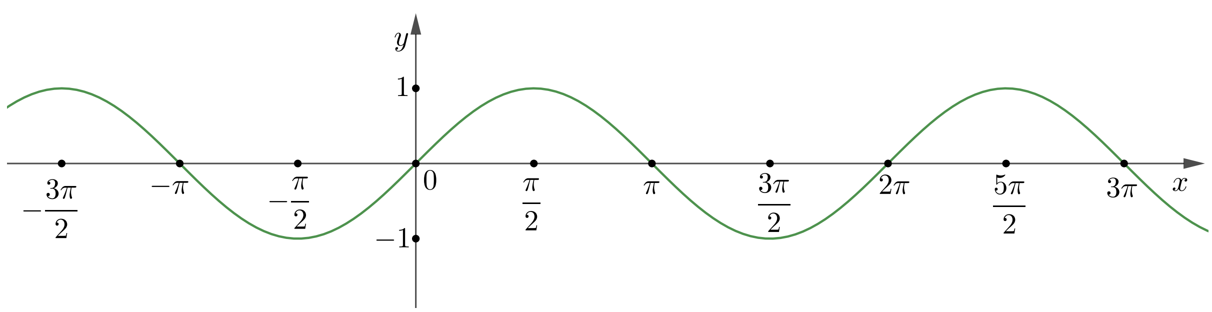
Trigonometrijske funkcije

Sinus

$\sin : \mathbb{R} \to [-1, 1]\,, f(x) = \sin x$

Funkcija $f(x) = \sin x$ je

- **periodična**: $\sin x = \sin(x + 2k\pi), k \in \mathbb{Z}$
- **neparna**: $\sin(-x) = -\sin x$

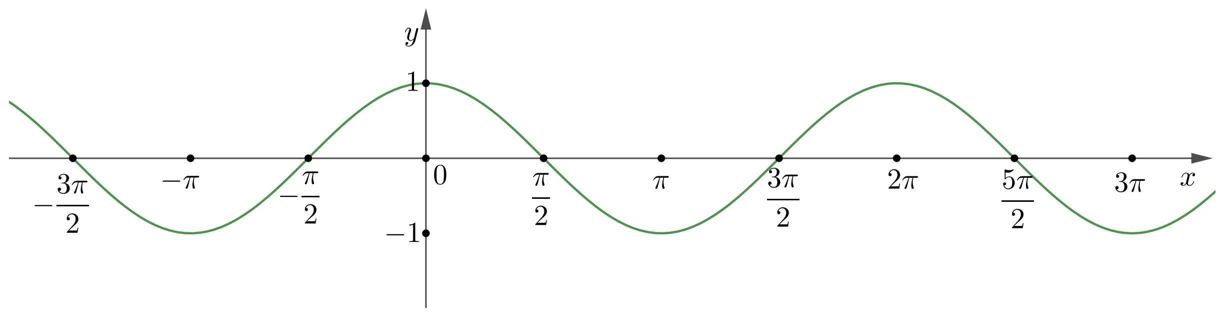


Kosinus

$\cos : \mathbb{R} \to [-1, 1]\,, f(x) = \cos x$

Funkcija $f(x) = \cos x$ je

- **periodična**: $\cos x = \cos(x + 2k\pi), k \in \mathbb{Z}$
- **parna**: $\cos(-x) = \cos x$

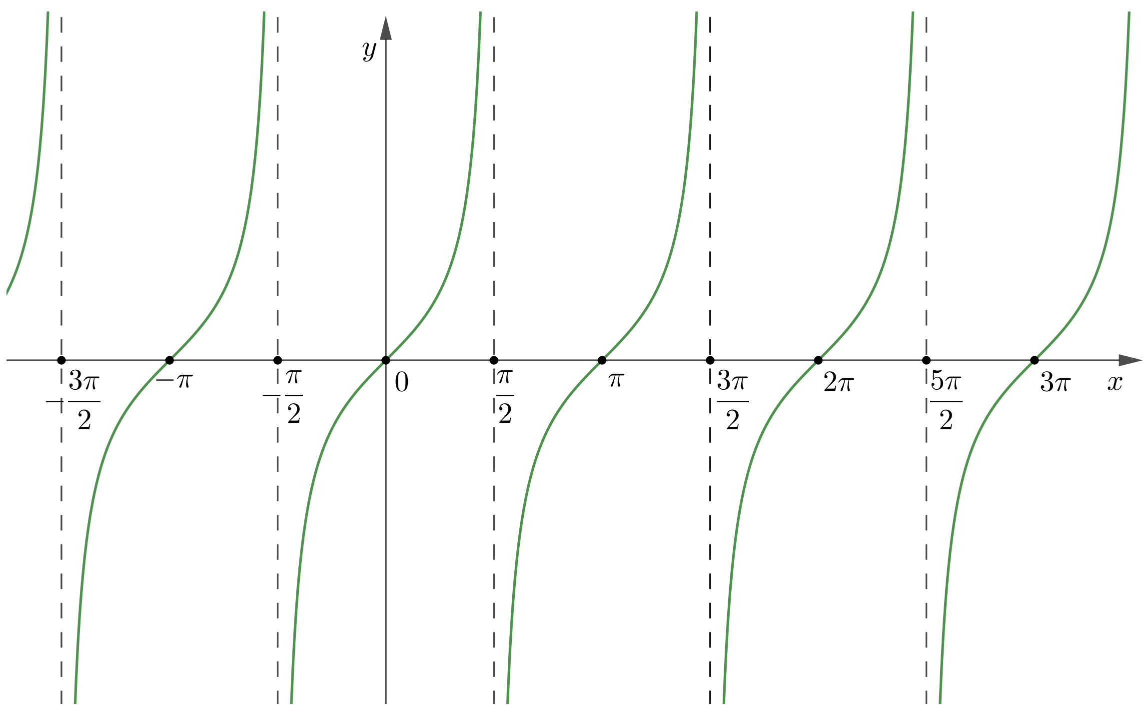


Tangens

tg: $\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\} \to \mathbb{R}, f(x) = \operatorname{tg} x$

Funkcija $f(x) = \operatorname{tg} x$ je

- **periodična**: $\operatorname{tg} x = \operatorname{tg}(x + k\pi), k \in \mathbb{Z}$
- **neparna**: $\operatorname{tg}(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x} = -\operatorname{tg} x$



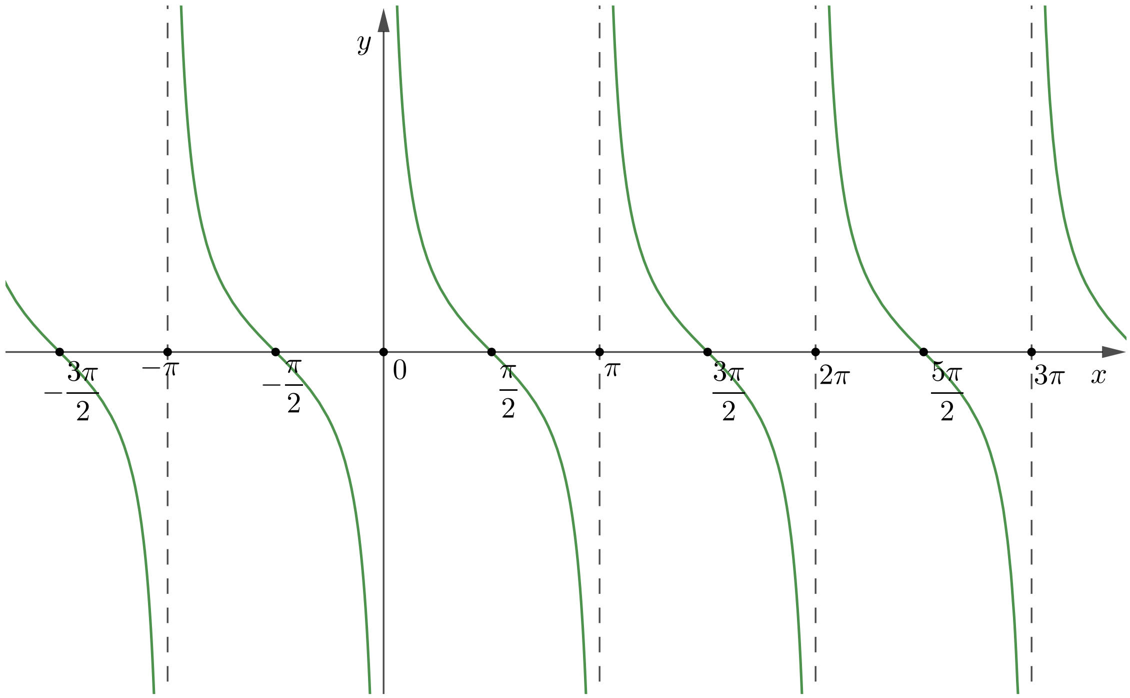
Kotangens

ctg: $\mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\} \to \mathbb{R}, f(x) = \operatorname{ctg} x$

Funkcija $f(x) = \operatorname{ctg} x$ je

- **periodična**: $\operatorname{ctg} x = \operatorname{ctg}(x + k\pi), k \in \mathbb{Z}$

- **neparna**: $\operatorname{ctg}(-x) = \dfrac{\cos(-x)}{\sin(-x)} = \dfrac{\cos x}{-\sin x} = -\operatorname{ctg} x$



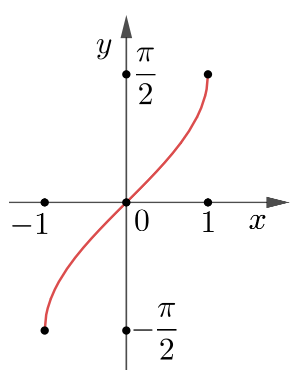
Arkus funkcije

Arkus Sinus

Funkcija $\sin : \mathbb{R} \to [-1,1]\,, f(x) = \sin x$ nije injekcija, pa nije ni bijekcija, no restrikcija $\sin\Big|_{[-\frac{\pi}{2},\frac{\pi}{2}]} \to [-1,1]$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju $\arcsin : [-1,1] \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$\arcsin(\sin x) = x, \forall x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\sin(\arcsin x) = x, \forall x \in [-1,1]$

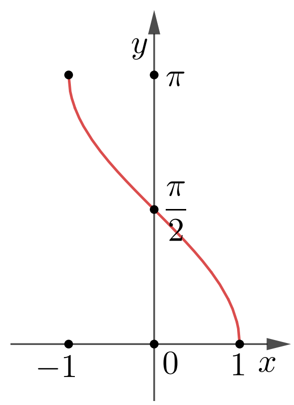


Arkus Kosinus

Funkcija $\cos:\mathbb{R}\to[-1,1]$ nije injekcija, pa nije ni bijekcija, no restrikcija $\cos\Big|_{[0,\pi]}\to[-1,1]$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju $\arccos:[-1,1]\to[0,\pi]$.

$\arccos(\cos x)=x,\forall x\in[0,\pi]$

$\cos(\arccos x)=x,\forall x\in[-1,1]$

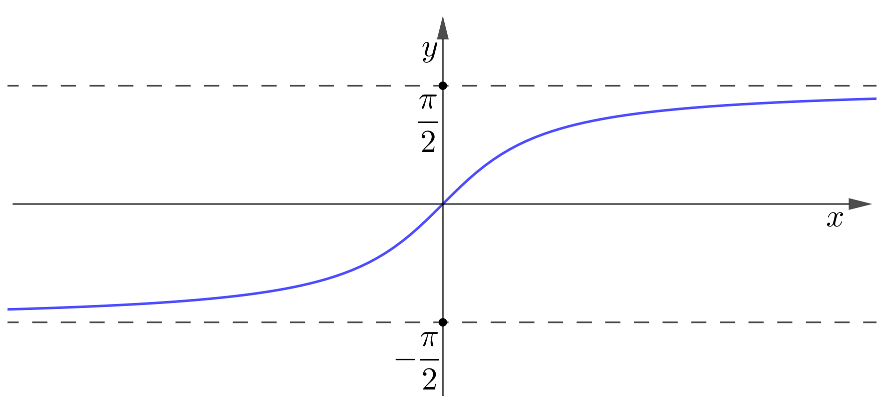


Arkus Tangens

Funkcija tg: $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi,k\in\mathbb{Z}\right\}\to\mathbb{R}$ nije injekcija, pa nije ni bijekcija, no restrikcija $\mathrm{tg}\Big|_{\left\langle-\frac{\pi}{2},\frac{\pi}{2}\right\rangle}\to\mathbb{R}$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju arctg: $\mathbb{R}\to\left\langle-\frac{\pi}{2},\frac{\pi}{2}\right\rangle$.

$\mathrm{arctg}(\mathrm{tg}\,x)=x,\forall x\in\left\langle-\frac{\pi}{2},\frac{\pi}{2}\right\rangle$

$\mathrm{tg}(\mathrm{arctg}\,x)=x,\forall x\in\mathbb{R}$

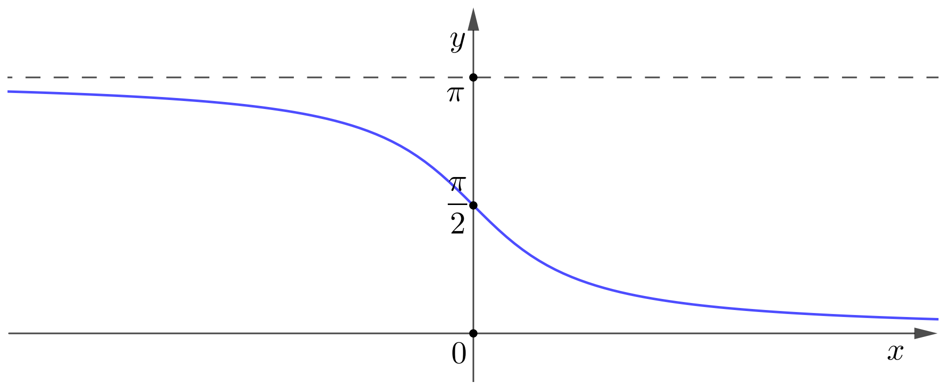


Arkus Kotangens

Funkcija ctg: $\mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\} \to \mathbb{R}$ nije injekcija, pa nije ni bijekcija, no restrikcija $\operatorname{ctg}\Big|_{\langle 0,\pi\rangle} \to \mathbb{R}$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju arcctg: $\mathbb{R} \to \langle 0, \pi\rangle$.

$\operatorname{arcctg}(\operatorname{ctg} x) = x, \forall x \in \langle 0, \pi\rangle$

$\operatorname{ctg}(\operatorname{arcctg} x) = x, \forall x \in \mathbb{R}$



Hiperbolne funkcije

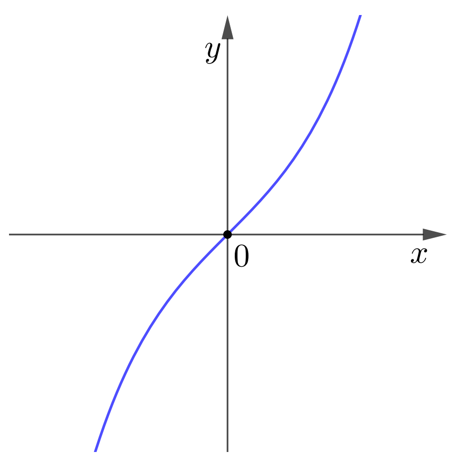
Sinus hiperbolni

Funkcija sh: $\mathbb{R} \to \mathbb{R}$ definirana sa

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

naziva se sinus hiperbolni.

Funkcija $f(x) = \operatorname{sh} x$ je **neparna**: $\operatorname{sh}(-x) = \dfrac{e^{-x} - e^x}{2} = -\dfrac{e^x - e^{-x}}{2} = -\operatorname{sh} x$



Inverz funkcije:

$$\begin{aligned} f(x) &= \frac{e^x - e^{-x}}{2} \\ y &= \frac{e^x - e^{-x}}{2} \quad / \cdot 2 \\ 2y &= e^x - e^{-x} \quad / \cdot e^x \\ 0 &= e^{2x} - 2ye^x - 1, \qquad t = e^x \\ 0 &= t^2 - 2yt - 1 \end{aligned}$$

$$\begin{aligned} t_{1,2} &= \frac{2y \pm \sqrt{(2y)^2 - 4 \cdot 1 \cdot (-1)}}{2} \\ t_{1,2} &= \frac{2y \pm \sqrt{4y^2 + 4}}{2} \\ t_{1,2} &= \frac{2y \pm \sqrt{4\,(y^2 + 1)}}{2} \\ t_{1,2} &= \frac{2y \pm 2\sqrt{y^2 + 1}}{2} \end{aligned}$$

$e^x = y + \sqrt{y^2+1}$ ili $e^x = y - \sqrt{y^2+1}$
$e^x = y - \sqrt{y^2+1}$ odbacujemo jer je $e^x > 0$

$e^x = y + \sqrt{y^2+1} \quad / \ln$
Arsh: $\mathbb{R} \to \mathbb{R}, \quad \operatorname{Arsh} y = \ln\left(y + \sqrt{y^2+1}\right)$

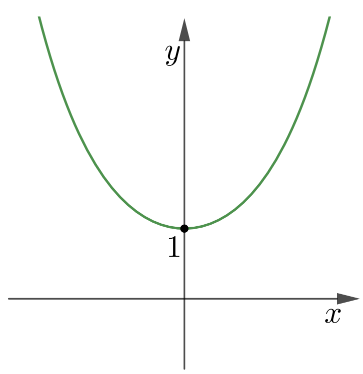
Kosinus hiperbolni

Funkcija ch: $\mathbb{R} \to [1, \infty\rangle$ definirana sa

$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

naziva se kosinus hiperbolni.

Funkcija $f(x) = \operatorname{ch} x$ je **parna**: $\operatorname{ch}(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \operatorname{ch} x$



Inverz funkcije:

$$\begin{aligned} f(x) &= \frac{e^x + e^{-x}}{2} \\ y &= \frac{e^x + e^{-x}}{2} \quad / \cdot 2 \\ 2y &= e^x + e^{-x} \quad / \cdot e^x \\ 0 &= e^{2x} - 2ye^x + 1, \qquad t = e^x \\ 0 &= t^2 - 2yt + 1 \end{aligned}$$

$$\begin{aligned} t_{1,2} &= \frac{2y \pm \sqrt{(2y)^2 - 4 \cdot 1 \cdot 1}}{2} \\ t_{1,2} &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\ t_{1,2} &= \frac{2y \pm \sqrt{4\,(y^2 - 1)}}{2} \\ t_{1,2} &= \frac{2y \pm 2\sqrt{y^2 - 1}}{2} \end{aligned}$$

$e^x = y + \sqrt{y^2 - 1}$ ili $e^x = y - \sqrt{y^2 - 1}$

$e^x = y - \sqrt{y^2 - 1}$ odbacujemo jer je $e^x > 0$

$e^x = y + \sqrt{y^2 - 1} \quad / \ln$

Arch: $[1, \infty\rangle \to \mathbb{R}$, $\operatorname{Arch} y = \ln\left(y + \sqrt{y^2 - 1}\right)$

Vrijedi: $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$

$$\begin{aligned} \operatorname{ch}^2 x - \operatorname{sh}^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\ &= 1 \end{aligned}$$

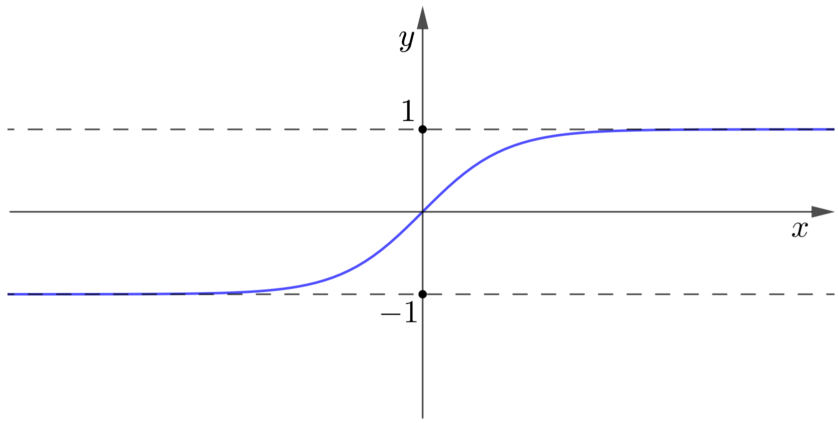
Tangens hiperbolni

Funkcija th: $\mathbb{R} \to \langle -1, 1 \rangle$ definirana sa

$$\operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

naziva se tangens hiperbolni.

Funkcija $f(x) = \operatorname{th} x$ je **neparna**: $\operatorname{th}(-x) = \dfrac{\operatorname{sh}(-x)}{\operatorname{ch}(-x)} = \dfrac{-\operatorname{sh} x}{\operatorname{ch} x} = -\operatorname{th} x$



Inverz funkcije:

$$\begin{aligned} f(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad / \cdot (e^x + e^{-x}) \\ ye^x + ye^{-x} &= e^x - e^{-x} \quad / \cdot e^x \\ ye^{2x} + y &= e^{2x} - 1, \end{aligned}$$

$$\begin{aligned} 1 + y &= e^{2x}(1 - y) \\ e^{2x} &= \frac{1+y}{1-y} \quad / \ln \\ 2x &= \ln\left(\frac{1+y}{1-y}\right) \\ \mathrm{Arth}(y) &= \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \end{aligned}$$

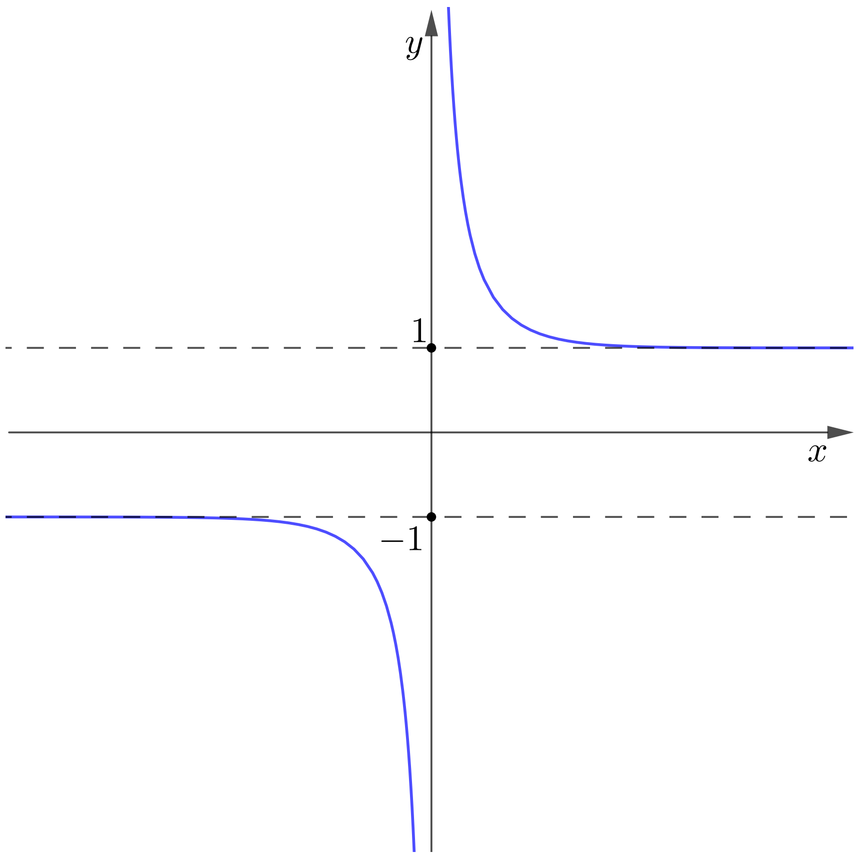
Kotangens hiperbolni

Funkcija cth: $\mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus [-1, 1]$ definirana sa

$$\operatorname{cth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

naziva se kotangens hiperbolni.

Funkcija $f(x) = \operatorname{cth} x$ je **neparna**: $\operatorname{cth}(-x) = \dfrac{\operatorname{ch}(-x)}{\operatorname{sh}(-x)} = \dfrac{\operatorname{ch} x}{-\operatorname{sh} x} = -\operatorname{cth} x$



Inverz funkcije:

$$\begin{aligned} f(x) &= \frac{e^x+e^{-x}}{e^x-e^{-x}} \\ y &= \frac{e^x+e^{-x}}{e^x-e^{-x}} \quad / \cdot (e^x-e^{-x}) \\ ye^x-ye^{-x} &= e^x+e^{-x} \quad / \cdot e^x \\ ye^{2x}-y &= e^{2x}+1, \end{aligned}$$

$$\begin{aligned} e^{2x}(y-1) &= y+1 \\ e^{2x} &= \frac{y+1}{y-1} \quad / \ln \\ 2x &= \ln\left(\frac{y+1}{y-1}\right) \\ \mathrm{Arcth}(y) &= \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right) \end{aligned}$$

Zadatak 5.4. Odredite prirodnu domenu funkcije:

a) $f(x) = e^{\frac{1}{x^2-4}} + \frac{1}{x}$

b) $f(x) = \mathrm{arctg}\frac{1}{x-3}$

c) $f(x) = \frac{\sqrt{x-2}}{\ln(x+1)}$

d) $f(x) = \sqrt{1-10^x}$

e) $f(x) = \arcsin\sqrt{\ln x}$

f) $f(x) = \ln(2\sin x - 1)$

g) $f(x) = \ln\sqrt{1-x} + \sqrt{\ln(1-x)}$

h) $f(x) = \frac{1}{4^x - 3\cdot 2^x + 2}$

i) $f(x) = \ln\frac{x+3}{2-x}$

j) $f(x) = e^{\frac{1}{x}} + \dfrac{\sqrt{x^2-1}}{x+2}$

k) $f(x) = \arccos\dfrac{2x}{1+x}$

Rješenje: a) $f(x) = e^{\frac{1}{x^2-4}} + \dfrac{1}{x}$

$x \neq 0 \qquad x^2 - 4 \neq 0$

$x^2 \neq 4$

$x \neq 2$ i $x \neq -2$

$D_f = \mathbb{R} \setminus \{-2, 0, 2\}$

b) $f(x) = \text{arctg}\dfrac{1}{x-3}$

$D_{\text{arctg}} = \mathbb{R} \qquad x - 3 \neq 0$

$x \neq 3$

$D_f = \mathbb{R} \setminus \{3\}$

c) $f(x) = \dfrac{\sqrt{x-2}}{\ln(x+1)}$

$x - 2 \geq 0 \qquad x + 1 > 0 \qquad \ln(x+1) \neq 0$

$x \geq 2 \qquad x > -1 \qquad x + 1 \neq e^0$

$x + 1 \neq 1$

$x \neq 0$

$D_f = [2, \infty\rangle$

d) $f(x) = \sqrt{1 - 10^x}$

$1 - 10^x \geq 0$

1. način:	2. način
$1 \geq 10^x \quad / \log$	$10^0 \geq 10^x$
$\log 1 \geq x$	$0 \geq x$
$0 \geq x$	

$D_f = \langle -\infty, 0]$

e) $f(x) = \arcsin\sqrt{\ln x}$

$$x > 0 \qquad \ln x \geq 0 \qquad -1 \leq \sqrt{\ln x} \leq 1$$

$$x \geq e^0 \qquad \sqrt{\ln x} \leq 1 \quad /^2$$

$$x \geq 1 \qquad \ln x \leq 1$$

$$x \leq e^1 = e$$

$$D_f = [1, e]$$

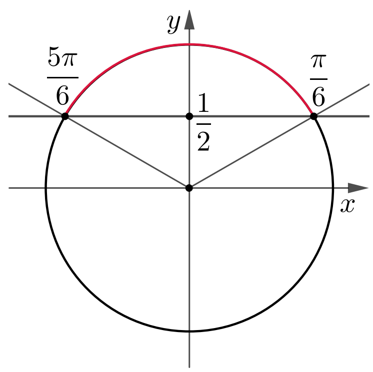
f) $f(x) = \ln(2\sin x - 1)$

$$2\sin x - 1 > 0$$

$$2\sin x > 1$$

$$\sin x > \frac{1}{2}$$

$$D_f = \bigcup_{k\in\mathbb{Z}} \left\langle \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \right\rangle$$



g) $f(x) = \ln\sqrt{1-x} + \sqrt{\ln(1-x)}$

$$1 - x > 0 \qquad \ln(1-x) \geq 0$$

$$x < 1 \qquad 1 - x \geq 1$$

$$x \leq 0$$

$$D_f = \langle -\infty, 0]$$

h) $f(x) = \dfrac{1}{4^x - 3\cdot 2^x + 2}$

$$4^x - 3\cdot 2^x + 2 \neq 0 \qquad t = 2^x$$

$$t^2 - 3t + 2 = 0$$

$$2^x \neq 1 \qquad 2^x \neq 2 \qquad t_{1,2} = \frac{3 \pm \sqrt{3^2 - 4\cdot 2}}{2}$$

$$2^x \neq 2^0 \qquad 2^x \neq 2^1 \qquad t_{1,2} = \frac{3 \pm \sqrt{1}}{2}$$

$$x \neq 0 \qquad x \neq 1 \qquad t_1 = 1, \quad t_2 = 2$$

$$D_f = \mathbb{R} \setminus \{0, 1\}$$

i) $f(x) = \ln\dfrac{x+3}{2-x}$

$$\frac{x+3}{2-x} > 0$$

$$D_f = \langle -3, 2\rangle$$

	$-\infty$ to -3	-3 to 2	2 to $+\infty$
$x+3$	$-$	$+$	$+$
$2-x$	$+$	$+$	$-$
$\frac{x+3}{2-x}$	$-$	$+$	$-$

j) $f(x) = e^{\frac{1}{x}} + \dfrac{\sqrt{x^2-1}}{x+2}$

$x \neq 0$ $\qquad x^2 - 1 \geq 0$ $\qquad x+2 \neq 0$

$x^2 \geq 1$ $\qquad x \neq -2$

$x \leq -1$ ili $x \geq 1$

$$D_f = \langle -\infty, -1] \bigcup [1, \infty\rangle \setminus \{-2\} = \langle -\infty, -2\rangle \bigcup \langle -2, -1] \bigcup [1, \infty\rangle$$

k) $f(x) = \arccos \dfrac{2x}{1+x}$

$$-1 \leq \frac{2x}{1+x} \leq 1$$

$$-1 \leq \frac{2x}{1+x} \qquad\qquad \frac{2x}{1+x} \leq 1$$

$$0 \leq \frac{2x}{1+x} + 1 \qquad\qquad \frac{2x}{1+x} - 1 \leq 0$$

$$0 \leq \frac{2x+1+x}{1+x} \qquad\qquad \frac{2x-1-x}{1+x} \leq 0$$

$$0 \leq \frac{3x+1}{1+x} \qquad\qquad \frac{x-1}{1+x} \leq 0$$

	$-\infty$ to -1	-1 to $-\frac{1}{3}$	$-\frac{1}{3}$ to $+\infty$
$3x+1$	$-$	$-$	$+$
$1+x$	$-$	$+$	$+$
$\frac{3x+1}{1+x}$	$+$	$-$	$+$

	$-\infty$ to -1	-1 to 1	1 to $+\infty$
$x-1$	$-$	$-$	$+$
$1+x$	$-$	$+$	$+$
$\frac{x-1}{1+x}$	$+$	$-$	$+$

$$x \in \langle -\infty, -1\rangle \bigcup \left[-\frac{1}{3}, \infty\right\rangle \qquad\qquad x \in \langle -1, 1]$$

$$D_f = \left(\langle -\infty, -1\rangle \bigcup \left[-\frac{1}{3}, \infty\right\rangle\right) \bigcap \langle -1, 1]$$

$$D_f = \left[-\frac{1}{3}, 1\right]$$

Zadatak 5.5. Odredite inverz i sliku funkcije

a) $f(x) = \ln(2\sin x - 1)$

b) $f(x) = \text{arctg}\sqrt{e^x + 1}$

c) $f(x) = 3 \cdot 2^{1-x} + 1$

d) $f(x) = 2^{x^3}$

e) $f(x) = (x-1)^3$

Rješenje: a) $f(x) = \ln(2\sin x - 1)$

$$\begin{aligned}
f(x) &= \ln(2\sin x - 1) \\
y &= \ln(2\sin x - 1) \quad /e^{\cdot} \\
e^y &= 2\sin x - 1 \\
\sin x &= \frac{e^y + 1}{2} \quad /\arcsin \\
x &= \arcsin\frac{e^y + 1}{2} \\
f^{-1}(y) &= \arcsin\frac{e^y + 1}{2}
\end{aligned}$$

$$\text{Im} = D_{f^{-1}}$$

$$\begin{aligned}
-1 \leq \frac{e^x + 1}{2} &\leq 1 \\
\frac{e^x + 1}{2} &\leq 1 \\
e^x + 1 &\leq 2 \\
e^x &\leq 1 \\
x &\leq 0
\end{aligned}$$

$$\text{Im} = D_{f^{-1}} = \langle -\infty, 0]$$

b) $f(x) = \text{arctg}\sqrt{e^x + 1}$

$$\begin{aligned}
f(x) &= \text{arctg}\sqrt{e^x + 1} \\
y &= \text{arctg}\sqrt{e^x + 1} \quad /\text{tg} \\
\text{tg}\, y &= \sqrt{e^x + 1} \quad /^2 \\
\text{tg}^2 y &= e^x + 1 \\
e^x &= \text{tg}^2 y - 1 \quad /\ln \\
x &= \ln\left(\text{tg}^2 y - 1\right) \\
f^{-1}(x) &= \ln\left(\text{tg}^2 x - 1\right)
\end{aligned}$$

$$\text{Im} f = D_{f^{-1}} \subseteq \left[0, \frac{\pi}{2}\right\rangle$$

$$\begin{aligned}
\text{tg}^2 x - 1 &> 0 \\
\text{tg}^2 x &> 1
\end{aligned}$$

$$\text{tg}\, x \leq -1 \text{ ili } \text{tg}\, x > 1$$

$$x \in \left\langle -\frac{\pi}{2}, -\frac{\pi}{4}\right\rangle \bigcup \left\langle \frac{\pi}{4}, \frac{\pi}{2}\right\rangle$$

$$\text{Im} f = \left(\left\langle -\frac{\pi}{2}, -\frac{\pi}{4}\right\rangle \bigcup \left\langle \frac{\pi}{4}, \frac{\pi}{2}\right\rangle\right) \bigcap \left[0, \frac{\pi}{2}\right\rangle$$

$$\text{Im} f = \left\langle \frac{\pi}{4}, \frac{\pi}{2}\right\rangle$$

c) $f(x) = 3 \cdot 2^{1-x} + 1$

$$\begin{aligned}
f(x) &= 3\cdot 2^{1-x}+1 \\
y &= 3\cdot 2^{1-x}+1 \\
3\cdot 2^{1-x} &= y-1 \\
2^{1-x} &= \frac{y-1}{3} \quad /\log_2 \\
1-x &= \log_2\frac{y-1}{3} \\
x &= 1-\log_2\frac{y-1}{3} \\
f^{-1}(x) &= 1-\log_2\frac{x-1}{3}
\end{aligned}$$

$$\mathrm{Im} f = D_{f^{-1}}$$

$$\begin{aligned}
\frac{x-1}{3} &> 0 \\
x-1 &> 0 \\
x &> 1
\end{aligned}$$

$$\mathrm{Im} f = D_{f^{-1}} = \langle 1,\infty\rangle$$

d) $f(x)=2^{x^3}$

$$\begin{aligned}
f(x) &= 2^{x^3} \\
y &= 2^{x^3} \quad /\log_2 \\
\log_2 y &= x^3 \quad /\sqrt[3]{} \\
x &= \sqrt[3]{\log_2 y} \\
f^{-1}(x) &= \sqrt[3]{\log_2 x}
\end{aligned}$$

$$\mathrm{Im} f = D_{f^{-1}}$$

$$x > 0$$

$$\mathrm{Im} f = D_{f^{-1}} = \langle 0,\infty\rangle$$

e) $f(x)=(x-1)^3$

$$\begin{aligned}
f(x) &= (x-1)^3 \\
y &= (x-1)^3 \quad /\sqrt[3]{} \\
x-1 &= \sqrt[3]{y} \\
x &= \sqrt[3]{y}+1 \\
f^{-1}(x) &= \sqrt[3]{x}+1
\end{aligned}$$

$$\mathrm{Im} f = D_{f^{-1}} = \mathbb{R}$$

Zadatak 5.6. Ispitajte (ne)parnost sljedećih funkcija:

a) $f(x)=\sqrt{x^2-4}$

b) $f(x)=3^{\frac{1}{x}}$

c) $f(x)=\frac{1}{x}$

d) $f(x)=\frac{1}{x^2}$

e) $f(x)=\frac{1}{x-1}$

Rješenje:

$$\begin{aligned}
\text{a) } f(-x) &= \sqrt{(-x)^2-4}\\
&= \sqrt{x^2-4}\\
&= f(x) \Longrightarrow f \text{ je parna.}\\
\text{b) } f(-x) &= 3^{\frac{1}{-x}}\\
&= 3^{-\frac{1}{x}}
\end{aligned}$$

$\Longrightarrow f$ nije ni parna ni neparna.

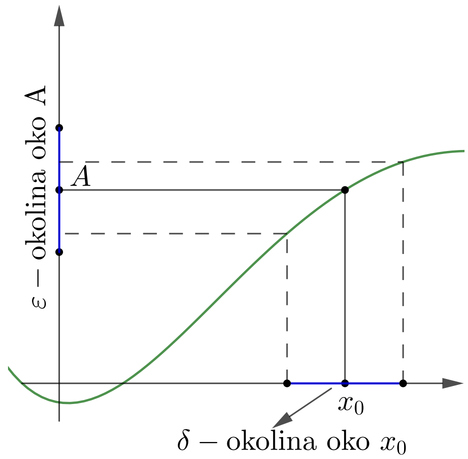
$$\begin{aligned}
\text{c) } f(-x) &= \frac{1}{-x}\\
&= -\frac{1}{x}\\
&= -f(x) \Longrightarrow f \text{ je neparna.}\\
\text{d) } f(-x) &= \frac{1}{(-x)^2}\\
&= \frac{1}{x^2}\\
&= f(x) \Longrightarrow f \text{ je parna.}
\end{aligned}$$

$$\text{e) } f(-x) = \frac{1}{-x-1}$$

$\Longrightarrow f$ nije ni parna ni neparna.

5.3 Limes funkcije

Definicija 5.16. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ funkcija i neka je $x_0 \in I$. Ako postoji $A \in \mathbb{R}$ takav da vrijedi: $\forall \varepsilon > 0, \exists \delta > 0$ tako da $|x - x_0| < \delta \Longrightarrow |f(x) - A| < \varepsilon$, onda kažemo da f ima limes u točki x_0 i pišemo $\lim\limits_{x \to x_0} f(x) = A$.



Definicija 5.17. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ funkcija i neka je $x_0 \in I$. Ako postoji $A \in \mathbb{R}$ takav da vrijedi: $\forall \varepsilon > 0, \exists \delta > 0$ tako da $x \in \langle x_0 - \delta, x_0 \rangle \Longrightarrow |f(x) - A| < \varepsilon$, onda kažemo da f ima limes u točki x_0 **slijeva** i pišemo $\lim\limits_{x \to x_0^-} f(x) = A$.

Definicija 5.18. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ funkcija i neka je $x_0 \in I$. Ako postoji $A \in \mathbb{R}$ takav da vrijedi: $\forall \varepsilon > 0, \exists \delta > 0$ tako da $x \in \langle x_0, x_0 + \delta \rangle \Longrightarrow |f(x) - A| < \varepsilon$, onda kažemo da f ima limes u točki x_0 **zdesna** i pišemo $\lim\limits_{x \to x_0^+} f(x) = A$.

Funkcija f ima limes u točki x_0 ako postoje limesi slijeva i zdesna u toj točki i jednaki su.

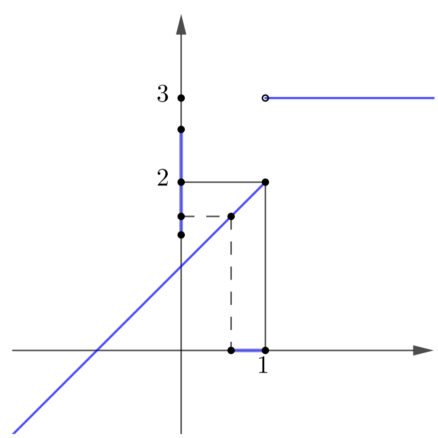
Zadatak 5.7. Neka je $f : \mathbb{R} \to \mathbb{R}$ zadana s $f(x) = \begin{cases} x+1, & x \leq 1 \\ 3, & x > 1 \end{cases}$. Postoji li

a) $\lim\limits_{x \to 1^-} f(x)$

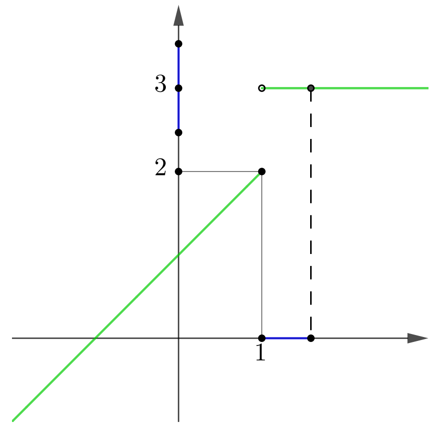
b) $\lim\limits_{x \to 1^+} f(x)$

c) $\lim\limits_{x \to 1} f(x)$

a) Vrijedi da je $\lim\limits_{x \to 1^-} f(x) = 2$. Naime, za bilo koji $\varepsilon > 0$ odaberimo $\delta = \varepsilon$. Tada lako vidimo da čim je $x \in \langle 1 - \delta, 1 \rangle$ imamo da je $f(x) = x + 1 \in \langle 2 - \varepsilon, 2 + \varepsilon \rangle$. Dakle, svaka točka iz lijeve δ-okoline od 1 preslika se u ε-okolinu oko 2.



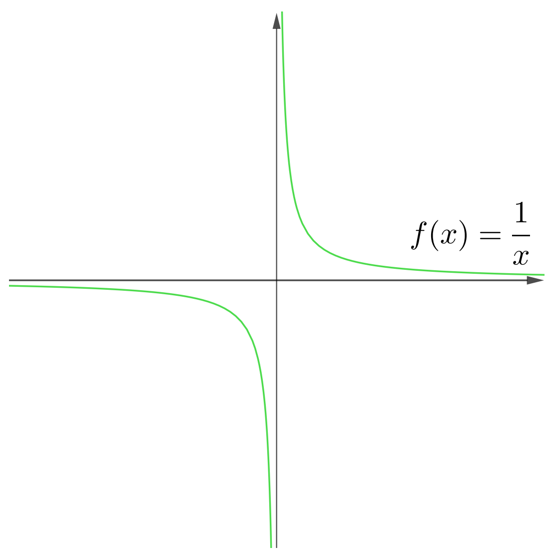
b) Vrijedi da je $\lim\limits_{x\to 1^+} f(x) = 3$. Slično kao pod a), za bilo koji $\varepsilon > 0$ odaberimo $\delta = \varepsilon$, pa čim je $x \in \langle 1, 1+\delta\rangle$, očito je $f(x) = 3 \in \langle 3-\varepsilon, 3+\varepsilon\rangle$, odnosno svaka točka iz desne δ-okoline od 1 preslika se u ε-okolinu oko 3.



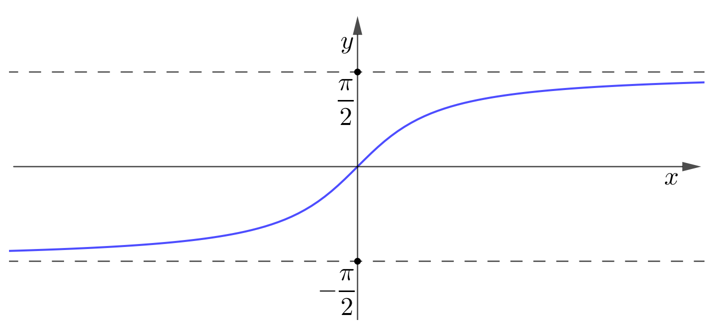
c) Kako je $\lim\limits_{x\to 1^-} f(x) = 2$ i $\lim\limits_{x\to 1^+} f(x) = 3$, pa imamo da je $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$, tj. $\lim\limits_{x\to 1} f(x)$ ne postoji.

Primjer:

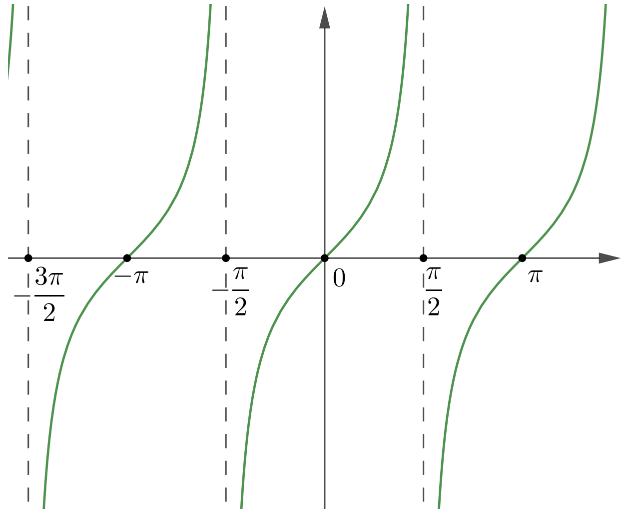
- $\lim\limits_{x\to +\infty} \dfrac{1}{x} = 0$
- $\lim\limits_{x\to -\infty} \dfrac{1}{x} = 0$
- $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$
- $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$



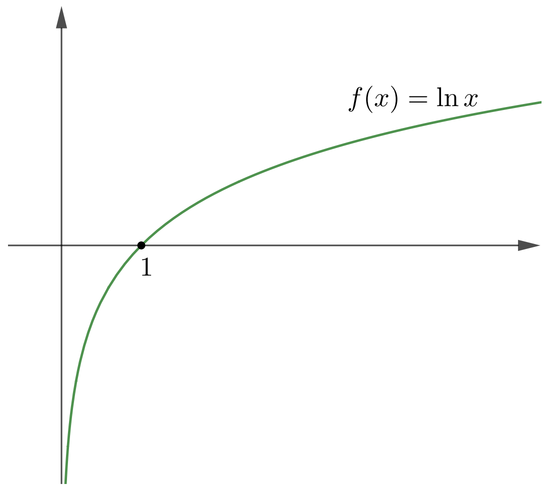
- $\lim\limits_{x\to +\infty} \operatorname{arctg} x = \dfrac{\pi}{2}$
- $\lim\limits_{x\to -\infty} \operatorname{arctg} x = -\dfrac{\pi}{2}$



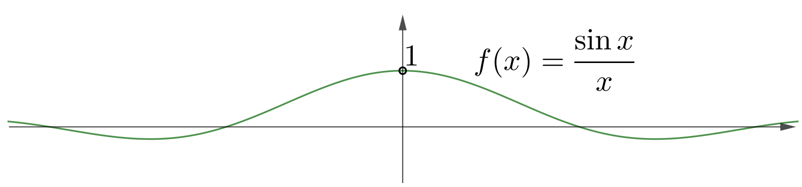
- $\lim\limits_{x\to\frac{\pi}{2}^-} \operatorname{tg} x = +\infty$
- $\lim\limits_{x\to\frac{\pi}{2}^+} \operatorname{tg} x = -\infty$



- $\lim\limits_{x\to+\infty} \ln x = +\infty$
- $\lim\limits_{x\to 0^+} \ln x = -\infty$



- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x\to\pm\infty} \dfrac{\sin x}{x} = 0$
- $\lim\limits_{x\to+\infty} \left(1 + \dfrac{1}{x}\right)^x = e$
- $\lim\limits_{x\to+\infty} \left(1 + \dfrac{k}{x}\right)^x = e^k$



Zadatak 5.8. Odredite sljedeće limese:

a) $\lim\limits_{x\to+\infty} \dfrac{(x+1)^2}{x^2+1}$

b) $\lim\limits_{x\to+\infty} \dfrac{100x}{x^2-1}$

c) $\lim\limits_{x\to+\infty} \dfrac{x^2-5x+1}{3x+7}$

d) $\lim\limits_{x\to+\infty} \dfrac{x^2-1}{1-x}$

e) $\lim\limits_{x\to+\infty} \dfrac{2x^2-3x-4}{\sqrt{x^4+1}}$

f) $\lim\limits_{x\to -1} \dfrac{x^3+1}{x^2+1}$

g) $\lim\limits_{x\to 5} \dfrac{x^2-7x+10}{x^2-25}$

h) $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$

i) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$

j) $\lim\limits_{x\to 7} \dfrac{2-\sqrt{x-3}}{x^2-49}$

k) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

l) $\lim\limits_{x\to 2} \dfrac{\sin x}{x}$

m) $\lim\limits_{x\to +\infty} \dfrac{\sin x}{x}$

n) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\sin 2x}$

o) $\lim\limits_{x\to 0} \left(\dfrac{x^2-2x+3}{x^2-3x+2}\right)^{\frac{\sin x}{x}}$

p) $\lim\limits_{x\to +\infty} \left(\dfrac{x-1}{x+1}\right)^{x}$

r) $\lim\limits_{x\to 0} \left(\dfrac{2+x}{3-x}\right)^{x}$

s) $\lim\limits_{x\to 1} \left(\dfrac{x-1}{x^2-1}\right)^{x+1}$

t) $\lim\limits_{x\to +\infty} \sin \dfrac{1}{\sqrt{x}}$

u) $\lim\limits_{x\to +\infty} \dfrac{x+1}{\sqrt{x^2+x+1}}$

v) $\lim\limits_{x\to +\infty} \left(\dfrac{x^2+3x+1}{2x^2-1}\right)^{x}$

z) $\lim\limits_{x\to +\infty} \left(\dfrac{2x^2+1}{x^2-3}\right)^{x}$

Rješenje:

a)
$$\begin{aligned}
\lim_{x\to+\infty} \frac{(x+1)^2}{x^2+1} &= \lim_{x\to+\infty} \frac{x^2+2x+1}{x^2+1} : \frac{x^2}{x^2} \\
&= \lim_{x\to+\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}} \\
&= \frac{1}{1} = 1
\end{aligned}$$

b)
$$\begin{aligned}
\lim_{x\to+\infty} \frac{100x}{x^2-1} : \frac{x^2}{x^2} &= \lim_{x\to+\infty} \frac{\frac{100}{x}}{1-\frac{1}{x^2}} \\
&= \frac{0}{1} = 0
\end{aligned}$$

c)
$$\lim_{x\to+\infty} \frac{x^2-5x+1}{3x+7} : \frac{x^2}{x^2} = \lim_{x\to+\infty} \frac{1-\frac{5}{x}+\frac{1}{x^2}}{\frac{3}{x}+\frac{7}{x^2}}$$

$$= \frac{1}{0^+} = +\infty$$

d) $$\lim_{x\to+\infty} \frac{x^2-1}{1-x} : \frac{x^2}{x^2} = \lim_{x\to+\infty} \frac{1-\frac{1}{x^2}}{\frac{1}{x^2}-\frac{1}{x}}$$

$$= \frac{1}{0^-} \qquad \frac{1}{x^2} \xrightarrow{x\to+\infty} 0,\ \frac{1}{x} \xrightarrow{x\to+\infty} 0 \text{ i } \frac{1}{x} > \frac{1}{x^2}$$

$$= -\infty \qquad \text{pa } \tfrac{1}{x^2} - \tfrac{1}{x} \xrightarrow{x\to+\infty} 0^-$$

e) $$\lim_{x\to+\infty} \frac{2x^2-3x-4}{\sqrt{x^4+1}} : \frac{x^2}{x^2} = \lim_{x\to+\infty} \frac{2-\frac{3}{x}-\frac{4}{x^2}}{\sqrt{1+\frac{1}{x^4}}}$$

$$= \frac{2}{1}$$

$$= 2$$

f) $$\lim_{x\to-1} \frac{x^3+1}{x^2+1} = \lim_{x\to-1} \frac{(-1)^3+1}{(-1)^2+1}$$

$$= \frac{0}{2}$$

$$= 0$$

g) $$\lim_{x\to5} \frac{x^2-7x+10}{x^2-25} = \left(\frac{25-35+10}{25-25} = \frac{0}{0}\right)$$

$$= \lim_{x\to5} \frac{x^2-5x-2x+10}{(x-5)(x+5)}$$

$$= \lim_{x\to5} \frac{x(x-5)-2(x-5)}{(x-5)(x+5)}$$

$$= \lim_{x\to5} \frac{(x-5)(x-2)}{(x-5)(x+5)}$$

$$= \lim_{x\to5} \frac{x-2}{x+5}$$

$$= \lim_{x\to5} \frac{5-2}{5+5}$$

$$= \frac{3}{10}$$

$$\text{h)} \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \left(\frac{1-1}{1-1} = \frac{0}{0}\right)$$

$$= \left\{\begin{array}{l} x = t^2 \\ \\ x \to 1 \Rightarrow t \to 1 \end{array}\right\}$$

$$= \lim_{t\to 1} \frac{t-1}{t^2-1}$$

$$= \lim_{t\to 1} \frac{t-1}{(t-1)(t+1)}$$

$$= \lim_{t\to 1} \frac{1}{t+1}$$

$$= \frac{1}{2}$$

$$\text{i)} \lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} = \left(\frac{1-1}{1-1} = \frac{0}{0}\right)$$

$$= \left\{\begin{array}{l} 1+x = t^6 \\ \\ x \to 0 \Rightarrow t \to 1 \end{array}\right.$$

$$= \lim_{t\to 1} \frac{\sqrt{t^6}-1}{\sqrt[3]{t^6}-1}$$

$$= \lim_{t\to 1} \frac{t^3-1}{t^2-1}$$

$$= \lim_{t\to 1} \frac{(t-1)(t^2+t+1)}{(t-1)(t+1)}$$

$$= \lim_{t\to 1} \frac{t^2+t+1}{t+1}$$

$$= \frac{3}{2}$$

$$\text{j)} \lim_{x\to 7} \frac{2-\sqrt{x-3}}{x^2-49} = \left(\frac{2-2}{49-49} = \frac{0}{0}\right)$$

$$
\begin{aligned}
&= \lim_{x\to 7} \frac{2-\sqrt{x-3}}{x^2-49} \cdot \frac{2+\sqrt{x-3}}{2+\sqrt{x-3}} \\
&= \lim_{x\to 7} \frac{2^2-\left(\sqrt{x-3}\right)^2}{(x^2-49)\left(2+\sqrt{x-3}\right)} \\
&= \lim_{x\to 7} \frac{7-x}{(x-7)(x+7)\left(2+\sqrt{x-3}\right)} \\
&= \frac{-1}{(7+7)\left(2+\sqrt{7-3}\right)} \\
&= -\frac{1}{56}
\end{aligned}
$$

k) $$
\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \left(\frac{1-1}{0} = \frac{0}{0}\right) \\
&= \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \\
&= \lim_{x\to 0} \frac{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} \\
&= \lim_{x\to 0} \frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} \\
&= \lim_{x\to 0} \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} \\
&= \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} \\
&= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} \\
&= 1
\end{aligned}
$$

l) $$\lim_{x\to 2} \frac{\sin x}{x} = \frac{\sin 2}{2}$$

m) $\lim\limits_{x\to+\infty}\dfrac{\sin x}{x}=0$

$$\begin{aligned}
-1 &\leq \sin x \leq 1 \quad \Big/ : x\\
-\frac{1}{x} &\leq \frac{\sin x}{x} \leq \frac{1}{x} \quad \Big/ \lim_{x\to+\infty}\\
\lim_{x\to+\infty} -\frac{1}{x} &\leq \lim_{x\to+\infty}\frac{\sin x}{x} \leq \lim_{x\to+\infty}\frac{1}{x}\\
0 &\leq \lim_{x\to+\infty}\frac{\sin x}{x} \leq 0\\
\lim_{x\to+\infty}\frac{\sin x}{x} &= 0
\end{aligned}$$

n) $$\begin{aligned}
\lim_{x\to 0}\frac{\sin 5x}{\sin 2x} &= \left(\frac{0}{0}\right)\\
&= \lim_{x\to 0}\frac{\frac{\sin 5x}{5x}}{\frac{\sin 2x}{2x}}\cdot\frac{5x}{2x}\\
&= \frac{5}{2}\lim_{x\to 0}\frac{\frac{\sin 5x}{5x}}{\frac{\sin 2x}{2x}}\\
&= \frac{5}{2}\cdot\frac{1}{1}\\
&= \frac{5}{2}
\end{aligned}$$

* Općenito, za $A\in\mathbb{R}$, $$\begin{aligned}
\lim_{x\to 0}\frac{\sin Ax}{Ax} &= \left\{\begin{matrix} Ax=t \\ x\to 0\Rightarrow t\to 0\end{matrix}\right\}\\
&= \lim_{t\to 0}\frac{\sin t}{t}\\
&= 1
\end{aligned}$$

$$\text{o) } \lim_{x\to 0}\left(\frac{x^2-2x+3}{x^2-3x+2}\right)^{\frac{\sin x}{x}} = \left(\lim_{x\to 0}\frac{x^2-2x+3}{x^2-3x+2}\right)^{\lim\limits_{x\to 0}\frac{\sin x}{x}}$$

$$= \left(\frac{3}{2}\right)^1$$

$$= \frac{3}{2}$$

$$\text{p) } \lim_{x\to+\infty}\left(\frac{x-1}{x+1}\right)^x = \lim_{x\to+\infty}\left(\frac{x+1-2}{x+1}\right)^x$$

$$= \lim_{x\to+\infty}\left(\frac{x+1}{x+1}+\frac{-2}{x+1}\right)^x$$

$$= \lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^x$$

$$= \lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{x+1-1}$$

$$= \lim_{x\to+\infty}\left[\left(1+\frac{-2}{x+1}\right)^{x+1}\left(1+\frac{-2}{x+1}\right)^{-1}\right]$$

$$= \underbrace{\lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{x+1}}_{e^{-2}}\underbrace{\lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{-1}}_{1}$$

$$= \frac{1}{e^2}$$

$$\text{r) } \lim_{x\to 0}\left(\frac{2+x}{3-x}\right)^x = \left(\frac{2+0}{3-0}\right)^0$$

$$= 1$$

$$\text{s) } \lim_{x\to 1}\left(\frac{x-1}{x^2-1}\right)^{x+1} = \left(\frac{1-1}{1-1}\right)^2$$

$$= \lim_{x\to 1}\left(\frac{x-1}{(x-1)(x+1)}\right)^{x+1}$$

$$= \lim_{x\to 1} \left(\frac{1}{x+1}\right)^{x+1}$$

$$= \frac{1}{4}$$

t) $$\lim_{x\to+\infty} \sin\frac{1}{\sqrt{x}} = \sin 0 = 0$$

u) $$\lim_{x\to+\infty} \frac{x+1}{\sqrt{x^2+x+1}} : \frac{x}{x} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \frac{1+0}{\sqrt{1+0+0}} = 1$$

v) $$\lim_{x\to+\infty} \left(\frac{x^2+3x+1}{2x^2-1}\right)^x = \left(\lim_{x\to+\infty} \frac{x^2+3x+1}{2x^2-1} : \frac{x^2}{x^2}\right)^{\lim\limits_{x\to+\infty} x} = \left(\lim_{x\to+\infty} \frac{1+\frac{3}{x}+\frac{1}{x^2}}{2-\frac{1}{x^2}}\right)^{\lim\limits_{x\to+\infty} x} = \left(\frac{1}{2}\right)^{+\infty} = 0$$

z) $$\lim_{x\to+\infty} \left(\frac{2x^2+1}{x^2-3}\right)^x = \left(\lim_{x\to+\infty} \frac{2x^2+1}{x^2-3} : \frac{x^2}{x^2}\right)^{\lim\limits_{x\to+\infty} x} = \left(\lim_{x\to+\infty} \frac{2+\frac{1}{x^2}}{1-\frac{3}{x^2}}\right)^{\lim\limits_{x\to+\infty} x} = \left(\frac{2}{1}\right)^{+\infty} = +\infty$$

Poglavlje 6

Derivacija

6.1 Definicija i osnovna svojstva

Definicija 6.1. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ funkcija i neka je $x_0 \in I$. Kažemo da je f **derivabilna** u točki x_0 ako postoji $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ i pišemo:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Zadatak 6.1. Neka je $f : \mathbb{R} \to \mathbb{R}$ zadana s :

a) $f(x) = c$ (konstantna funkcija)

b) $f(x) = x$

c) $f(x) = x^2$

d) $f(x) = \cos x$

Odredite $f'(x)$.

Rješenje:

a)
$$\begin{aligned} f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{c - c}{x - x_0} \\ &= \lim_{x \to x_0} \frac{0}{x - x_0} \\ &= 0 \end{aligned}$$

$$\text{b) } f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x-x_0}{x-x_0} = \lim_{x\to x_0} 1 = 1$$

$$\begin{aligned}\text{c) } f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0} \\ &= \lim_{x\to x_0} \frac{(x-x_0)(x+x_0)}{x-x_0} \\ &= \lim_{x\to x_0} (x+x_0) \\ &= 2x_0\end{aligned}$$

$$\begin{aligned}\text{d) } f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{\cos x-\cos x_0}{x-x_0} \\ &\quad \left(\cos x - cosy = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}\right) \\ &= \lim_{x\to x_0} \frac{-2\sin\frac{x+x_0}{2}\sin\frac{x-x_0}{2}}{x-x_0} \\ &= \lim_{x\to x_0} \frac{-\sin\frac{x+x_0}{2}\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} \\ &= \lim_{x\to x_0} \left(-\sin\tfrac{x+x_0}{2}\right) \lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} (*)\end{aligned}$$

$$\begin{aligned}\lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} &= \left\{\begin{matrix} \dfrac{x-x_0}{2} = t \\ x\to x_0 \Rightarrow t\to 0 \end{matrix}\right\} \\ &= \lim_{t\to 0}\frac{\sin t}{t} \\ &= 1\end{aligned}$$

$$\begin{aligned}(*) &= \lim_{x\to x_0} \left(-\sin\tfrac{x+x_0}{2}\right)\cdot 1 \\ &= -\sin\tfrac{x_0+x_0}{2} \\ &= -\sin x_0\end{aligned}$$

Nastavljajući ovaj postupak i za druge funkcije dobivamo:

Tablica derivacija:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\operatorname{tg} x$	$\dfrac{1}{\cos^2 x}$
x	1	$\operatorname{ctg} x$	$-\dfrac{1}{\sin^2 x}$
x^n	nx^{n-1}	$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\operatorname{arctg} x$	$\dfrac{1}{1+x^2}$
$\ln x$	$\dfrac{1}{x}$	$\operatorname{arcctg} x$	$-\dfrac{1}{1+x^2}$
$\sin x$	$\cos x$	$\operatorname{sh} x$	$\operatorname{ch} x$
$\cos x$	$-\sin x$	$\operatorname{ch} x$	$\operatorname{sh} x$

Pravila deriviranja:

- $[c \cdot f(x)]' = c \cdot f'(x)$
- $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
- $[f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$
- $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
- $[(f \circ g)(x)]' = [f(g(x))]' = f'(g(x)) \cdot g'(x)$

Zadatak 6.2. Odredite $f'(x)$ za

a) $f(x) = x^5 - 4x^3 + 2x - 3$

b) $f(x) = \dfrac{1}{4} - \dfrac{1}{3}x^3 + x^2 - \dfrac{1}{4}x^4$

c) $f(x) = \dfrac{ax^6 + bx + c}{\sqrt{a^2+b^2}}$

d) $f(x) = \dfrac{2x+3}{x^2 - 5x + 5}$

e) $f(x) = x^2 \cdot 2^{-x}$

f) $f(x) = x^2 + \dfrac{2}{2x-1} - \dfrac{1}{x}$

g) $f(x) = 5\sin x + 3\cos x$

h) $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

i) $f(x) = x \cdot \arcsin x$

j) $f(x) = (x-1) \cdot e^x$

k) $f(x) = \dfrac{e^x}{x^2}$

l) $f(x) = \cos x \cdot e^x$

m) $f(x) = \dfrac{x^2}{\ln x}$

n) $f(x) = \dfrac{1}{x} + 2\ln x + \dfrac{\ln x}{x}$

o) $f(x) = x \cdot \operatorname{sh} x$

Rješenje:

$$\text{a) } f(x) = x^5 - 4x^3 + 2x - 3$$

$$f'(x) = 5x^4 - 4 \cdot 3x^2 + 2$$

$$= 5x^4 - 12x^2 + 2$$

$$\text{b) } f(x) = \frac{1}{4} - \frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4$$

$$f'(x) = -\frac{3}{3} \cdot x^2 + 2x - \frac{4}{4} \cdot x^3$$

$$= -x^2 + 2x - x^3$$

$$\text{c) } f(x) = \frac{ax^6 + bx + c}{\sqrt{a^2 + b^2}}$$

$$= \frac{1}{\sqrt{a^2 + b^2}} (ax^6 + bx + c)$$

$$f'(x) = \frac{1}{\sqrt{a^2 + b^2}} (6ax^5 + b)$$

$$\text{d) } f(x) = \frac{2x + 3}{x^2 - 5x + 5}$$

$$f'(x) = \frac{(2x+3)'(x^2 - 5x + 5) - (2x+3)(x^2 - 5x + 5)'}{(x^2 - 5x + 5)^2}$$

$$= \frac{2(x^2 - 5x + 5) - (2x+3)(2x-5)}{(x^2 - 5x + 5)^2}$$

$$= \frac{2x^2 - 10x + 10 - 4x^2 + 10x - 6x + 15}{(x^2 - 5x + 5)^2}$$

$$= \frac{-2x^2 - 6x + 25}{(x^2 - 5x + 5)^2}$$

$$\text{e) } f(x) = x^2 \cdot 2^{-x} \qquad \text{ili} \qquad f(x) = \frac{x^2}{2^x}$$

$$f'(x) = 2x \cdot 2^{-x} + x^2 \cdot 2^{-x} \cdot \ln 2 \cdot (-1) \qquad f'(x) = \frac{2x \cdot 2^x - x^2 \cdot 2^x \ln 2}{2^{2x}}$$

$$= \frac{2x}{2^x} - \frac{x^2 \cdot \ln 2}{2^x} \qquad = \frac{2^x (2x - x^2 \ln 2)}{2^{2x}}$$

$$= \frac{2x - x^2 \ln 2}{2^x} \qquad = \frac{2x - x^2 \ln 2}{2^x}$$

$$\begin{aligned}
\text{f) } f(x) &= x^2 + \frac{2}{2x-1} - \frac{1}{x} \\
&= x^2 + 2(2x-1)^{-1} - x^{-1} \\
f'(x) &= 2x + 2\cdot(-1)\cdot(2x-1)^{-2}\cdot 2 - (-1)\cdot x^{-2} \\
&= 2x - 4(2x-1)^{-2} + x^{-2} \\
&= 2x - \frac{4}{(2x-1)^2} + \frac{1}{x^2}
\end{aligned}$$

$$\begin{aligned}
\text{g) } f(x) &= 5\sin x + 3\cos x \\
f'(x) &= 5\cos x - 3\sin x
\end{aligned}$$

$$\begin{aligned}
\text{h) } f(x) &= \frac{\sin x + \cos x}{\sin x - \cos x} \\
f'(x) &= \frac{(\cos x - \sin x)(\sin x - \cos x) - (\cos x + \sin x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \\
&= \frac{-(\cos x - \sin x)^2 - (\cos x + \sin x)^2}{(\sin x - \cos x)^2} \\
&= \frac{-\cos^2 x + 2\cos x \sin x - \sin^2 x - \cos^2 x - 2\cos x\sin x - \sin^2 x}{(\sin x - \cos x)^2} \\
&= \frac{-2(\cos^2 x + \sin^2 x)}{(\sin x - \cos x)^2} \\
&= \frac{-2}{(\sin x - \cos x)^2}
\end{aligned}$$

$$\begin{aligned}
\text{i) } f(x) &= x \cdot \arcsin x \\
f'(x) &= \arcsin x + \frac{x}{\sqrt{1-x^2}}
\end{aligned}$$

$$\begin{aligned}
\text{j) } f(x) &= (x-1)\cdot e^x \\
f'(x) &= 1\cdot e^x + (x-1)e^x \\
&= (x-1+1)e^x \\
&= xe^x
\end{aligned}$$

k) $$\begin{aligned} f(x) &= \frac{e^x}{x^2} \\ f'(x) &= \frac{e^x \cdot x^2 - e^x \cdot 2x}{x^4} \\ &= \frac{xe^x(x-2)}{x^4} \\ &= \frac{e^x(x-2)}{x^3} \end{aligned}$$

l) $$\begin{aligned} f(x) &= e^x \cos x \\ f'(x) &= e^x \cos x + e^x(-\sin x) \\ &= e^x(\cos x - \sin x) \end{aligned}$$

m) $$\begin{aligned} f(x) &= \frac{x^2}{\ln x} \\ f'(x) &= \frac{2x \ln x - x^2 \cdot \frac{1}{x}}{\ln^2 x} \\ &= \frac{x(2\ln x - 1)}{\ln^2 x} \end{aligned}$$

n) $$\begin{aligned} f(x) &= \frac{1}{x} + 2\ln x + \frac{\ln x}{x} \\ f'(x) &= -\frac{1}{x^2} + \frac{2}{x} + \frac{\frac{1}{x} \cdot x - \ln x}{x^2} \\ &= \frac{2}{x} - \frac{\ln x}{x^2} \end{aligned}$$

o) $$\begin{aligned} f(x) &= x \cdot \operatorname{sh} x \\ f'(x) &= \operatorname{sh} x + x \operatorname{ch} x \end{aligned}$$

Zadatak 6.3. Odredite $f'(x)$ za

a) $f(x) = \operatorname{arctg}\left(1 + \frac{1}{x}\right)$

b) $f(x) = \sqrt{1 - x^2}$

c) $f(x) = 2x + 5\cos^3 x$

d) $f(x) = \frac{x-2}{\sqrt{x^2+1}}$

e) $f(x) = \sqrt{\arcsin x}$

f) $f(x) = \sqrt{xe^x + x}$

g) $f(x) = \arcsin \dfrac{1}{x^2}$

h) $f(x) = \ln(1 - x^2)$

i) $f(x) = \sqrt{\ln x + 1} + \ln\left(\sqrt{x} + 1\right)$

Rješenje:

a)
$$\begin{aligned} f(x) &= \operatorname{arctg}\left(1 + \frac{1}{x}\right) \\ f'(x) &= \frac{1}{1 + \left(1 + \frac{1}{x}\right)^2} \cdot \left(1 + \frac{1}{x}\right)' \\ &= \frac{1}{1 + 1 + \frac{2}{x} + \frac{1}{x^2}} \cdot \frac{-1}{x^2} \\ &= \frac{1}{\frac{2x^2+2x+1}{x^2}} \cdot \frac{-1}{x^2} \\ &= \frac{x^2}{2x^2 + 2x + 1} \cdot \frac{-1}{x^2} \\ &= \frac{-1}{2x^2 + 2x + 1} \end{aligned}$$

$g(x) = \operatorname{arctg} x, \quad h(x) = 1 + \dfrac{1}{x}, \quad f = g \circ h$

b)
$$\begin{aligned} f(x) &= \sqrt{1 - x^2} \\ f'(x) &= \frac{1}{2\sqrt{1 - x^2}} \cdot (1 - x^2)' \\ &= \frac{-2x}{2\sqrt{1 - x^2}} \\ &= \frac{-x}{\sqrt{1 - x^2}} \end{aligned}$$

$g(x) = \sqrt{x}, \quad h(x) = 1 - x^2, \quad f = g \circ h$

c)
$$\begin{aligned} f(x) &= 2x + 5\cos^3 x \\ f'(x) &= 2 + 5 \cdot 3\cos^2 x \cdot (-\sin x) \\ &= 2 - 15\cos^2 x \sin x \end{aligned}$$

$g(x) = x^3, \quad h(x) = \cos x, \quad f(x) = 2x + 5(g \circ h)(x)$

$$\begin{aligned}
\text{d) } f(x) &= \frac{x-2}{\sqrt{x^2+1}}\\
f'(x) &= \frac{\sqrt{x^2+1}-(x-2)\frac{1}{2\sqrt{x^2+1}}\cdot 2x}{x^2+1}\\
&= \frac{\frac{x^2+1-x^2+2x}{\sqrt{x^2+1}}}{x^2+1}\\
&= \frac{2x+1}{(x^2+1)(x^2+1)^{\frac{1}{2}}}\\
&= \frac{2x+1}{\sqrt{(x^2+1)^3}}
\end{aligned}$$

$$\begin{aligned}
\text{e) } f(x) &= \sqrt{\arcsin x}\\
f'(x) &= \frac{1}{2\sqrt{\arcsin x}}\cdot\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

$$\begin{aligned}
\text{f) } f(x) &= \sqrt{xe^x+x}\\
f'(x) &= \frac{1}{2\sqrt{xe^x+x}}\left(e^x+xe^x+1\right)\\
&= \frac{e^x+xe^x+1}{2\sqrt{xe^x+x}}
\end{aligned}$$

$$\begin{aligned}
\text{g) } f(x) &= \arcsin\frac{1}{x^2}\\
f'(x) &= \frac{1}{\sqrt{1-\left(\frac{1}{x^2}\right)^2}}\cdot\frac{-2}{x^3}\\
&= \frac{-2}{x^3\sqrt{1-\frac{1}{x^4}}}\\
&= \frac{-2}{x\sqrt{x^4-1}}
\end{aligned}$$

$$\begin{aligned}
\text{h) } f(x) &= \ln(1-x^2)\\
f'(x) &= \frac{1}{1-x^2}\cdot(-2x)\\
&= \frac{2x}{x^2-1}
\end{aligned}$$

$$\begin{aligned}
\text{i) } f(x) &= \sqrt{\ln x + 1} + \ln\left(\sqrt{x} + 1\right) \\
f'(x) &= \frac{1}{2\sqrt{\ln x + 1}} \cdot \frac{1}{x} + \frac{1}{\sqrt{x} + 1} \cdot \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2x\sqrt{\ln x + 1}} + \frac{1}{2\sqrt{x}(\sqrt{x} + 1)} \\
&= \frac{1}{2x\sqrt{\ln x + 1}} + \frac{1}{2x + 2\sqrt{x}}
\end{aligned}$$

Logaritamsko deriviranje koristimo kada se argument funkcije x nalazi i u bazi i u eksponentu.

$$\begin{aligned}
f(x) &= g(x)^{h(x)} && / \ln \\
\ln\left(f\left(x\right)\right) &= h(x) \ln\left(g\left(x\right)\right) && /' \\
\frac{1}{f(x)} f'(x) &= h'(x) \ln\left(g\left(x\right)\right) + h(x) \frac{1}{g(x)} g'(x) && / \cdot f(x) \\
f'(x) &= f(x) \left[h'(x) \ln\left(g\left(x\right)\right) + h(x) \frac{g'(x)}{g(x)} \right]
\end{aligned}$$

Zadatak 6.4. Odredite $f'(x)$ za

a) $f(x) = x^x$

b) $f(x) = (\sin x)^x$

c) $f(x) = \left(1 + \dfrac{1}{x}\right)^x$

Rješenje:

$$\begin{aligned}
\text{a) } f(x) &= x^x && / \ln \\
\ln f(x) &= x \ln x && /' \\
\frac{1}{f(x)} f'(x) &= \ln x + x \cdot \frac{1}{x} && / \cdot f(x) \\
f'(x) &= x^x \left(\ln x + 1\right)
\end{aligned}$$

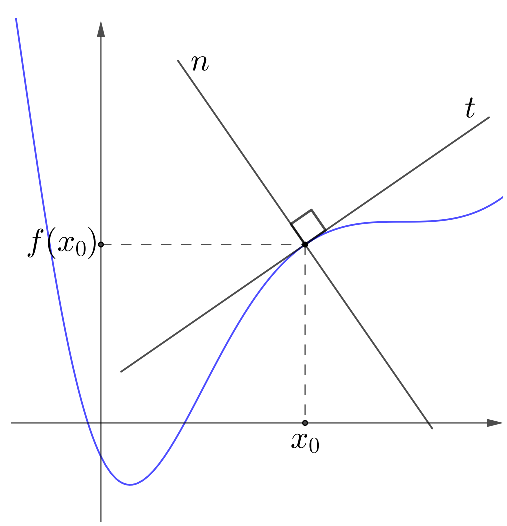
$$
\begin{aligned}
\text{b) } f(x) &= (\sin x)^x && / \ln \\
\ln f(x) &= x \ln(\sin x) && /' \\
\frac{1}{f(x)} f'(x) &= \ln(\sin x) + x \, \frac{1}{\sin x} \cos x && / \cdot f(x) \\
f'(x) &= (\sin x)^x \left[\ln(\sin x) + x \cdot \operatorname{ctg} x \right]
\end{aligned}
$$

$$
\begin{aligned}
\text{c) } f(x) &= \left(1 + \frac{1}{x}\right)^x && / \ln \\
\ln f(x) &= x \ln\left(1 + \frac{1}{x}\right) && /' \\
\frac{1}{f(x)} f'(x) &= \ln\left(1 + \frac{1}{x}\right) + x \cdot \frac{1}{1 + \frac{1}{x}} \cdot \frac{-1}{x^2} \\
\frac{1}{f(x)} f'(x) &= \ln\left(1 + \frac{1}{x}\right) + \frac{-x}{x^2 + x} && / \cdot f(x) \\
f'(x) &= \left(1 + \frac{1}{x}\right)^x \left[\ln\left(1 + \frac{1}{x}\right) + \frac{-1}{x+1} \right]
\end{aligned}
$$

6.2 Tangenta i normala krivulje

Definicija 6.2. Tangenta kruvulje $(x, f(x))$ u točki x_0 je pravac koji dodiruje krivulju u točno jednoj točki u nekoj okolini točke x_0.

Definicija 6.3. Normala kruvulje $(x, f(x))$ u točki x_0 je pravac okomit na tangentu te krivulje u točki x_0.



Derivacija funkcije f u x_0 jednaka je nagibu tangente na graf $f(x)$ u točki x_0, odnosno

$$f'(x_0) = k.$$

Uočimo i da je $y_0 = f(x_0)$ pa izlazi da je jednadžba tangente t

$$t...y - y_0 = k(x - x_0).$$

Ako vrijedi da je $p_1 \perp p_2$, onda su njihovi koeficijenti smjera negativne recipročne vrijednosti, $k_1 = \dfrac{-1}{k_2}$, pa je jednadžba normale u točki x_0

$$n...y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0).$$

Zadatak 6.5. Napišite jednadžbu tangente i normale na krivulju $y = \sqrt{x}$ u točki s apscisom $x = 4$.

$x = 4 \Longrightarrow y = \sqrt{4} = 2$

$T(x_0, y_0) = T(4, 2)$

$f'(x) = \dfrac{1}{2\sqrt{x}} \Longrightarrow f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

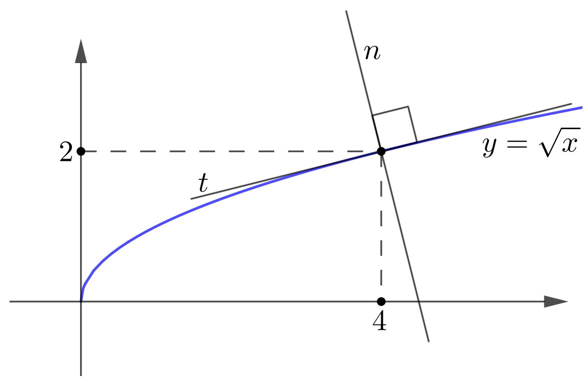
$k_t = \dfrac{1}{4}$ i $k_n = -4$

$y - 2 = \dfrac{1}{4}(x - 4)$

$t...y = \dfrac{1}{4}x + 1$

$y - 2 = -4(x - 4)$

$n...y = -4x + 18$



Zadatak 6.6. Odredite jednadžbe svih tangenti na graf funkcije $f(x) = x^4 - x + 3$ koje prolaze ishodištem.

Rješenje:

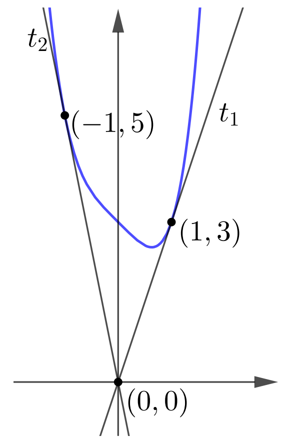
$f'(x) = 4x^3 - 1$

Pronađimo koordinate dirališta $D(x_0, y_0)$.

$$
\begin{aligned}
t...y - y_0 &= f'(x_0)(x - x_0)
\end{aligned}
$$

Kako je $O(0,0) \in t$ imamo:

$$
\begin{aligned}
0 - (x_0^4 - x_0 + 3) &= (4x_0^3 - 1)(0 - x_0) \\
-x_0^4 + x_0 - 3 &= -4x_0^4 + x_0 \\
3x_0^4 &= 3 \\
x_0^4 &= 1 \\
x_0 = 1 \Rightarrow y_0 = 3 \quad &\text{i} \quad x_0 = -1 \Rightarrow y_0 = 5
\end{aligned}
$$



$$y - y_0 = f'(x_0)(x - x_0)$$

$y - 3 = 3(x - 1) \qquad y - 5 = -5(x + 1)$

$t_1 ... y = 3x \qquad t_2 ... y = -5x$

Zadatak 6.7. Odredite jednadžbe tangenti povučenih iz $T(9,-2)$ na graf funkcije $f(x) = \dfrac{1}{x-5}$.

Rješenje:

$$f'(x) = \frac{-1}{(x-5)^2}$$

Pronađimo koordinate dirališta $D(x_0, y_0)$.

$$t...y - y_0 = f'(x_0)(x - x_0)$$

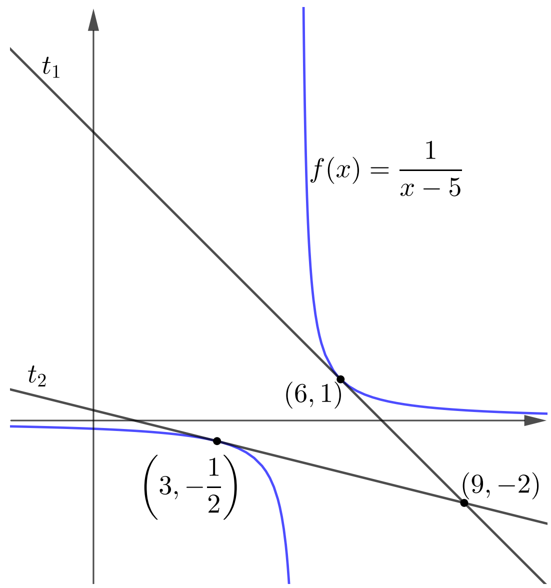
Kako je $T(9,-2) \in t$ imamo:

$$
\begin{aligned}
-2 - \frac{1}{x_0 - 5} &= \frac{-1}{(x_0-5)^2}(9 - x_0) \qquad / \cdot (x_0 - 5)^2 \\
-2(x_0 - 5)^2 - x_0 + 5 &= -9 + x_0 \\
-2x_0^2 + 20x_0 - 50 - x_0 + 5 &= -9 + x_0 \\
-2x_0^2 + 18x_0 - 36 &= 0 \quad / : (-2) \\
x_0^2 - 9x_0 + 18 &= 0 \\
x_0 = 6 \Rightarrow y_0 = 1 \quad &\text{i} \quad x_0 = 3 \Rightarrow y_0 = -\frac{1}{2}
\end{aligned}
$$

$$y - y_0 = f'(x_0)(x - x_0)$$

$y - 1 = -1(x - 6) \qquad y + \dfrac{1}{2} = -\dfrac{1}{4}(x - 3)$

$t_1 ... y = -x + 7 \qquad t_2 ... y = -\dfrac{1}{4}x + \dfrac{1}{4}$



Zadatak 6.8. Odredite jednadžbe tangenti na graf funkcije $f(x) = \dfrac{x-1}{x+3}$ paralelnih s pravcem $p \ldots y = x - 2$.

Rješenje:

$$\begin{aligned} f'(x) &= \frac{(x-1)'(x+3) - (x-1)(x+3)'}{(x+3)^2} \\ &= \frac{x+3-x+1}{(x+3)^2} \\ &= \frac{4}{(x+3)^2} \end{aligned}$$

Pronađimo koordinate dirališta $D(x_0, y_0)$.

$$t \ldots y - y_0 = f'(x_0)(x - x_0)$$

Kako je $t || p$ imamo da je $k_t = 1 = f'(x_0)$

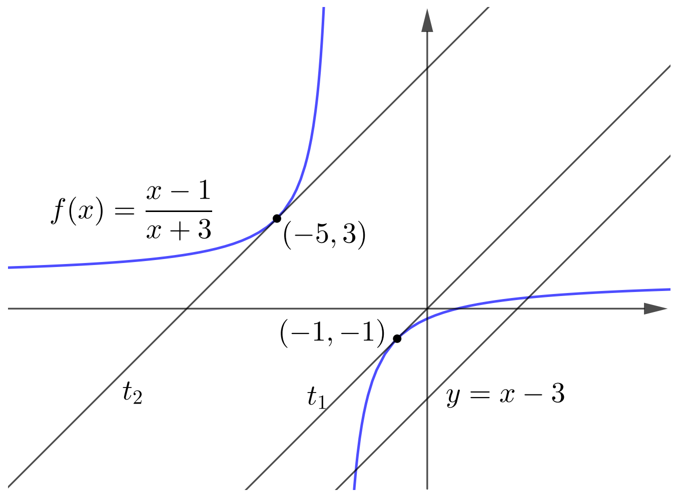
$$\begin{aligned} \frac{4}{(x_0+3)^2} &= 1 \qquad / \cdot (x_0+3)^2 \\ (x_0+3)^2 &= 4 \\ x_0 + 3 &= \pm 2 \end{aligned}$$

$$x_0 = -5 \Rightarrow y_0 = 3 \quad \text{i} \quad x_0 = -1 \Rightarrow y_0 = -1$$

$$y - y_0 = f'(x_0)(x - x_0)$$

$y - 3 = x + 5 \qquad y + 1 = x + 1$

$t_1 \ldots y = x + 8 \qquad t_2 \ldots y = x$



Zadatak 6.9. Odredite jednadžbe tangenti na graf funkcije $f(x) = x^3 - 3x^2 + 3x - 4$ paralelnih s pravcem $p ... y = 12x - 12$.

Rješenje:

$f'(x) = 3x^2 - 6x + 3$

$$t ... y - y_0 = f'(x_0)(x - x_0)$$

Pronađimo koordinate dirališta $D(x_0, y_0)$.

Kako je $t || p$ imamo da je $k_t = 12 = f'(x_0)$

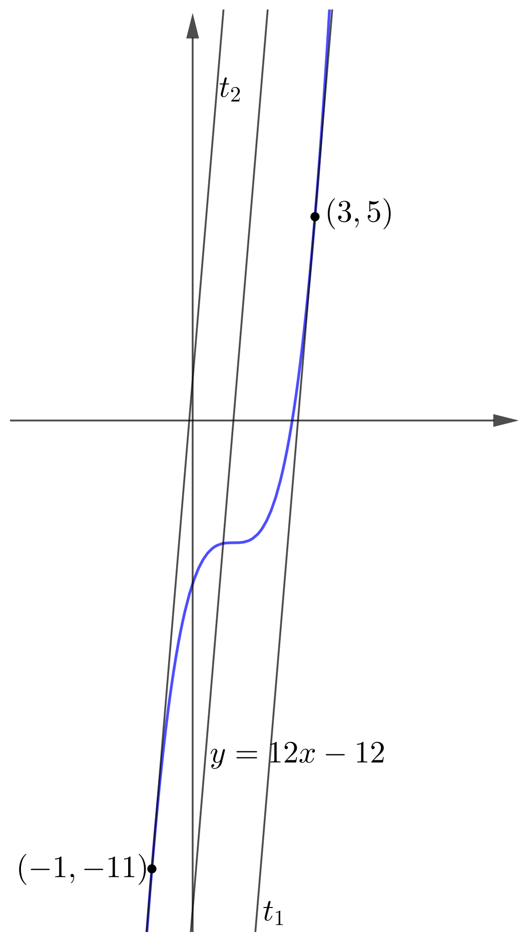
$$\begin{aligned} 3x_0^2 - 6x_0 + 3 &= 12 \quad / : 3 \\ x_0^2 - 2x_0 - 3 &= 0 \end{aligned}$$

$x_0 = 3 \Rightarrow y_0 = 5 \quad$ i $\quad x_0 = -1 \Rightarrow y_0 = -11$

$$y - y_0 = f'(x_0)(x - x_0)$$

$y - 5 = 12(x - 3) \qquad y + 11 = 12(x + 1)$

$t_1 ... y = 12x - 31 \qquad t_2 ... y = 12x + 1$



Zadatak 6.10. Odredite a i b tako da pravac $y = -2x + 13$ bude tangenta parabole $f(x) = -x^2 + ax + b$ u točki a) $D(3, 7)$ i b) $D(1, 11)$.

Rješenje:

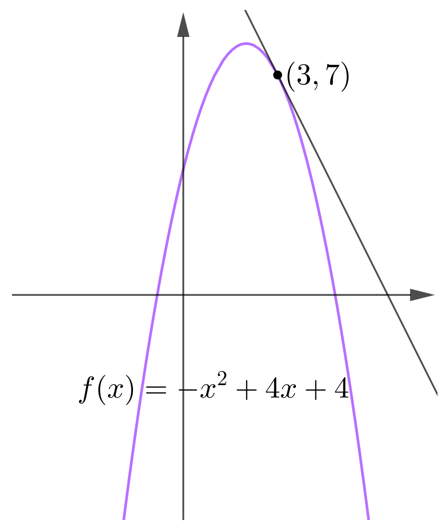
$f'(x) = -2x + a$

a)

$$\begin{aligned} f'(x_0) &= k \\ f'(3) &= -2 \\ -2 \cdot 3 + a &= -2 \\ a &= -2 + 6 \\ a &= 4 \end{aligned}$$

$$\begin{aligned} y_0 &= f(x_0) \\ 7 &= -3^2 + 4 \cdot 3 + b \\ 7 + 9 - 12 &= b \\ b &= 4 \end{aligned}$$

$$f(x) = -x^2 + 4x + 4$$

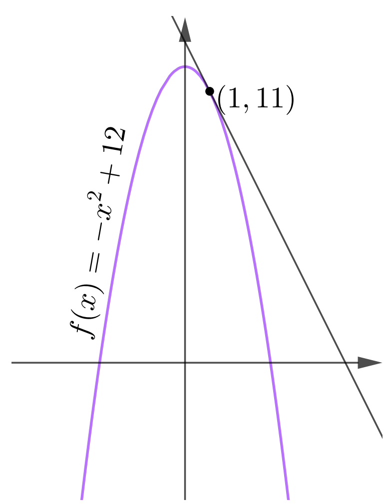


b)

$$\begin{aligned} f'(x_0) &= k \\ f'(1) &= -2 \\ -2 \cdot 1 + a &= -2 \\ a &= 0 \end{aligned}$$

$$\begin{aligned} y_0 &= f(x_0) \\ 11 &= -1^2 + 0 \cdot 1 + b \\ 11 + 1 &= b \\ b &= 12 \end{aligned}$$

$$f(x) = -x^2 + 12$$



Zadatak 6.11. Odredite jednadžbu normale na graf $f(x) = e^{2x} + e^x - 2$ u točki u kojoj graf siječe os x.

Rješenje:

Odredimo koordinate $D(x_0, 0)$.

$$\begin{aligned}
f(x_0) &= e^{2x_0} + e^{x_0} - 2 \qquad (t = e^{x_0}) \\
t^2 + t - 2 &= 0 \\
t^2 + 2t - t - 2 &= 0 \\
(t+2)(t-1) &= 0 \\
t = -2 \quad &\text{ili} \quad t = 1 \\
e^{x_0} = 1 \quad &\text{no} \quad e^{x_0} \neq -2, \text{ jer je } e^x > 0 \\
x_0 = 0 & \\
& D(0,0). \\
f'(x) &= 2e^{2x} + e^x \\
f'(0) &= 3
\end{aligned}$$

$$y - 0 = -\frac{1}{3}(x - 0)$$

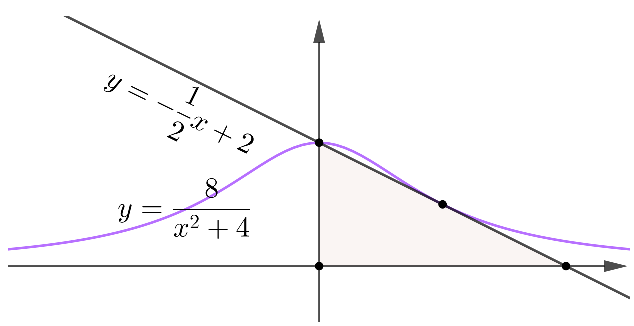
$$n...y = -\frac{1}{3}x$$

Zadatak 6.12. Zadana je funkcija $f(x) = \dfrac{8}{x^2+4}$ i $T(2,1)$ točka na grafu funkcije. Odredite površinu trokuta što ga zatvara tangenta na graf u toj točki s koordinatnim osima.

Rješenje:

Odredimo jednadžbu tangente u $D(2,1)$.

$$\begin{aligned}
f'(x_0) &= \frac{-16x}{(x^2+4)^2} \\
f'(2) &= \frac{-16 \cdot 2}{(2^2+4)^2} = -\frac{1}{2} \\
y - 1 &= -\frac{1}{2}(x - 2) \\
t...y &= -\frac{1}{2}x + 2
\end{aligned}$$



Odredimo sjecišta t s koordinatnim osima:

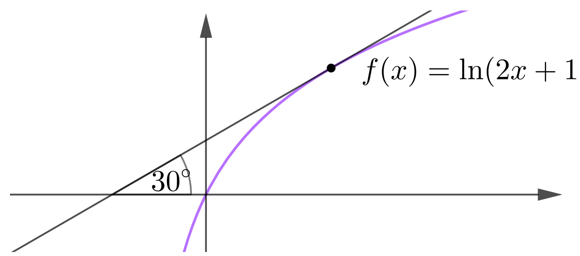
$$\begin{aligned}
0 &= -\frac{1}{2}x + 2 & y &= -\frac{1}{2} \cdot 0 + 2 & P_\Delta &= \frac{2 \cdot 4}{2} \\
x &= 4 & y &= 2 & &= 4 \\
&T_x(4,0) & &T_y(0,2) & &
\end{aligned}$$

Zadatak 6.13. U kojoj točki krivulje $y = \ln(2x+1)$ treba postaviti tangentu tako da ona zatvara kut $\alpha = 30°$ s osi x?

Rješenje:

Tražimo diralište $D(x_0, y_0)$ pri čemu je $f'(x_0) = k = \operatorname{tg}\alpha$.

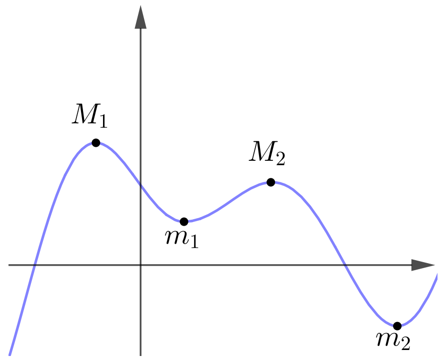
$$\begin{aligned}
\operatorname{tg} 30° &= \frac{1}{\sqrt{3}} \\
k &= \frac{1}{\sqrt{3}} \\
f'(x) &= \frac{2}{2x+1} \\
\frac{1}{\sqrt{3}} &= \frac{2}{2x_0+1} \\
2x_0+1 &= 2\sqrt{3} \\
2x_0 &= 2\sqrt{3}-1 \\
x_0 &= \sqrt{3}-\frac{1}{2} \\
f(x_0) &= \ln\left(2\left(\sqrt{3}-\frac{1}{2}\right)+1\right) \\
&= \ln 2\sqrt{3} \\
t ... y - \ln 2\sqrt{3} &= \frac{1}{\sqrt{3}}\left(x - \sqrt{3} + \frac{1}{2}\right)
\end{aligned}$$



6.3 Lokalni ekstremi

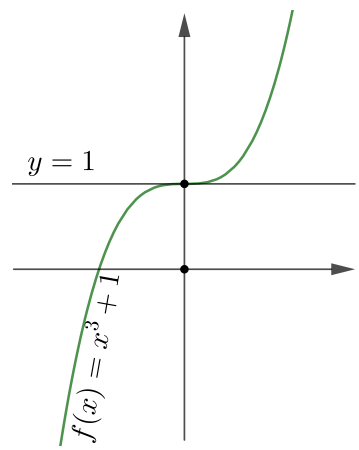
Definicija 6.4. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ i neka je $x_0 \in I$. Ako postoji $\varepsilon > 0$ takav da je $f(x_0) \leq f(x), \forall x \in \langle x_0 - \varepsilon, x_0 + \varepsilon\rangle$ onda kažemo da f u x_0 ima **lokalni minimum**.

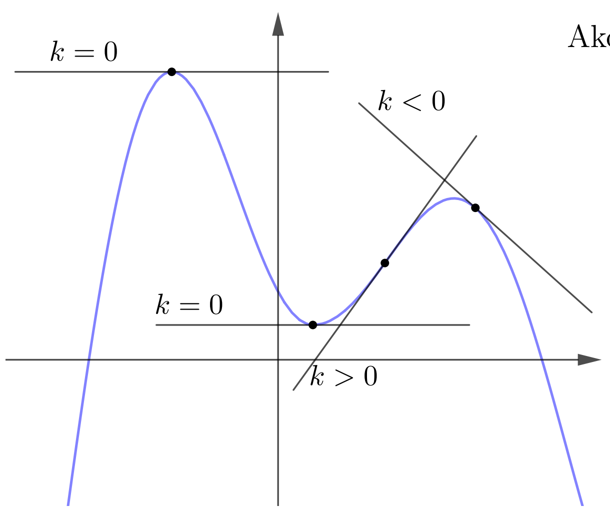
Definicija 6.5. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ i neka je $x_0 \in I$. Ako postoji $\varepsilon > 0$ takav da je $f(x_0) \geq f(x), \forall x \in \langle x_0 - \varepsilon, x_0 + \varepsilon\rangle$ onda kažemo da f u x_0 ima **lokalni maksimum**.



Nužan uvjet za lokalni ekstrem u x_0: $f'(x_0) = 0$. Ako f ima lokalni ekstrem u x_0 onda je $f'(x_0) = 0$ i za x_0 kažemo da je **stacionarna točka**.

No, to nije dovoljan uvijet: za npr. $f(x) = x^3 + 1$ u $x_0 = 0$ vrijedi $f'(x) = 3x^2$ i $f'(0) = 0$, no funkcija nema ekstrem u 0.





Ako funkcija f na intervalu I

- raste, onda je $f'(x) \geq 0, \forall x \in I$
- pada, onda je $f'(x) \leq 0, \forall x \in I$
- ima ekstrem u $x_0 \in I$, onda je $f'(x_0) = 0$

Neka je x_0 stacionarna točka, tj. $f'(x_0) = 0$. Ako je

- $f''(x_0) > 0$, onda f u x_0 ima lokalni minimum
- $f''(x_0) < 0$, onda f u x_0 ima lokalni maksimum
- $f''(x_0) = 0$, onda računamo derivacije dok ne dobijemo k takav da je $f^k(x_0) \neq 0$. Ako je
  - k paran, i
    - $f^k(x_0) < 0$, onda f u x_0 ima lokalni maksimum
    - $f^k(x_0) > 0$, onda f u x_0 ima lokalni minimum
  - k neparan, onda f u x_0 nema ekstrem

Zadatak 6.14. Odredite ekstreme i intervale rasta i pada funkcije:

a) $f(x) = x^2(x-12)^2$

b) $f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 1$

c) $f(x) = \dfrac{x^2-2x+2}{x-1}$

d) $f(x) = \dfrac{x^2}{1+x^2}$

e) $f(x) = \dfrac{4}{\sqrt{x^2+8}}$

f) $f(x) = x - \ln(1+x)$

g) $f(x) = x - \mathrm{arctg}x$

Rješenje: a)

$$\begin{aligned} f(x) &= x^2(x-12)^2 \\ &= x^2(x^2-24x+144) \\ &= x^4 - 24x^3 + 144x^2 \end{aligned}$$

$$f'(x) = 4x^3 - 72x^2 + 288x$$

Odredimo stacionarne točke:

$$\begin{aligned} 4x^3 - 72x^2 + 288x &= 0 \quad / : 4 \\ x^3 - 18x^2 + 72x &= 0 \\ x(x^2 - 18x + 72) &= 0 \\ x(x^2 - 12x - 6x + 72) &= 0 \\ x(x-12)(x-6) &= 0 \end{aligned}$$

$x_1 = 0,\ x_2 = 12,\ x_3 = 6$

	$-\infty$	0	6	12	$+\infty$
f'	$-$	$+$	$-$	$+$	
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	
		m	M	m	

Funkcija f ima u 0 i 12 lokalni minimum, a u 6 lokalni maksimum. Karakter ekstrema možemo odrediti i preko druge derivacije:

$$\begin{aligned}
f''(x) &= 12x^2 - 144x + 288 \\
f''(0) &= 288 > 0, \text{ pa } f \text{ u } 0 \text{ ima lokalni minimum.} \\
f''(6) &= -144 < 0, \text{ pa } f \text{ u } 6 \text{ ima lokalni maksimum.} \\
f''(12) &= 288 > 0, \text{ pa } f \text{ u } 12 \text{ ima lokalni minimum.}
\end{aligned}$$

b)

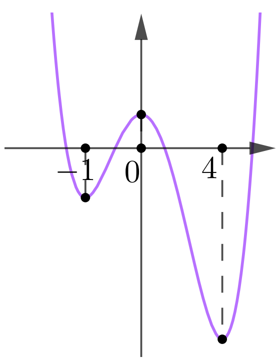
$$\begin{aligned}
f(x) &= \frac{1}{4}x^4 - x^3 - 2x^2 + 1 \\
f'(x) &= x^3 - 3x^2 - 4x
\end{aligned}$$

Odredimo stacionarne točke:

$$\begin{aligned}
x^3 - 3x^2 - 4x &= 0 \\
x(x^2 - 3x - 4) &= 0 \\
x(x^2 - 4x + x - 4) &= 0 \\
x(x-4)(x+1) &= 0
\end{aligned}$$

$x_1 = 0$, $x_2 = 4$, $x_3 = -1$

	$\langle -\infty, -1 \rangle$	$\langle -1, 0 \rangle$	$\langle 0, 4 \rangle$	$\langle 4, +\infty \rangle$
f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$
	m (u -1)	M (u 0)	m (u 4)	



Funkcija f ima u -1 i 4 lokalni minimum, a u 0 lokalni maksimum. Intervali rasta su $\langle -1, 0 \rangle$ i $\langle 4, +\infty \rangle$, a intervali pada su $\langle -\infty, -1 \rangle$ i $\langle 0, 4 \rangle$

c)

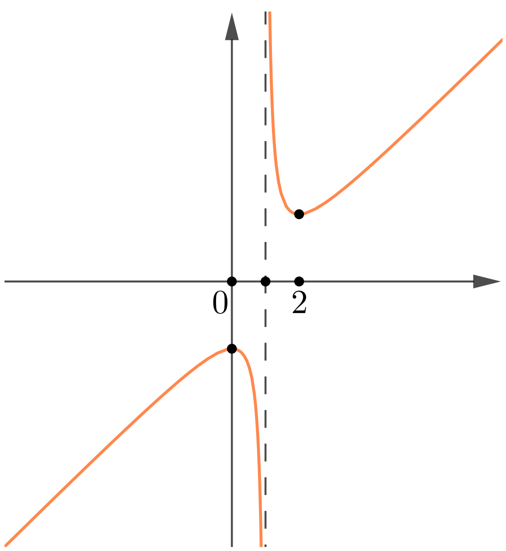
$$
\begin{aligned}
f(x) &= \frac{x^2-2x+2}{x-1} \\
D_f &= \mathbb{R}\setminus\{1\} \\
f'(x) &= \frac{(2x-2)(x-1)-(x^2-2x+2)}{(x-1)^2} \\
&= \frac{2x^2-2x-2x+2-x^2+2x-2}{(x-1)^2} \\
&= \frac{x^2-2x}{(x-1)^2}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
\frac{x^2-2x}{(x-1)^2} &= 0 \\
x^2-2x &= 0 \\
x(x-2) &= 0
\end{aligned}
$$

$x_1 = 0,\ x_2 = 2$

	$-\infty$	0	1	2	$+\infty$
f'	$+$	$-$	$-$	$+$	
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	
		M		m	



Funkcija f ima u 2 lokalni minimum, a u 0 lokalni maksimum. Intervali rasta su $\langle -\infty, 0\rangle$ i $\langle 2, +\infty\rangle$, a intervali pada su $\langle 0, 1\rangle$ i $\langle 1, 2\rangle$

d)

$$
\begin{aligned}
f(x) &= \frac{x^2}{1+x^2} \\
D_f &= \mathbb{R} \\
f'(x) &= \frac{2x}{(1+x^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{2x}{(1+x^2)^2} &= 0 \\
x_1 &= 0
\end{aligned}
$$

	$-\infty$	0	$+\infty$
f'	$-$	$+$	
f	$\searrow$	$\nearrow$	
		m	

Interval rasta je $\langle 0, +\infty\rangle$, a interval pada je $\langle -\infty, 0\rangle$. U točki $x = 0$ funkcija ima lokalni minimum.

e)

$$
\begin{aligned}
f(x) &= \frac{4}{\sqrt{x^2+8}} \\
D_f &= \mathbb{R} \\
f'(x) &= \frac{-4 \cdot \frac{1}{2\sqrt{x^2+8}} \cdot 2x}{x^2+8} \\
&= \frac{-4x}{(x^2+8)\sqrt{x^2+8}}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
\frac{-4x}{(x^2+8)\sqrt{x^2+8}} &= 0 \\
-4x &= 0 \\
x_1 &= 0
\end{aligned}
$$

	$-\infty$	0	$+\infty$
f'	$+$	$-$	
f	$\nearrow$	$\searrow$	
		M	

Interval pada je $\langle 0, +\infty\rangle$, a interval rasta je $\langle -\infty, 0\rangle$. U točki $x = 0$ funkcija ima lokalni maksimum.

f)

$$
\begin{aligned}
f(x) &= x - \ln(1+x) \\
D_f &= \langle -1, +\infty\rangle \\
f'(x) &= 1 - \frac{1}{1+x} \\
&= \frac{x}{1+x}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
\frac{x}{1+x} &= 0 \\
x_1 &= 0
\end{aligned}
$$

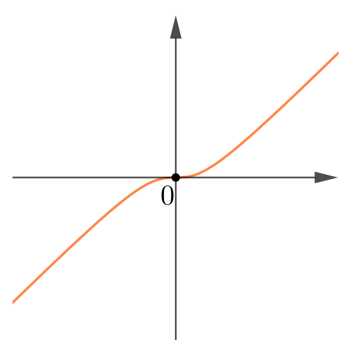
	-1	0	$+\infty$
f'	$-$	$+$	
f	$\searrow$	$\nearrow$	
		m	

g)

$$
\begin{aligned}
f(x) &= x - \operatorname{arctg} x \\
D_f &= \mathbb{R} \\
f'(x) &= 1 - \frac{1}{1+x^2} \\
&= \frac{x^2}{1+x^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{x^2}{1+x^2} &= 0 \\
x_1 &= 0
\end{aligned}
$$

	$-\infty$	0	$+\infty$
f'	$+$	$+$	
f	$\nearrow$	$\nearrow$	
		m	



Raste na cijeloj domeni. Nema lokalnih esktrema.

Zadatak 6.15. Ispitajte ekstreme funkcije $f(x) = x^4 \cdot e^{-x^2}$ pomoću f''.

Rješenje:

$$\begin{aligned} f(x) &= x^4 \cdot e^{-x^2}, \qquad D_f = \mathbb{R} \\ f'(x) &= 4x^3 \cdot e^{-x^2} + x^4 \cdot e^{-x^2} \cdot (-2x) \\ &= e^{-x^2}(4x^3 - 2x^5) \end{aligned}$$

Odredimo stacionarne točke:

$$\begin{aligned} e^{-x^2}(4x^3 - 2x^5) &= 0 / : e^{-x^2} \neq 0 \\ x^3(4 - 2x^2) &= 0 \end{aligned}$$

Kandidati: $x_1 = 0$, $x_2 = -\sqrt{2}$, $x_3 = \sqrt{2}$

$$\begin{aligned} f''(x) &= e^{-x^2} \cdot (-2x) \cdot (4x^3 - 2x^5) + e^{-x^2} \cdot (12x^2 - 10x^4) \\ &= e^{-x^2} \cdot (-8x^4 + 4x^6) + e^{-x^2} \cdot (12x^2 - 10x^4) \\ &= e^{-x^2}(4x^6 - 18x^4 + 12x^2) \end{aligned}$$

$$\begin{aligned} f''(\sqrt{2}) &= e^{-2} \cdot (-16) < 0 \\ f''(-\sqrt{2}) &= e^{-2} \cdot (-16) < 0 \end{aligned}$$

f u $\pm\sqrt{2}$ ima lok.maksimum

$$\begin{aligned} f''(0) &= 0 \\ f'''(x) &= e^{-x^2}(-8x^7 + 60x^5 - 96x^3 + 24x) \\ f'''(0) &= 0 \\ f^{IV}(x) &= e^{-x^2}(16x^8 - 176x^6 + 492x^4 - 336x^2 + 24) \\ f^{IV}(0) &= 24 > 0 \end{aligned}$$

f u 0 ima lok. minimum

6.4 Konveksnost i konkavnost

Definicija 6.6. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ i neka je $J \subseteq I$. Ako vrijedi: $\forall x_1, x_2 \in J$ je

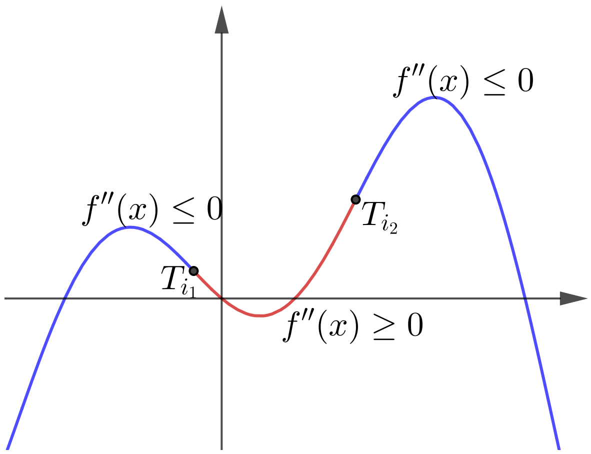
$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2},$$

onda za f kažemo da je **konveksna** na J.

Definicija 6.7. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ i neka je $J \subseteq I$. Ako vrijedi: $\forall x_1, x_2 \in J$ je

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2},$$

onda za f kažemo da je **konkavna** na J.



Točka T u kojoj graf mjenja oblik iz konveksnog u konkavni i obratno naziva se **točka infleksije**. Ako je:

- $f''(x) \geq 0$, onda je f konveksna;
- $f''(x) \leq 0$, onda je f konkavna;
- $f''(x_0) = 0$, onda je x_0 kandidat za točku infleksije.

Nužan uvjet za točku infleksije u x_0 je $f''(x_0) = 0$, no to nije i dovoljan uvijet.

Zadatak 6.16. Odredite točke infleksije i intervale konveksnosti i konkavnosti za:

a) $f(x) = (x^2 - 3x + 2)e^x$

b) $f(x) = x \ln^2 x$

c) $f(x) = \dfrac{1}{3x^2 + 1}$

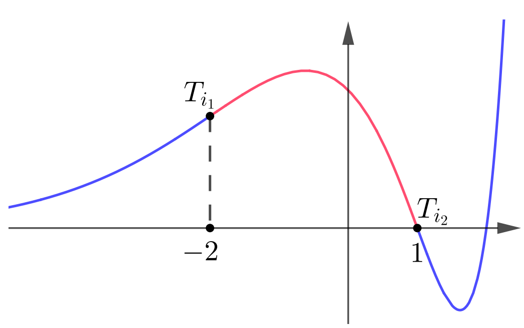
Rješenje: a)

$$\begin{aligned} f(x) &= (x^2 - 3x + 2)e^x, \qquad D_f = \mathbb{R} \\ f'(x) &= (2x - 3)e^x + (x^2 - 3x + 2)e^x \\ f''(x) &= 2e^x + (2x - 3)e^x + (2x - 3)e^x + (x^2 - 3x + 2)e^x \\ &= (2 + 2x - 3 + 2x - 3 + x^2 - 3x + 2)e^x \\ &= (x^2 + x - 2)e^x \end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned} (x^2 + x - 2)e^x &= 0 / : e^x \neq 0 \\ x^2 + 2x - x - 2 &= 0 \\ x(x + 2) - (x + 2) &= 0 \\ (x - 1)(x + 2) &= 0 \end{aligned}$$

Kandidati: $x_1 = 1$, $x_2 = -2$



	$-\infty$	-2	1	$+\infty$
f''	$+$	$-$	$+$	
f	$\cup$	$\cap$	$\cup$	

Intervali konveksnosti su $\langle -\infty, -2\rangle$ i $\langle 1, +\infty\rangle$, a interval konkavnosti je $\langle -2, 1\rangle$.Točke infleksije su $T_{i_1}(-2, 12e^{-2})$ i $T_{i_2}(1, 0)$.

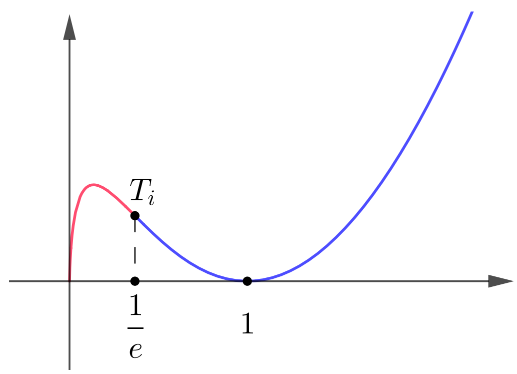
b)

$$\begin{aligned}
f(x) &= x\ln^2 x, \qquad D_f = \langle 0, +\infty\rangle\\
f'(x) &= \ln^2 x + 2x\ln x\cdot\frac{1}{x}\\
&= \ln^2 x + 2\ln x\\
f''(x) &= 2\ln x\cdot\frac{1}{x}+\frac{2}{x}\\
&= \frac{2}{x}(\ln x + 1)
\end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned}
\frac{2}{x}(\ln x+1) &= 0 / \cdot\frac{x}{2} \neq 0\\
\ln x &= -1
\end{aligned}$$

Kandidati: $x_1 = \frac{1}{e}$



	0	$\frac{1}{e}$	$+\infty$
f''	$-$	$+$	
f	$\cap$	$\cup$	

Interval konkavnosti je $\left\langle 0, \frac{1}{e}\right\rangle$, a interval konveksnosti je $\left\langle \frac{1}{e}, +\infty\right\rangle$. Točka infleksije je $T_i\left(\frac{1}{e}, \frac{1}{e}\right)$.

c)

$$\begin{aligned}
f(x) &= \frac{1}{3x^2+1}, \qquad D_f = \mathbb{R}\\
f'(x) &= \frac{-6x}{(3x^2+1)^2}\\
f''(x) &= \frac{-6(3x^2+1)^2 + 6x\cdot 2(3x^2+1)\cdot 6x}{(3x^2+1)^4}\\
&= \frac{(3x^2+1)\left[-6(3x^2+1)+72x^2\right]}{(3x^2+1)^4}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-18x^2 - 6 + 72x^2}{(3x^2+1)^3} \\
&= \frac{54x^2 - 6}{(3x^2+1)^3}
\end{aligned}
$$

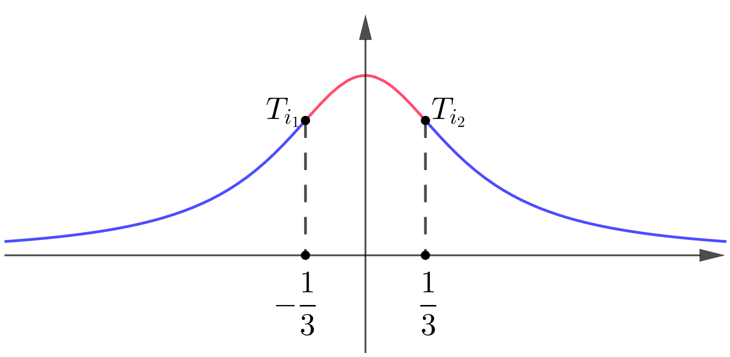
Odredimo točke infleksije:

$$
\begin{aligned}
\frac{54x^2-6}{(3x^2+1)^3} &= 0 / \cdot \frac{(3x^2+1)^3}{6} \neq 0 \\
9x^2 &= 1 \\
x^2 &= \frac{1}{9}
\end{aligned}
$$

Kandidati: $x_1 = \frac{1}{3}$, $x_2 = -\frac{1}{3}$

	$-\infty$ do $-\frac{1}{3}$	$-\frac{1}{3}$ do $\frac{1}{3}$	$\frac{1}{3}$ do $+\infty$
f''	$+$	$-$	$+$
f	$\cup$	$\cap$	$\cup$

Intervali konveksnosti su $\left\langle -\infty, -\frac{1}{3} \right\rangle$ i $\left\langle -\frac{1}{3}, +\infty \right\rangle$, a interval konkavnosti je $\left\langle -\frac{1}{3}, \frac{1}{3} \right\rangle$. Točke infleksije su $T_{i_1}\left(-\frac{1}{3}, \frac{3}{4}\right)$ i $T_{i_2}\left(\frac{1}{3}, \frac{3}{4}\right)$.



6.5 Primjena ekstrema

Zadatak 6.17. Konopcem duljine 12 m treba omeđiti dio ravnog terena oblika kružnog isječka najveće moguće površine. Kolika je ta površina?

Rješenje:

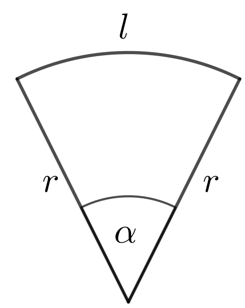
$l = \frac{\alpha}{2\pi} \cdot O_\circ = \frac{\alpha}{2\pi} \cdot 2r\pi = \alpha r$

$$P_{KI} = \frac{\alpha}{2\pi} \cdot P_\circ = \frac{\alpha}{2\pi} r^2\pi = \frac{\alpha r^2}{2} = \frac{rl}{2}$$

Kako je, prema uvjetu zadatka $l + 2r = 12$, odnosno $l = 12 - 2r$ imamo:

$$\begin{aligned} P_{KI} &= \frac{rl}{2} \\ P_{KI}(r) &= \frac{r(12-2r)}{2} \\ &= 6r - r^2 \end{aligned}$$

Maksimum ove funkcije je najveća površina koju tražimo.



$$\begin{aligned} P'_{KI}(r) &= 6 - 2r \\ 0 &= 6 - 2r \\ r &= 3 \\ P''_{KI}(r) &= -2 \\ P''_{KI}(3) &= -2 < 0 \end{aligned}$$

P u 3 zaista ima maksimum.

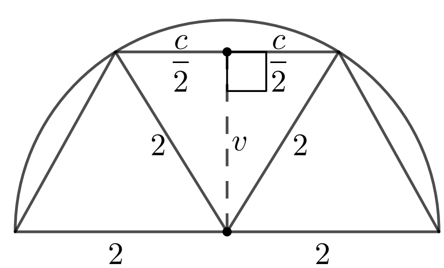
Tražena površina je:

$$\begin{aligned} P_{KI}(3) &= \frac{3(12 - 2 \cdot 3)}{2} \\ &= 9 \text{ cm}^2 \end{aligned}$$

Zadatak 6.18. U polukrug polumjera $r = 2$ cm upišite jednakokračni trapez maksimalne površine takav da je $a = 2r$. Kolika je površina tog trapeza?

Rješenje:

$$\begin{aligned} a &= 4 \\ v^2 &= 2^2 - \left(\frac{c}{2}\right)^2 \\ v &= \sqrt{4 - \frac{c^2}{4}} \end{aligned}$$



$$
\begin{aligned}
P &= \frac{(a+c)}{2}v \\
&= \frac{(4+c)}{2}\sqrt{4-\frac{c^2}{4}} \\
&= \frac{(4+c)}{2}\sqrt{\frac{16-c^2}{4}} \\
&= \frac{1}{4}(4+c)\sqrt{16-c^2}
\end{aligned}
$$

Maksimum ove funkcije je najveća površina koju tražimo.

$$
\begin{aligned}
P' &= \frac{1}{4}\left(\sqrt{16-c^2}+(4+c)\frac{1}{2\sqrt{16-c^2}}(-2c)\right) \\
&= \frac{1}{4}\left(\sqrt{16-c^2}-\frac{c(4+c)}{\sqrt{16-c^2}}\right) \\
&= \frac{1}{4}\left(\frac{16-c^2-4c-c^2}{\sqrt{16-c^2}}\right) \\
&= \frac{1}{4}\left(\frac{16-4c-2c^2}{\sqrt{16-c^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{1}{4}\left(\frac{16-4c-2c^2}{\sqrt{16-c^2}}\right) \qquad / \cdot 4\sqrt{16-c^2} \\
0 &= 16-4c-2c^2 \\
0 &= c^2+2c-8
\end{aligned}
$$

$c_1 = -4$ i $c_2 = 2$

$c_1 = -4$ odbacujemo jer $c > 0$

	-4		2		$+\infty$
P'		$+$		$-$	
P		$\nearrow$	M	$\searrow$	

P u 2 zaista ima maksimum.

Tražena površina je:

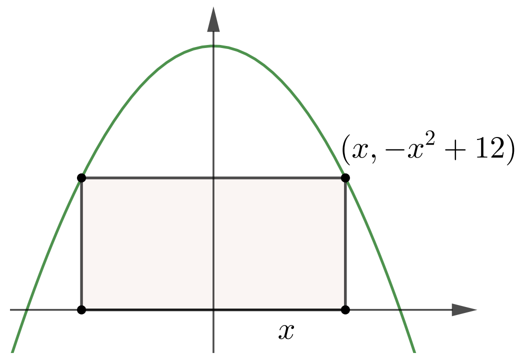
$$
\begin{aligned}
P(2) &= \frac{1}{4}(4+2)\sqrt{16-2^2} \\
&= 3\sqrt{3}
\end{aligned}
$$

Zadatak 6.19. U lik omeđen parabolom $y = -x^2 + 12$ i osi x upisujemo pravokutnik maksimalne površine. Kolika je površina tog pravokutnika?

Rješenje:

$$
\begin{aligned}
P &= ab \\
P(x) &= 2x(-x^2+12) \\
&= -2x^3+24x
\end{aligned}
$$

Maksimum ove funkcije je najveća površina koju tražimo.



$$
\begin{aligned}
P'(x) &= -6x^2+24 \\
0 &= -6x^2+24 \\
x^2 &= 4
\end{aligned}
$$

$x_1 = -2$ i $x_2 = 2$

$x_1 = -2$ odbacujemo jer $x > 0$

$$
\begin{aligned}
P''(x) &= -12x \\
P''(2) &= -24 < 0
\end{aligned}
$$

P u 2 zaista ima maksimum.

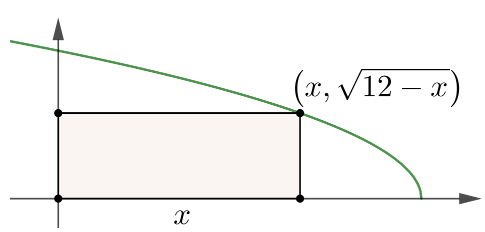
$$
\begin{aligned}
P(2) &= -2 \cdot 2^3 + 24 \cdot 2 \\
P(2) &= 32
\end{aligned}
$$

Zadatak 6.20. U lik omeđen grafom funkcije $f(x) = \sqrt{12-x}$ i koordinatnim osima upisujemo pravokutnik maksimalne površine. Kolika je površina tog pravokutnika?

Rješenje:

$$
\begin{aligned}
P &= ab \\
P(x) &= x\sqrt{12-x}
\end{aligned}
$$

Maksimum ove funkcije je najveća površina koju tražimo.



$$
\begin{aligned}
P'(x) &= \sqrt{12-x} + x \cdot \frac{1}{2\sqrt{12-x}} \cdot (-1) \\
&= \frac{2(12-x)-x}{2\sqrt{12-x}} \\
&= \frac{24-3x}{2\sqrt{12-x}}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{24-3x}{2\sqrt{12-x}} \quad / \cdot 2\sqrt{12-x} \\
0 &= 24-3x \\
x &= 8
\end{aligned}
$$

$$
\begin{array}{c||c|c|}
 & 0 \quad & 8 \quad & 12 \\
\hline
P' & + & - & \\
P & \nearrow & \searrow & \\
 & & M &
\end{array}
$$

$$
\begin{aligned}
P(8) &= 8\sqrt{12-8} \\
&= 16
\end{aligned}
$$

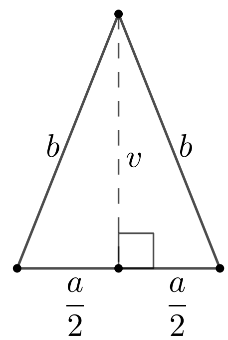
P u 8 zaista ima maksimum.

Zadatak 6.21. Žicu duljine 12 m savijamo u oblik jednakokračnog trokuta. Kolika je maksimalna površina tog trokuta?

Rješenje:

$$
\begin{aligned}
P &= \frac{av}{2} \\
v^2 &= b^2 - \frac{a^2}{4} \\
v &= \sqrt{\frac{4b^2-a^2}{4}}
\end{aligned}
$$

$$
\begin{aligned}
o &= a+2b \\
12 &= a+2b \\
a &= 12-2b
\end{aligned}
$$



$$
\begin{aligned}
P(b) &= \frac{(12-2b)\left(\sqrt{\frac{4b^2-(12-2b)^2}{4}}\right)}{2} \\
&= \frac{2\,(6-b)\left(\sqrt{\frac{4b^2-144+48b-4b^2}{4}}\right)}{2} \\
&= (6-b)\sqrt{\frac{4(12b-36)}{4}} \\
&= (6-b)\sqrt{12b-36}
\end{aligned}
$$

Maksimum ove funkcije je najveća površina koju tražimo.

$$
\begin{aligned}
P'(b) &= -\sqrt{12b-36} + \frac{12(6-b)}{2\sqrt{12b-36}} \\
&= -\sqrt{12b-36} + \frac{6(6-b)}{\sqrt{12b-36}} \\
&= \frac{-(12b-36)+6(6-b)}{\sqrt{12b-36}} \\
&= \frac{-18b+72}{\sqrt{12b-36}}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{-18b+72}{\sqrt{12b-36}} \quad / \cdot \sqrt{12b-36} \\
0 &= -18b+72 \\
b &= 4
\end{aligned}
$$

	3		4		12
P'		$+$		$-$	
P		$\nearrow$	M	$\searrow$	

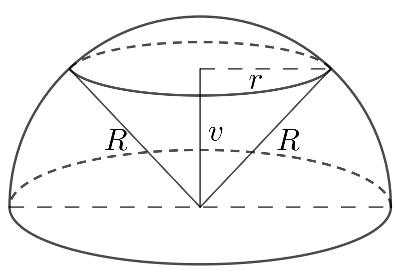
P u 4 zaista ima maksimum.

$$
\begin{aligned}
P(4) &= (6-4)\sqrt{12\cdot 4-36} \\
&= 2\sqrt{12} \\
&= 4\sqrt{3}
\end{aligned}
$$

Zadatak 6.22. U polukuglu radijusa R upisan je stožac s vrhom u centru baze polukugle. Odredite radijus baze stošca maksimalnog volumena.

Rješenje:

$$
\begin{aligned}
R^2 &= v^2 + r^2 \\
r^2 &= R^2 - v^2
\end{aligned}
$$



$$
\begin{aligned}
V_S &= \frac{1}{3} B \cdot v \\
&= \frac{1}{3} r^2 \pi \cdot v \\
&= \frac{1}{3}\left(R^2 - v^2\right)\pi \cdot v \\
V(v) &= \frac{\pi}{3}\left(R^2 v - v^3\right)
\end{aligned}
$$

Maksimum ove funkcije je najveća površina koju tražimo.

$$\begin{aligned}
V'(v) &= \frac{\pi}{3}\left(R^2 - 3v^2\right) \\
0 &= \frac{\pi}{3}\left(R^2 - 3v^2\right) \\
v^2 &= \frac{R^2}{3} \\
v &= \frac{R\sqrt{3}}{3} \\
V''(v) &= \frac{\pi}{3}(-6v) \\
&= -2\pi v \\
V''\left(\frac{R\sqrt{3}}{3}\right) &= -2\pi\frac{R\sqrt{3}}{3} < 0
\end{aligned}$$

V u $\dfrac{R\sqrt{3}}{3}$ ima maksimum.

Traženi radijus je:

$$\begin{aligned}
r^2 &= R^2 - v^2 \\
r^2 &= R^2 - \left(\frac{R\sqrt{3}}{3}\right)^2 \\
&= R^2 - \frac{3R^2}{9} \\
&= \frac{2}{3}R^2 \\
r &= \frac{\sqrt{2}}{\sqrt{3}}R \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{\sqrt{6}}{3}R
\end{aligned}$$

Zadatak 6.23. Na paraboli $y = x^2 + 1$ odredite onu točku koja je nabliža točki $A\left(0, \dfrac{5}{2}\right)$.

Rješenje:

Neka je $T(x, y)$ točka na paraboli $y = x^2 + 1$.

$$\begin{aligned}
\mathrm{d}(T, A) &= \sqrt{(x-0)^2 + (y - \tfrac{5}{2})^2} \\
&= \sqrt{x^2 + (x^2 + 1 - \tfrac{5}{2})^2} \\
&= \sqrt{x^2 + (x^2 - \tfrac{3}{2})^2} \\
&= \sqrt{x^2 + x^4 - 3x^2 + \tfrac{9}{4}} \\
d(x) &= \sqrt{x^4 - 2x^2 + \tfrac{9}{4}}
\end{aligned}$$

Maksimum ove funkcije je najveća površina koju tražimo.

$$\begin{aligned}
d'(x) &= \frac{4x^3 - 4x}{2\sqrt{x^4 - 2x^2 + \frac{9}{4}}} \\
0 &= \frac{4x^3 - 4x}{2\sqrt{x^4 - 2x^2 + \frac{9}{4}}} \\
0 &= 4x(x^2 - 1)
\end{aligned}$$

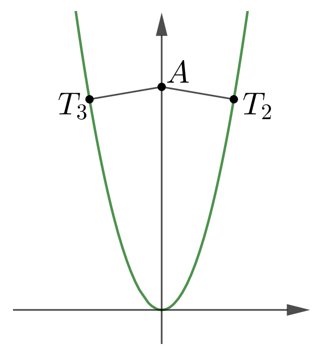
Kandidati: $x_1 = 0$, $x_2 = 1$, $x_3 = -1$

$$T_1(0,1) \implies d(T_1, A) = \sqrt{(0-0)^2 + (1-\tfrac{5}{2})^2} = \frac{3}{2}$$

$$T_2\,(1,2) \implies d(T_2, A) = \sqrt{(1-0)^2 + \left(2 - \frac{5}{2}\right)^2} = \frac{\sqrt{5}}{2}$$

$$T_3\,(-1,2) \implies d(T_2, A) = \sqrt{(-1-0)^2 + \left(2 - \frac{5}{2}\right)^2} = \frac{\sqrt{5}}{2}$$

Najbliže paraboli su točke T_2 i T_3.



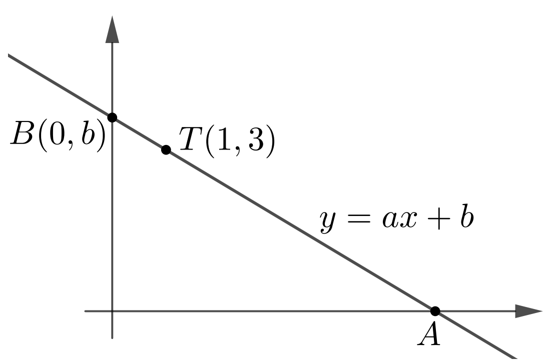
Zadatak 6.24. Točkom $T(1,3)$ polažemo padajući pravac koji s koordinatnim osima zatvara trokut najmanje površine. Koliki je koeficijent smjera tog pravca?

Rješenje:

Točka B je sjecište pravca p i osi $y \implies B(0,b)$.

Točka A je sjecište pravca p i osi $x \implies A\left(-\frac{b}{a}, 0\right)$.

Kako je $T \in p$ imamo $3 = a \cdot 1 + b$, odnosno $b = 3 - a$.



$$\begin{aligned}
P_\Delta &= \frac{1}{2}\cdot\frac{-b}{a}\cdot b\\
&= \frac{1}{2}\cdot\frac{-b^2}{a}\\
P(a) &= \frac{-(3-a)^2}{2a}\\
&= \frac{-9+6a-a^2}{2a}\\
&= \frac{-9}{2a}+3-\frac{a}{2}
\end{aligned}$$

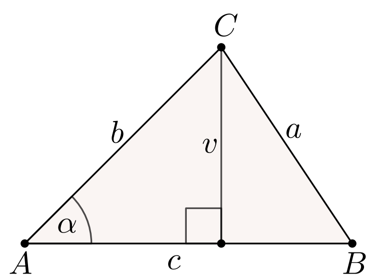
$$\begin{aligned}
P'(a) &= \frac{9}{2a^2}-\frac{1}{2}\\
0 &= \frac{9}{2a^2}-\frac{1}{2}\\
\frac{1}{2} &= \frac{9}{2a^2}\\
a^2 &= 9\\
a &= -3
\end{aligned}$$

$$\begin{aligned}
P''(a) &= -\frac{9}{a^3}\\
P''(-3) &= -\frac{9}{(-3)^3}\\
&= \frac{1}{3}>0
\end{aligned}$$

Minimalna površina dobije se za pravac $y=-3x+6$. $k=-3$

Zadatak 6.25. Neka je trokut ΔABC takav da je $\alpha=\frac{\pi}{6}$, te $b+c=100$ cm. Odredite duljinu stranice c tako da površina trokuta bude maksimalna.

Rješenje:



$$\begin{aligned}
\sin\frac{\pi}{6} &= \frac{v}{b}\\
\frac{1}{2} &= \frac{v}{b}\\
v &= \frac{1}{2}b
\end{aligned}$$

$$\begin{aligned}
P_\Delta &= \frac{1}{2}\cdot c\cdot v\\
&= \frac{1}{2}\cdot c\cdot\frac{1}{2}b\\
&= \frac{1}{4}\cdot c\cdot(100-c)\\
P(c) &= \frac{1}{4}\cdot(100c-c^2)
\end{aligned}$$

$$\begin{aligned}
P'(c) &= \frac{1}{4}\cdot(100-2c)\\
0 &= \frac{1}{4}\cdot(100-2c)\\
2c &= 100\\
c &= 50
\end{aligned}$$

$$\begin{aligned}
P''(c) &= \frac{1}{4}\cdot(-2)\\
&= -\frac{1}{2}\\
P''(50) &= -\frac{1}{2}<0
\end{aligned}$$

P u 50 ima maksimum.

$$\begin{aligned}
P(50) &= \frac{1}{4}\cdot(100\cdot 50-50^2)\\
&= \frac{1}{4}\cdot(5000-2500)\\
&= \frac{2500}{4}\\
&= 625\text{cm}^2
\end{aligned}$$

6.6 L'Hospitalovo pravilo

Neodređeni oblici limesa su:

- $\frac{\infty}{\infty}$, $\frac{0}{0}$

- $0 \cdot \infty$
- $\infty - \infty$
- 0^0, 1^∞, ∞^0

Ako je $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ oblika $\dfrac{0}{0}$ ili $\dfrac{\infty}{\infty}$, onda je

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Ako je $\lim\limits_{x\to x_0} f(x)\cdot g(x)$ oblika $0\cdot\infty$, onda je $\lim\limits_{x\to x_0} \dfrac{f(x)}{\frac{1}{g(x)}}$ oblika $\dfrac{0}{0}$, a $\lim\limits_{x\to x_0} \dfrac{g(x)}{\frac{1}{f(x)}}$ oblika $\dfrac{\infty}{\infty}$.

Zadatak 6.26. Izračunajte sljedeće limese:

a) $\lim\limits_{x\to 0} \dfrac{(1+x)^n - 1}{x}$

b) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$

c) $\lim\limits_{x\to 0} \dfrac{\operatorname{ch} x - 1}{1 - \cos x}$

d) $\lim\limits_{x\to +\infty} \dfrac{x^3}{e^{2x}}$

e) $\lim\limits_{x\to +\infty} \dfrac{\ln x}{x^n}$

f) $\lim\limits_{x\to 4} \dfrac{x^2 - 16}{x^2 + x - 20}$

g) $\lim\limits_{x\to 1} \dfrac{\ln x}{x - 1}$

h) $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\sin x}$

i) $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x}$

j) $\lim\limits_{x\to t} \dfrac{\sin x - \sin t}{x - t}$

k) $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{\sin^3 x}$

l) $\lim\limits_{x\to 0} \dfrac{\ln 2 - \ln x}{\ln 2 + \ln x}$

m) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

n) $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{x}{\ln x}\right)$

o) $\lim\limits_{x\to 0^+} (\operatorname{sh} x \cdot \ln x)$

p) $\lim\limits_{x\to 0} \left(x \cdot \ln^2 x\right)$

r) $\lim\limits_{x\to +\infty} \left(x \cdot \ln\left(e + \dfrac{1}{x}\right) - x\right)$

s) $\lim\limits_{x\to +\infty} \left(x \cdot e^{-\frac{1}{x^2}} - x\right)$

t) $\lim\limits_{x\to +\infty} (x^2 \cdot 2^{-x})$

Rješenje:

$$\text{a) } \lim_{x\to 0} \frac{(1+x)^n - 1}{x} = \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{n(1+x)^{n-1}}{1}$$

$$
\begin{aligned}
&= n \cdot \lim_{x\to 0} \frac{(1+x)^{n-1}}{1} \\
&= n
\end{aligned}
$$

$$
\begin{aligned}
\text{b) } \lim_{x\to 0} \frac{x-\sin x}{x^3} &= \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{\sin x}{6x} \\
&= \frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} \\
&= \frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
\text{c) } \lim_{x\to 0} \frac{\operatorname{ch} x - 1}{1-\cos x} &= \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{\operatorname{sh} x}{\sin x} = \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{\operatorname{ch} x}{\cos x} \\
&= \frac{1}{1} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\text{d) } \lim_{x\to +\infty} \frac{x^3}{e^{2x}} &= \left(\frac{+\infty}{+\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to +\infty} \frac{3x^2}{2e^{2x}} = \left(\frac{+\infty}{+\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to +\infty} \frac{6x}{4e^{2x}} = \left(\frac{+\infty}{+\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to +\infty} \frac{6}{8e^{2x}} \\
&= \frac{3}{4} \lim_{x\to +\infty} \frac{1}{e^{2x}} \\
&= \frac{3}{4} \cdot 0 = 0
\end{aligned}
$$

$$\text{e) } \lim_{x\to+\infty}\frac{\ln x}{x^n} \;=\; \left(\frac{+\infty}{+\infty}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to+\infty}\frac{\frac{1}{x}}{nx^{n-1}}$$

$$= \frac{1}{n}\lim_{x\to+\infty}\frac{1}{x^n}$$

$$= \frac{1}{n}\cdot 0$$

$$= 0$$

$$\text{f) } \lim_{x\to4}\frac{x^2-16}{x^2+x-20} \;=\; \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to4}\frac{2x}{2x+1}$$

$$= \frac{8}{9}$$

$$\text{g) } \lim_{x\to1}\frac{\ln x}{x-1} \;=\; \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to1}\frac{\frac{1}{x}}{1}$$

$$= 1$$

$$\text{h) } \lim_{x\to0}\frac{e^x-e^{-x}}{\sin x} \;=\; \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to0}\frac{e^x+e^{-x}}{\cos x}$$

$$= \frac{2}{1}$$

$$= 2$$

$$\text{i) } \lim_{x\to0}\frac{a^x-b^x}{x} \;=\; \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to0}\frac{a^x\ln a-b^x\ln b}{1}$$

$$= \ln\frac{a}{b}$$

j) $\displaystyle\lim_{x\to t}\frac{\sin x-\sin t}{x-t} = \left(\frac{0}{0}\right)$

$$=^{\text{L'H}} \lim_{x\to t}\frac{\cos x}{1}$$

$$= \cos t$$

k) $\displaystyle\lim_{x\to 0}\frac{\operatorname{tg} x-\sin x}{\sin^3 x} = \left(\frac{0}{0}\right)$

$$=^{\text{L'H}} \lim_{x\to 0}\frac{\frac{1}{cos^2 x}-\cos x}{3\sin^2 x\cos x}$$

$$= \lim_{x\to 0}\frac{1-\cos^3 x}{3\sin^2 x\cos^3 x} = \left(\frac{0}{0}\right)$$

$$=^{\text{L'H}} \lim_{x\to 0}\frac{3\cos^2 x\sin x}{6\sin x\cos^4 x-9\sin^3 x\cos^2 x}$$

$$= \lim_{x\to 0}\frac{3\cos^2 x\sin x}{3\cos^2 x\sin x(2\cos^2 x-3\sin^2 x)}$$

$$= \lim_{x\to 0}\frac{1}{2\cos^2 x-3\sin^2 x}$$

$$= \frac{1}{2}$$

l) $\displaystyle\lim_{x\to 0}\frac{\ln 2-\ln x}{\ln 2+\ln x} = \left(\frac{\infty}{\infty}\right)$

$$=^{\text{L'H}} \lim_{x\to 0}\frac{-\frac{1}{x}}{\dfrac{1}{x}}$$

$$= -1$$

m) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = (+\infty-(+\infty))$

$$= \lim_{x\to 0^+}\frac{\sin x-x}{x\sin x} = \left(\frac{0}{0}\right)$$

$$=^{\text{L'H}} \lim_{x\to 0^+}\frac{\cos x-1}{\sin x+x\cos x} = \left(\frac{0}{0}\right)$$

$$=^{\text{L'H}} \lim_{x\to 0^+}\frac{-\sin x}{\cos x+\cos x-x\sin x}$$

$$
\begin{aligned}
&= \frac{0}{2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{n) } \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right) &= (\infty-\infty) \\
&= \lim_{x\to 1}\frac{1-x}{\ln x} = \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to 1}\frac{-1}{\frac{1}{x}} \\
&= \lim_{x\to 1}(-x) \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
\text{o) } \lim_{x\to 0^+}(\operatorname{sh} x\cdot \ln x) &= (0\cdot\infty) \\
&= \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\operatorname{sh} x}} = \left(\frac{\infty}{\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-\operatorname{ch} x}{\operatorname{sh}^2 x}} \\
&= \lim_{x\to 0^+}\frac{\operatorname{sh}^2 x}{-x\operatorname{ch} x} \\
&= -\lim_{x\to 0^+}\frac{\operatorname{sh} x}{\operatorname{ch} x}\cdot \underbrace{\lim_{x\to 0^+}\frac{\operatorname{sh} x}{x}}_{1} \\
&= -\lim_{x\to 0^+}\operatorname{th} x \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{p) } \lim_{x\to 0}\left(x\ln^2 x\right) &= (0\cdot\infty) \\
&= \lim_{x\to 0}\frac{\ln^2 x}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to 0}\frac{\frac{2\ln x}{x}}{-\frac{1}{x^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{x\to 0}\frac{2\ln x}{-\frac{1}{x}} = \left(\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to 0}\frac{\frac{2}{x}}{\frac{1}{x^2}}\\
&= \lim_{x\to 0}(2x)\\
&= 0
\end{aligned}
$$

r) $$
\begin{aligned}
\lim_{x\to+\infty}\left[x\ln\left(e+\frac{1}{x}\right)-x\right] &= \lim_{x\to+\infty}\left[x\left(\ln\left(e+\frac{1}{x}\right)-1\right)\right]\\
&= (\infty\cdot 0)\\
&= \lim_{x\to+\infty}\frac{\ln\left(e+\frac{1}{x}\right)-1}{\frac{1}{x}} = \left(\frac{0}{0}\right)\\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{\dfrac{1}{e+\frac{1}{x}}\cdot\left(-\dfrac{1}{x^2}\right)}{-\dfrac{1}{x^2}}\\
&= \lim_{x\to+\infty}\frac{1}{e+\frac{1}{x}}\\
&= \frac{1}{e}
\end{aligned}
$$

s) $$
\begin{aligned}
\lim_{x\to+\infty}\left(xe^{-\frac{1}{x^2}}-x\right) &= \lim_{x\to+\infty}\left[x\left(e^{-\frac{1}{x^2}}-1\right)\right]\\
&= (\infty\cdot 0)\\
&= \lim_{x\to+\infty}\frac{e^{-\frac{1}{x^2}}-1}{\frac{1}{x}} = \left(\frac{0}{0}\right)\\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{e^{-\frac{1}{x^2}}\cdot\dfrac{2}{x^3}}{-\dfrac{1}{x^2}}\\
&= \lim_{x\to+\infty}\frac{-2e^{-\frac{1}{x^2}}}{x}\\
&= \frac{-2}{\infty}
\end{aligned}
$$

$$= \quad 0$$

$$\begin{aligned}
\text{t) } \lim_{x\to+\infty}\left(x^2 2^{-x}\right) &= \lim_{x\to+\infty}\frac{x^2}{2^x} = \left(\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{2x}{2^x\ln 2} = \left(\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \frac{2}{\ln^2 2}\lim_{x\to+\infty}\frac{1}{2^x}\\
&= 0
\end{aligned}$$

Limese oblika $0^0, 1^\infty$ i ∞^0 svodimo na L'Hospitalovo pravilo logaritmiranjem:

$$\begin{aligned}
\lim_{x\to x_0} f(x)^{g(x)} &= A \quad /\ln\\
\lim_{x\to x_0} \ln\left(f(x)^{g(x)}\right) &= \ln A\\
\lim_{x\to x_0} g(x)\ln f(x) &= \ln A
\end{aligned}$$

Zadatak 6.27. Izračunajte limes $\lim\limits_{x\to 0}(\cos 2x)^{\frac{3}{x^2}}$.

$$\begin{aligned}
\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}} &= (1^\infty) = A \quad /\ln\\
\ln A &= \lim_{x\to 0}\left(\ln(\cos 2x)^{\frac{3}{x^2}}\right)\\
&= \lim_{x\to 0}\frac{3}{x^2}\ln(\cos 2x) = (\infty\cdot 0)\\
&= \lim_{x\to 0}\frac{3\ln\cos 2x}{x^2} = \left(\frac{0}{0}\right)\\
&=^{\text{L'H}} \lim_{x\to 0}\frac{3\frac{1}{\cos 2x}(-2\sin 2x)}{2x}\\
&= -6\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{1}{\cos 2x}\\
&= -6\cdot 1\cdot 1\\
\ln A &= -6\\
A &= e^{-6}
\end{aligned}$$

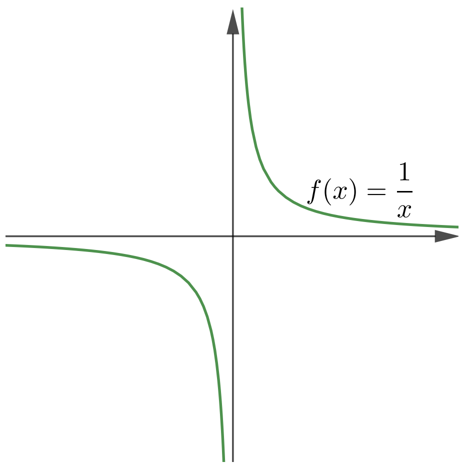
6.7 Asimptote

Vertikalne asimptote

Pravac $x = c$ je vertikalna asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{ili} \quad \lim_{x \to c^-} f(x) = \pm\infty$$

Primjer: $f(x) = \dfrac{1}{x}$, $D_f = \mathbb{R} \setminus \{0\}$



$$\left.\begin{array}{l} \displaystyle\lim_{x \to 0^+} \frac{1}{x} = +\infty \\ \displaystyle\lim_{x \to 0^-} \frac{1}{x} = -\infty \end{array}\right\} \Rightarrow x = 0 \text{ je vertikalna asimptota}$$

Napomena: Vertikalnu asimptotu tražimo u prekidima i na rubovima domene.

Horizontalne asimptote

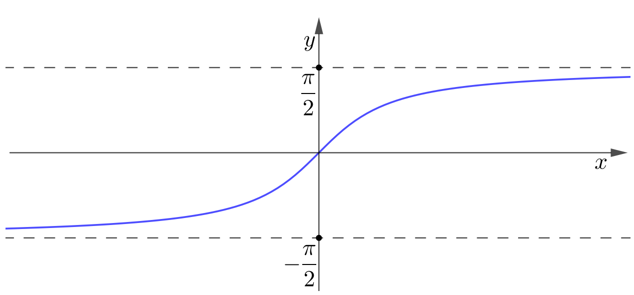
Pravac $y = c$ je lijeva horizontalna asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$\lim_{x \to -\infty} f(x) = c.$$

Pravac $y = c$ je desna horizontalna asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$\lim_{x \to +\infty} f(x) = c.$$

Primjer: $f(x) = \operatorname{arctg} x$, $D_f = \mathbb{R}$



$$\lim_{x \to +\infty} \operatorname{arctg} x = \frac{\pi}{2}$$
$$\lim_{x \to -\infty} \operatorname{arctg} x = -\frac{\pi}{2}$$

$x = \dfrac{\pi}{2}$ je desna horizontalna asimptota, a $x = -\dfrac{\pi}{2}$ je lijeva horizontalna asimptota.

Kose asimptote

Pravac $y = kx + l$ je lijeva kosa asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$k = \lim_{x\to-\infty} \frac{f(x)}{x} = k \text{ i } l = \lim_{x\to-\infty} [f(x) - kx]$$

Pravac $y = kx + l$ je desna kosa asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} \text{ i } l = \lim_{x\to+\infty} [f(x) - kx]$$

Napomena: Ako f ima lijevu(desnu) horizontalnu asimptotu, onda nema lijevu(desnu) kosu asimptotu i obratno.

Zadatak 6.28. Odredite sve asimptote funkcije:

a) $f(x) = \dfrac{x^3+1}{3x^2+1}$

b) $f(x) = \dfrac{x^3+2x-1}{2x^3+2}$

c) $f(x) = \operatorname{arctg} e^x$

d) $f(x) = \dfrac{1}{x^2}$

e) $f(x) = -\ln x + 2$

f) $f(x) = \dfrac{2x^3+5x^2+3x+2}{x^2+1}$

g) $f(x) = \sqrt[3]{x^2(6-x)}$

h) $f(x) = \dfrac{x^2-6x+3}{x-3}$

i) $f(x) = x + 2\operatorname{arcctg} x$

j) $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$

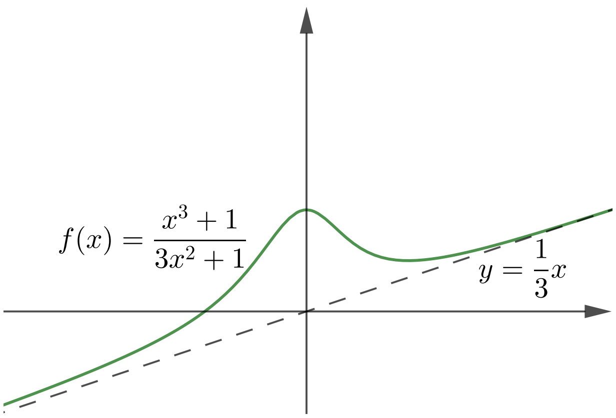
Rješenje: a) $D_f = \mathbb{R}$, pa f nema vertikalnih asimptota.

$$\begin{aligned} k &= \lim_{x\to\pm\infty} \frac{f(x)}{x} \\ &= \lim_{x\to\pm\infty} \frac{x^3+1}{x(3x^2+1)} \\ &= \lim_{x\to\pm\infty} \frac{x^3+1}{3x^3+x} : \frac{x^3}{x^3} \\ &= \lim_{x\to\pm\infty} \frac{1+\frac{1}{x^3}}{3+\frac{1}{x^2}} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} l &= \lim_{x\to\pm\infty} [f(x) - kx] \\ &= \lim_{x\to\pm\infty} \left[\frac{x^3+1}{3x^2+1} - \frac{1}{3}x\right] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to\pm\infty} \frac{3x^3+3-3x^3-x}{3(3x^2+1)} \\
&= \lim_{x\to\pm\infty} \frac{3-x}{9x^2+3} \\
&= 0
\end{aligned}
$$

$y = \frac{1}{3}x$ je kosa asimptota.



b) $f(x) = \dfrac{x^3+2x-1}{2x^3+2}$, $D_f = \mathbb{R} \setminus \{-1\}$

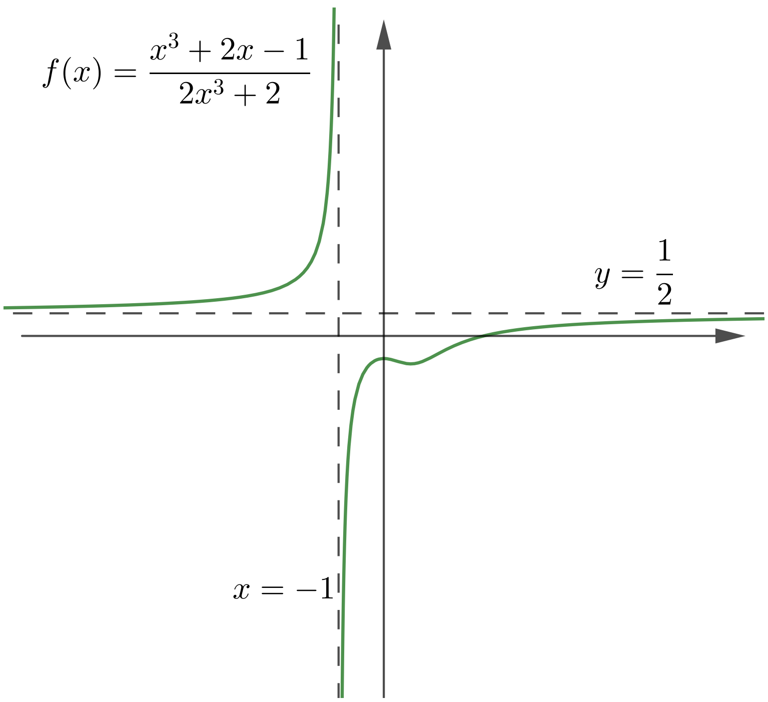
$$
\begin{aligned}
\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} \frac{x^3+2x-1}{2x^3+2} \\
&= \lim_{x\to -1^-} \frac{-4}{0^-} \\
&= +\infty
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \frac{x^3+2x-1}{2x^3+2} \\
&= \lim_{x\to -1^+} \frac{-4}{0^+} \\
&= -\infty
\end{aligned}
$$

$x = -1$ je vertikalna asimptota

$$
\begin{aligned}
\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{x^3+2x-1}{2x^3+2} \\
&= \frac{1}{2}
\end{aligned}
$$

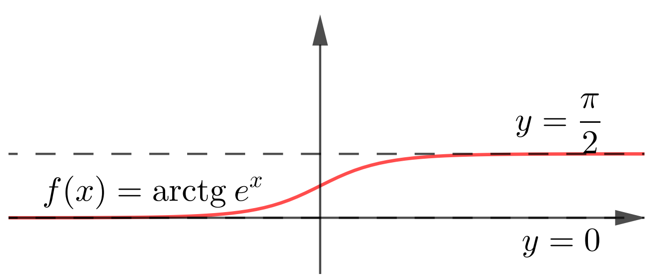
$y = \dfrac{1}{2}$ je horizontalna asimptota



c) $f(x) = \operatorname{arctg} e^x$, $D_f = \mathbb{R}$

Nema vertikalnih asimptota.

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \underbrace{\operatorname{arctg} \overbrace{e^x}^{+\infty}}_{\frac{\pi}{2}} \\
&= \frac{\pi}{2} \\
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \underbrace{\operatorname{arctg} \overbrace{e^x}^{0}}_{0} \\
&= 0
\end{aligned}$$



$y = \dfrac{\pi}{2}$ je desna horizontalna, a $y = 0$ lijeva horizontalna asimptota.

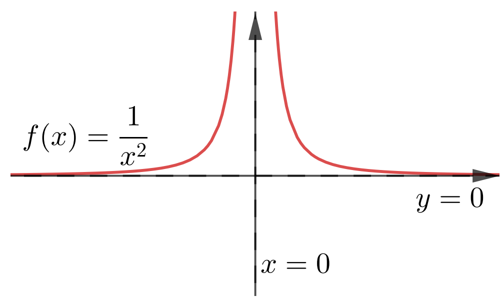
Kako f ima lijevu i desnu horizontalnu asimptotu, nema kosih.

d) $f(x) = \dfrac{1}{x^2}$, $D_f = \mathbb{R} \setminus \{0\}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2} = +\infty$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2} = +\infty$$

$x = 0$ je vertikalna asimptota



$$\lim_{x\to \pm\infty} f(x) = \lim_{x\to \pm\infty} \frac{1}{x^2} = 0$$

$y = 0$ je obostrana horizontalna asimptota

Kako f ima lijevu i desnu horizontalnu asimptotu, nema kosih.

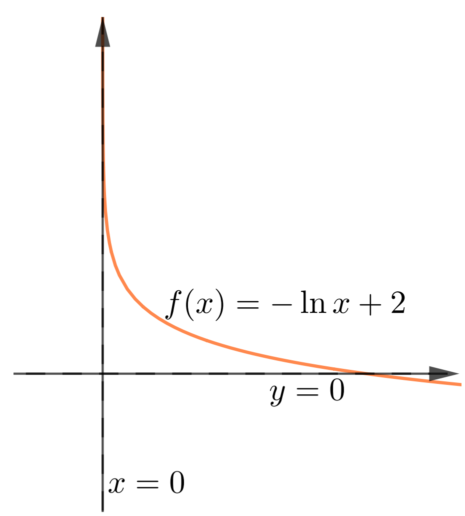
e) $f(x) = -\ln x + 2,\ D_f = \langle 0, +\infty\rangle$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-\ln x + 2) = -(-\infty) + 2 = +\infty$$

$x = 0$ je vertikalna asimptota

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} (-\ln x + 2) = -\infty$$

f nema horizontalnih asimptota.



$$\lim_{x\to +\infty} \frac{f(x)}{x} = \lim_{x\to +\infty} \frac{-\ln x + 2}{x} = \left(\frac{\infty}{\infty}\right) \stackrel{\text{L'H}}{=} \frac{-\frac{1}{x}}{1} = 0$$

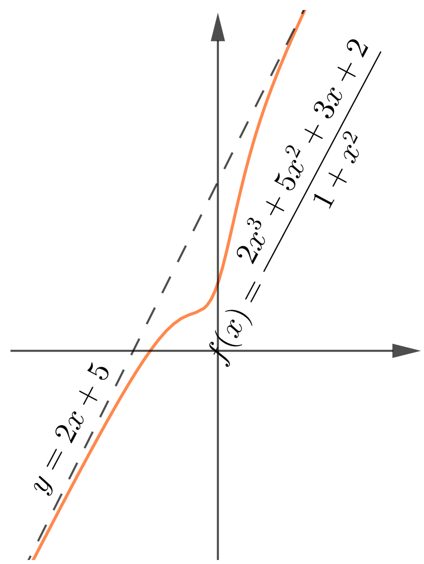
Kako 0 ne može biti koeficijent smjera kose asimptote, zaključujemo da f nema kosih asimptota.

f) $f(x)=\dfrac{2x^3+5x^2+3x+2}{x^2+1}$, $D_f=\mathbb{R}$

Nema vertikalnih asimptota.

$$\begin{aligned} k &= \lim_{x\to\pm\infty}\frac{f(x)}{x} \\ &= \lim_{x\to\pm\infty}\frac{\frac{2x^3+5x^2+3x+2}{x^2+1}}{x} \\ &= \lim_{x\to\pm\infty}\frac{2x^3+5x^2+3x+2}{x^3+x} : \frac{x^3}{x^3} \\ &= \lim_{x\to\pm\infty}\frac{2+\frac{5}{x}+\frac{3}{x^2}+\frac{2}{x^3}}{1+\frac{1}{x^2}} \\ &= \frac{2}{1}=2 \end{aligned}$$

$$\begin{aligned} l &= \lim_{x\to\pm\infty}[f(x)-kx] \\ &= \lim_{x\to\pm\infty}\left[\frac{2x^3+5x^2+3x+2}{x^2+1}-2x\right] \\ &= \lim_{x\to\pm\infty}\frac{2x^3+5x^2+3x+2-2x^3-2x}{x^2+1} \\ &= \lim_{x\to\pm\infty}\frac{5x^2+x+2}{x^2+1} : \frac{x^2}{x^2} \\ &= \lim_{x\to\pm\infty}\frac{5+\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}} \\ &= 5 \end{aligned}$$

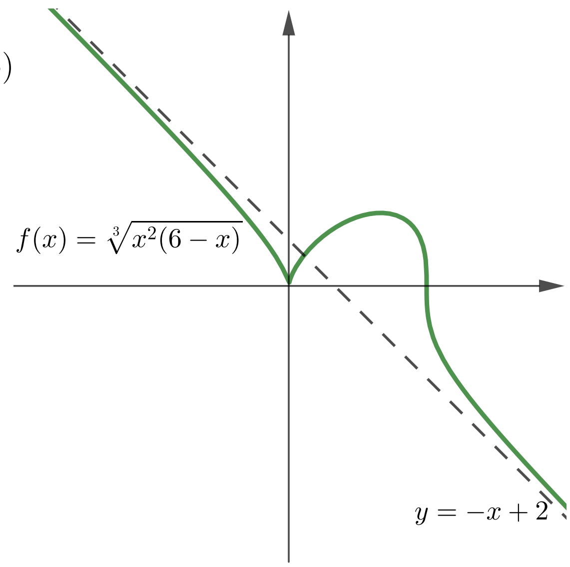


Pravac $y=2x+5$ je kosa asimptota funkcije f, pa zaključujemo da f nema horizontalnih asimptota.

g) $f(x)=\sqrt[3]{x^2(6-x)}$, $D_f=\mathbb{R}$, pa f nema vertikalnih asimptota.

$$\begin{aligned} k &= \lim_{x\to\pm\infty}\frac{f(x)}{x} \\ &= \lim_{x\to\pm\infty}\frac{\sqrt[3]{6x^2-x^3}}{x} \\ &= \lim_{x\to\pm\infty}\sqrt[3]{\frac{6}{x}-1} \\ &= -1 \end{aligned}$$

$$
\begin{aligned}
l &= \lim_{x\to\pm\infty}[f(x)-kx] \\
&= \lim_{x\to\pm\infty}\left[\sqrt[3]{6x^2-x^3}+x\right] = (-\infty+\infty) \\
&= \lim_{x\to\pm\infty}\left[\sqrt[3]{6x^2-x^3}+x\right]\cdot\frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\pm\infty}\frac{\sqrt[3]{\frac{6}{x}-1}+1}{\frac{1}{x}} = \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to\pm\infty}\frac{\frac{1}{3\sqrt[3]{\left(\frac{6}{x}-1\right)^2}}\cdot\left(-\frac{6}{x^2}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\pm\infty}\frac{2}{\sqrt[3]{\left(\frac{6}{x}-1\right)^2}} \\
&= 2
\end{aligned}
$$



Pravac $y=-x+2$ je kosa asimptota funkcije f, pa zaključujemo da f nema horizontalnih asimptota.

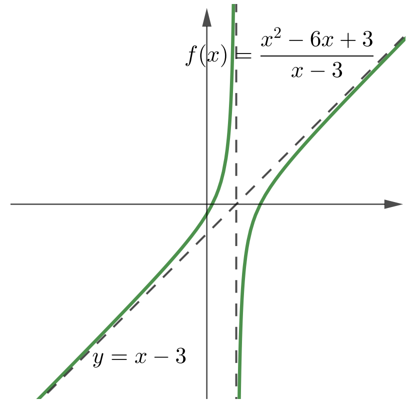
h) $f(x)=\dfrac{x^2-6x+3}{x-3}$, $D_f=\mathbb{R}\setminus\{3\}$.

$$
\begin{aligned}
\lim_{x\to 3^-}\frac{x^2-6x+3}{x-3} &= \frac{-6}{0^-} \qquad & \lim_{x\to 3^+}\frac{x^2-6x+3}{x-3} &= \frac{-6}{0^+} \\
&= +\infty & &= -\infty
\end{aligned}
$$

Pravac $x=3$ je vertikalna asimptota.

$$
\begin{aligned}
k &= \lim_{x\to\pm\infty}\frac{f(x)}{x} \\
&= \lim_{x\to\pm\infty}\frac{\frac{x^2-6x+3}{x-3}}{x} \\
&= \lim_{x\to\pm\infty}\frac{x^2-6x+3}{x^2-3x}:\frac{x^2}{x^2} \\
&= \lim_{x\to\pm\infty}\frac{1-\frac{6}{x}+\frac{3}{x^2}}{1-\frac{3}{x}} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
l &= \lim_{x\to\pm\infty}[f(x)-kx] \\
&= \lim_{x\to\pm\infty}\left[\frac{x^2-6x+3}{x-3}-x\right] \\
&= \lim_{x\to\pm\infty}\frac{x^2-6x+3-x^2+3x}{x-3} \\
&= \lim_{x\to\pm\infty}\frac{-3x+3}{x-3} \\
&= -3
\end{aligned}
$$



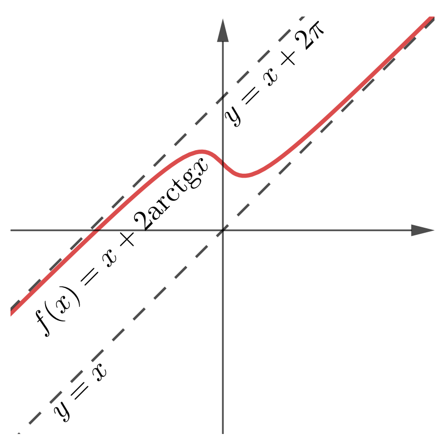
Pravac $y = x - 3$ je kosa asimptota funkcije f, pa zaključujemo da f nema horizontalnih asimptota.

i) $f(x) = x + 2\,\mathrm{arcctg}\,x$, $D_f = \mathbb{R}$, pa f nema vertikalnih asimptota.

$$
\begin{aligned}
k_1 &= \lim_{x\to+\infty}\frac{f(x)}{x} \\
&= \lim_{x\to+\infty}\frac{x+2\,\mathrm{arcctg}\,x}{x} \\
&= 1+\lim_{x\to+\infty}\frac{2\,\overbrace{\mathrm{arcctg}\,x}^{0}}{x} \\
&= 1
\end{aligned}
\qquad
\begin{aligned}
k_2 &= \lim_{x\to-\infty}\frac{f(x)}{x} \\
&= \lim_{x\to-\infty}\frac{x+2\,\mathrm{arcctg}\,x}{x} \\
&= 1+\lim_{x\to-\infty}\frac{2\,\overbrace{\mathrm{arcctg}\,x}^{\pi}}{x} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
l_1 &= \lim_{x\to+\infty}[f(x)-kx] \\
&= \lim_{x\to+\infty}[x+2\,\mathrm{arcctg}\,x-x] \\
&= \lim_{x\to+\infty}2\,\mathrm{arcctg}\,x \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
l_2 &= \lim_{x\to-\infty}[f(x)-kx] \\
&= \lim_{x\to-\infty}[x+2\,\mathrm{arcctg}\,x-x] \\
&= \lim_{x\to-\infty}2\,\mathrm{arcctg}\,x \\
&= 2\pi
\end{aligned}
$$

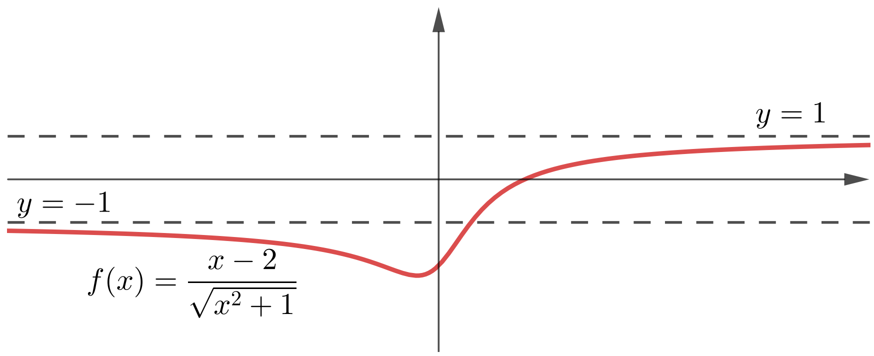


Pravac $y = x + 2\pi$ je lijeva kosa, a $y = x$ je desna kosa asimptota

j) $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$, $D_f = \mathbb{R}$, pa f nema vertikalnih asimptota.

$$\begin{aligned}\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x-2}{\sqrt{x^2+1}} : \frac{x}{x} \\ &= \lim_{x\to+\infty} \frac{1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} \\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= (x \leftrightarrow -x) \\ &= \lim_{x\to+\infty} \frac{-x-2}{\sqrt{x^2+1}} : \frac{x}{x} \\ &= \lim_{x\to+\infty} \frac{-1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} \\ &= -1\end{aligned}$$



Pravac $y = 1$ je desna horizontalna, a pravac $y = -1$ je lijeva horizontalna asimptota.

6.8 Tok funkcije

Zadatak 6.29. Odredite prirodnu domenu funkcije, nultočke, točke ekstrema, intervale rasta i pada, asimptote, te skicirajte graf funkcije:

a) $f(x) = \dfrac{x^3-4}{(x-1)^3}$

b) $f(x) = \dfrac{x^2-1}{e^{x^2}}$

c) $f(x) = \dfrac{\ln^2 x}{x}$

d) $f(x) = 2\arccos \dfrac{1}{x-1}$

e) $f(x) = \dfrac{x-2}{\sqrt{x^2+2}}$

Rješenje: a) $f(x) = \dfrac{x^3-4}{(x-1)^3}$

Domena: $D_f = \mathbb{R} \setminus \{1\}$

Nultočke:

$$\begin{aligned} f(x) &= 0 \\ \frac{x^3-4}{(x-1)^3} &= 0 \\ x^3-4 &= 0 \\ x^3 &= 4 \\ x_0 &= \sqrt[3]{4} \end{aligned}$$

Ekstremi:

$$\begin{aligned} f(x) &= \frac{x^3-4}{(x-1)^3} \\ f'(x) &= \frac{3x^2(x-1)^3-(x^3-4)3(x-1)^2}{(x-1)^6} \\ &= \frac{(x-1)^2\left(3x^2(x-1)-3(x^3-4)\right)}{(x-1)^6} \\ &= \frac{3x^3-3x^2-3x^3+12}{(x-1)^4} \\ &= \frac{-3x^2+12}{(x-1)^4} \end{aligned}$$

Odredimo stacionarne točke:

$$\begin{aligned} -3x^2+12 &= 0 \\ 3x^2 &= 12 \\ x^2 &= 4 \end{aligned}$$

$x_1 = -2$, $x_2 = 2$

	$-\infty$ — -2	-2 — 1	1 — 2	2 — $+\infty$
f'	$-$	$+$	$+$	$-$
f	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$
		m (u -2)		M (u 2)

Funkcija f u $x=-2$ ima lokalni minimum i $f(-2)=\dfrac{4}{9}$, a u $x=2$ lokalni maksimum i $f(2)=4$.

Asimptote:

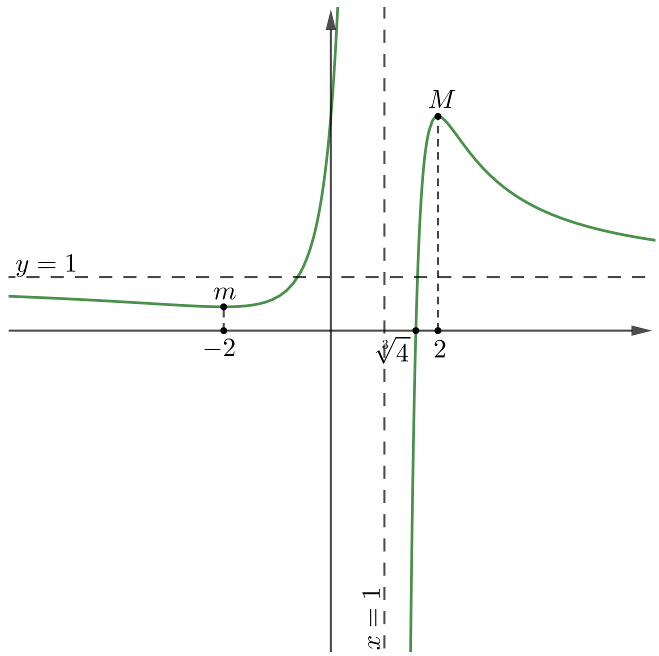
$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{x^3-4}{(x-1)^3} \\ &= \frac{-3}{0^-} \\ &= +\infty \end{aligned}$$

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{x^3-4}{(x-1)^3} \\ &= \frac{-3}{0^+} \\ &= -\infty \end{aligned}$$

Pravac $x=1$ je vertikalna asimptota

$$\begin{aligned} \lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{x^3-4}{(x-1)^3} : \frac{x^3}{x^3} \\ &= \lim_{x\to\pm\infty} \frac{1-\frac{4}{x^3}}{\left(1-\frac{1}{x}\right)^3} \\ &= 1 \end{aligned}$$

Pravac $y=1$ je lijeva i desna horizontalna asimptota, pa f nema kosih asimptota.



b) $f(x)=\dfrac{x^2-1}{e^{x^2}}$

Domena: $D_f=\mathbb{R}$

Nultočke:

$$\begin{aligned} f(x) &= 0\\ \frac{x^2-1}{e^{x^2}} &= 0\\ x^2-1 &= 0\\ x^2 &= 1\\ x_0 &= \pm 1 \end{aligned}$$

Ekstremi:

$$\begin{aligned} f'(x) &= \frac{2x\cdot e^{x^2}-(x^2-1)e^{x^2}\cdot 2x}{e^{2x^2}}\\ &= \frac{e^{x^2}\left(2x-2x^3+2x\right)}{e^{2x^2}}\\ &= \frac{4x-2x^3}{e^{x^2}} \end{aligned}$$

Odredimo stacionarne točke:

$$\begin{aligned} 4x-2x^3 &= 0\\ 2x(2-x^2) &= 0 \end{aligned}$$

$x_1=0,\ x_2=\sqrt{2},\ x_3=-\sqrt{2}$

	$-\infty$	$-\sqrt{2}$	0	$\sqrt{2}$	$+\infty$
f'	$+$	$-$	$+$	$-$	
f	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	
		M	m	M	

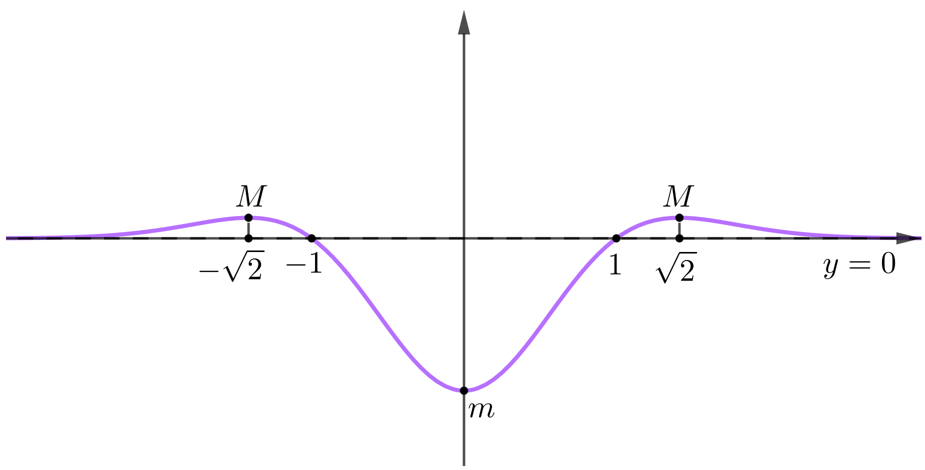
Funkcija f u $x = 0$ ima lokalni minimum, a u $x = -\sqrt{2}$ i $x = \sqrt{2}$ lokalni maksimum. $f(\sqrt{2}) = f(-\sqrt{2}) = \frac{1}{e^2}$ i $f(0) = -1$.

Asimptote:

Nema vertikalnih asimptota.

$$\begin{aligned}
\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{x^2-1}{e^{x^2}} = \left(\frac{\infty}{\infty}\right) \\
&\stackrel{\text{L'H}}{=} \lim_{x\to\pm\infty} \frac{2x}{2x\cdot e^{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{1}{e^{x^2}} \\
&= 0
\end{aligned}$$

Pravac $y = 0$ je lijeva i desna horizontalna asimptota, pa f nema kosih asimptota.



c) $f(x) = \frac{\ln^2 x}{x}$

Domena: $D_f = \langle 0, +\infty\rangle$

Nultočke:

$$\begin{aligned}
f(x) &= 0 \\
\ln^2 x &= 0 \\
\ln x &= 0 \\
x &= 1
\end{aligned}$$

Ekstremi:

$$\begin{aligned}
f(x) &= \frac{\ln^2 x}{x} \\
f'(x) &= \frac{2\ln x\cdot \frac{1}{x}\cdot x - \ln^2 x}{x^2} \\
&= \frac{\ln x(2-\ln x)}{x^2}
\end{aligned}$$

Odredimo stacionarne točke:

$$\begin{aligned}
\ln x(2-\ln x) &= 0\\
x_1 &= e^0\\
x_1 &= 1\\
x_2 &= e^2
\end{aligned}$$

	0	1	e^2	$+\infty$
f'	$-$	$+$	$-$	
f	$\searrow$	$\nearrow$	$\searrow$	
		m	M	

Funkcija f u $x=1$ ima lokalni minimum i $f(1)=0$, a u $x=e^2$ lokalni maksimum i $f(e^2)=\dfrac{4}{e^2}$.

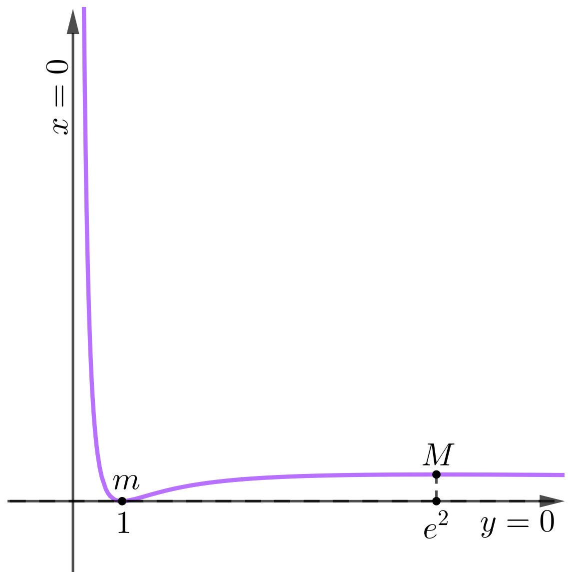
Asimptote:

$$\begin{aligned}
\lim_{x\to0^+} f(x) &= \lim_{x\to0^+}\frac{\ln^2 x}{x}\\
&= +\infty
\end{aligned}$$

Pravac $x=0$ je vertikalna asimptota

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty}\frac{\ln^2 x}{x}=\left(\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{2\ln x\cdot\frac{1}{x}}{1}\\
&= \lim_{x\to+\infty}\frac{2\ln x}{x}=\left(\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{2}{x}\\
&= 0
\end{aligned}$$

Pravac $y=0$ je desna horizontalna asimptota, pa f nema kosih asimptota.



d) $f(x) = 2\arccos\dfrac{1}{x-1}$

Domena:

$D_{\text{arccos}} = [-1,1] \Longrightarrow -1 \leq \dfrac{1}{x-1} \leq 1$

$$\begin{aligned} -1 &\leq \frac{1}{x-1} & \frac{1}{x-1} &\leq 1 \\ 0 &\leq \frac{x}{x-1} & \frac{2-x}{x-1} &\leq 0 \end{aligned}$$

$x \in \langle -\infty, 0] \bigcup \langle 1, +\infty\rangle \qquad x \in \langle -\infty, 1\rangle \bigcup [2, +\infty\rangle$

$$D_f = \langle -\infty, 0] \bigcup [2, +\infty\rangle$$

Nultočke:

$$\begin{aligned} f(x) &= 0 \\ 2\arccos\frac{1}{x-1} &= 0 \\ \frac{1}{x-1} &= 1 \\ x_0 &= 2 \end{aligned}$$

Ekstremi:

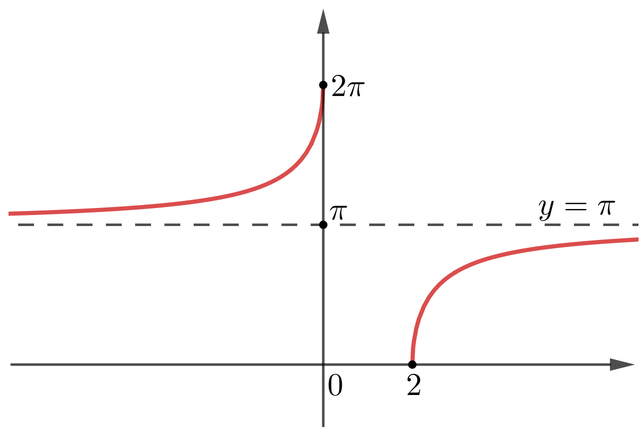
$$
\begin{aligned}
f(x) &= 2\arccos\frac{1}{x-1}\\
f'(x) &= -2\frac{1}{\sqrt{1-\left(\frac{1}{x-1}\right)^2}}\cdot\left(-\frac{1}{(x-1)^2}\right)\\
&= \frac{2}{\underbrace{(x-1)^2}_{>0}\underbrace{\sqrt{1-\left(\frac{1}{x-1}\right)^2}}_{>0}}\\
&> 0
\end{aligned}
$$

f nema ekstrema i raste na cijeloj domeni.

Asimptote:

Nema vertikalnih asimptota.

$$
\begin{aligned}
\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} 2\arccos\frac{1}{x-1}\\
&= 2\arccos 0\\
&= 2\cdot\frac{\pi}{2}\\
&= \pi
\end{aligned}
$$



Pravac $y=\pi$ je lijeva i desna horizontalna asimptota, pa f nema kosih asimptota.

$f(0) = 2\arccos(-1) = 2\pi$

$f(2) = 2\arccos 1 = 0$

e) $f(x) = \dfrac{x-2}{\sqrt{x^2+2}}$

Domena:

$D_f = \mathbb{R}$

Nultočke:

$$
\begin{aligned}
f(x) &= 0\\
\frac{x-2}{\sqrt{x^2+2}} &= 0\\
x-2 &= 0\\
x_0 &= 2
\end{aligned}
$$

Ekstremi:

$$
\begin{aligned}
f(x) &= \frac{x-2}{\sqrt{x^2+2}}\\
f'(x) &= \frac{\sqrt{x^2+2}-(x-2)\cdot\frac{1}{2\sqrt{x^2+2}}\cdot 2x}{x^2+2}\\
&= \frac{\frac{x^2+2-x^2+2x}{\sqrt{x^2+2}}}{x^2+2}\\
&= \frac{2+2x}{(x^2+2)\sqrt{x^2+2}}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
\frac{2+2x}{(x^2+2)\sqrt{x^2+2}} &= 0\\
2+2x &= 0\\
x_1 &= -1
\end{aligned}
$$

	$-\infty$	-1	$+\infty$
f'	$-$	$+$	
f	$\searrow$	$\nearrow$	
		m	

Funkcija f u $x=-1$ ima lokalni minimum i $f(-1)=-\sqrt{3}$.

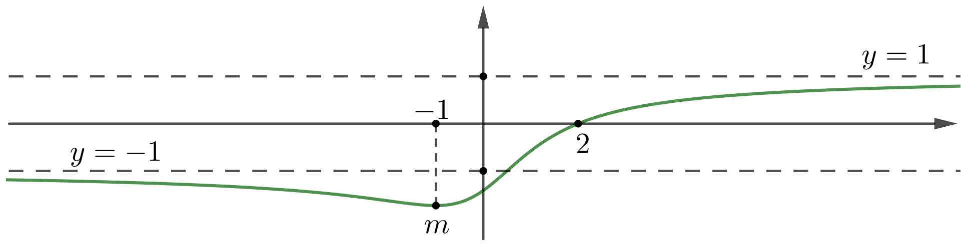
Asimptote:

Nema vertikalnih asimptota.

$$
\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty}\frac{x-2}{\sqrt{x^2+1}} : \frac{x}{x}\\
&= \lim_{x\to+\infty}\frac{1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}}\\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to-\infty} f(x) &= (x \leftrightarrow (-x))\\
&= \lim_{x\to+\infty}\frac{-x-2}{\sqrt{(-x)^2+1}} : \frac{x}{x}\\
&= \lim_{x\to+\infty}\frac{-1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}}\\
&= -1
\end{aligned}
$$

Pravac $y=-1$ je lijeva, a pravac $y=1$ desna horizontalna asimptota, pa f nema kosih asimptota.



Zadatak 6.30. Odredite prirodnu domenu funkcije, nultočke, točke ekstrema, intervale rasta i pada, točke infleksije, asimptote, te skicirajte graf funkcije:

a) $f(x) = x \cdot e^{-\frac{1}{x^2}}$

b) $f(x) = x - 2 \operatorname{arctg} x$

c) $f(x) = x^2 \cdot 2^{-x}$

d) $f(x) = \arcsin(e^x - 1)$

Rješenje: a) $f(x) = x \cdot e^{-\frac{1}{x^2}}$

Domena: $D_f = \mathbb{R} \setminus \{0\}$

Nultočke: Jedina moguća nultočka bi bila $x_0 = 0$, no nije jer 0 nije u domeni.

Ekstremi:

$$\begin{aligned} f(x) &= x \cdot e^{-\frac{1}{x^2}} \\ f'(x) &= e^{-\frac{1}{x^2}} + x \cdot e^{-\frac{1}{x^2}} \cdot \left(-\frac{-2}{x^3}\right) \\ &= \underbrace{e^{-\frac{1}{x^2}}}_{>0} \underbrace{\left(1 + \frac{2}{x^2}\right)}_{>0} \\ &> 0 \end{aligned}$$

Funkcija nema ekstreme i raste na cijeloj domeni.

Točke infleksije:

$$\begin{aligned} f''(x) &= e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} \cdot \left(1 + \frac{2}{x^2}\right) + e^{-\frac{1}{x^2}} \cdot \left(-\frac{4}{x^3}\right) \\ &= e^{-\frac{1}{x^2}} \left(\frac{2}{x^3} + \frac{4}{x^5} - \frac{4}{x^3}\right) \\ &= e^{-\frac{1}{x^2}} \left(\frac{-2x^2 + 4}{x^5}\right) \end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned} -2x^4+4 &= 0 \\ x^2 &= 2 \end{aligned}$$

Kandidati: $x_1=\sqrt{2}$, $x_2=-\sqrt{2}$

	$-\infty$ do $-\sqrt{2}$	$-\sqrt{2}$ do 0	0 do $\sqrt{2}$	$\sqrt{2}$ do $+\infty$
f''	$+$	$-$	$+$	$-$
f	$\cup$	$\cap$	$\cup$	$\cap$

Intervali konveksnosti su $\langle -\infty, -\sqrt{2}\rangle$ i $\langle 0, \sqrt{2}\rangle$, a intervali konkavnosti su $\langle -\sqrt{2}, 0\rangle$ i $\langle \sqrt{2}, +\infty\rangle$.

Asimptote:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} x\cdot e^{-\frac{1}{x^2}} \\ &= 0 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x\cdot e^{-\frac{1}{x^2}} \\ &= 0 \end{aligned}$$

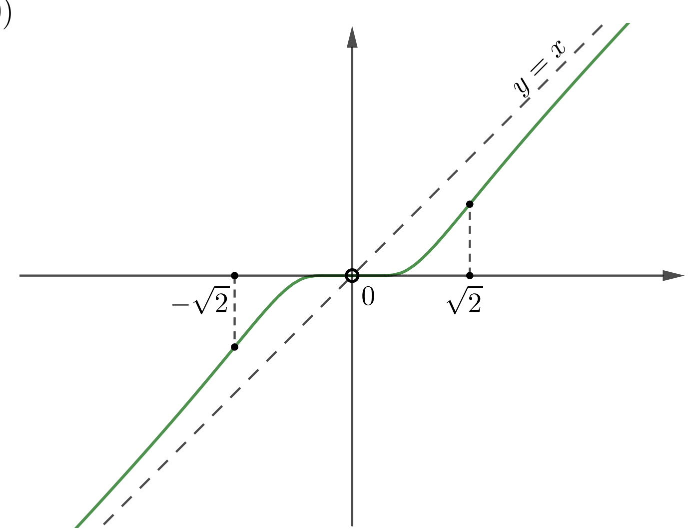
Nema vertikalnih asimptota.

$$\begin{aligned} \lim_{x\to \pm\infty} f(x) &= \lim_{x\to \pm\infty} x\cdot e^{-\frac{1}{x^2}} \\ &= \pm\infty \cdot 1 \\ &= \pm\infty \end{aligned}$$

Nema horizontalnih asimptota.

$$\begin{aligned} k &= \lim_{x\to\pm\infty} \frac{f(x)}{x} \\ &= \lim_{x\to\pm\infty} e^{-\frac{1}{x^2}} \\ &= 1 \\ l &= \lim_{x\to\pm\infty} [f(x)-kx] \\ &= \lim_{x\to\pm\infty} \left[x\cdot e^{-\frac{1}{x^2}} - x\right] = (\infty-\infty) \\ &= \lim_{x\to\pm\infty} x\left(e^{-\frac{1}{x^2}}-1\right) = (\infty\cdot 0) \\ &= \lim_{x\to\pm\infty} \frac{e^{-\frac{1}{x^2}}}{\frac{1}{x}} = \left(\frac{0}{0}\right) \\ &\overset{\text{L'H}}{=} \lim_{x\to\pm\infty} \frac{\frac{2}{x^3}\cdot e^{-\frac{1}{x^2}}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\pm\infty} \frac{-2}{xe^{\frac{1}{x^2}}} \\ &= 0 \end{aligned}$$

Pravac $y=x$ je kosa asimptota.



b) $f(x) = x - 2\operatorname{arctg} x$

Domena: $D_f = \mathbb{R}$

Nultočke: $x_0 = 0$

Ekstremi:

$$\begin{aligned} f(x) &= x - 2\operatorname{arctg} x \\ f'(x) &= 1 - \frac{2}{x^2+1} \\ &= \frac{1+x^2-2}{x^2+1} \\ &= \frac{x^2-1}{x^2+1} \end{aligned}$$

Odredimo stacionarne točke:

$$\begin{aligned} x^2 - 1 &= 0 \\ x^2 &= 1 \end{aligned}$$

$x_1 = -1$, $x_2 = 1$

	$-\infty$	-1	1	$+\infty$
f'	$+$	$-$	$+$	
f	$\nearrow$	$\searrow$	$\nearrow$	
		M	m	

Funkcija f u $x = 1$ ima lokalni minimum i $f(1) = 1 - \frac{\pi}{2}$, te u $x = -1$ ima lokalni maksimum i $f(-1) = \frac{\pi}{2} - 1$.

Točke infleksije:

$$\begin{aligned} f''(x) &= \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} \\ &= \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned} 4x &= 0 \\ x &= 0 \end{aligned}$$

	$-\infty$	0	$+\infty$
f''	$-$	$+$	
f	$\cap$	$\cup$	

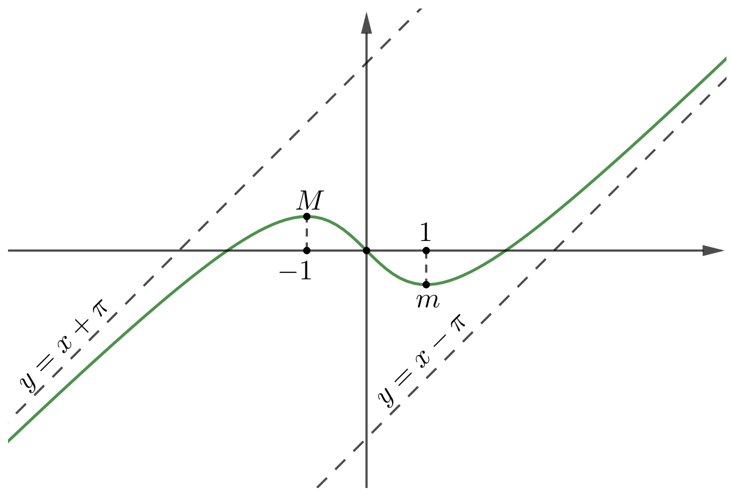
Interval konkavnosti je $\langle -\infty, 0\rangle$, a interval konveksnosti je $\langle 0, +\infty\rangle$.

Asimptote:

Nema vertikalnih asimptota.

$$
\begin{aligned}
k &= \lim_{x\to\pm\infty} \frac{x - 2\operatorname{arctg} x}{x} \\
&= \lim_{x\to\pm\infty} \left(1 - \frac{2\operatorname{arctg} x}{x}\right) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
l_1 &= \lim_{x\to+\infty} [f(x) - kx] \\
&= \lim_{x\to+\infty} [x - 2\operatorname{arctg} x - x] \\
&= \lim_{x\to+\infty} (-2\operatorname{arctg} x) \\
&= -2 \cdot \frac{\pi}{2} \\
&= -\pi
\end{aligned}
$$



$$
\begin{aligned}
l_2 &= \lim_{x\to-\infty} [f(x) - kx] \\
&= \lim_{x\to-\infty} [x - 2\operatorname{arctg} x - x] \\
&= \lim_{x\to-\infty} (-2\operatorname{arctg} x) \\
&= -2 \cdot \frac{-\pi}{2} \\
&= \pi
\end{aligned}
$$

Pravac $y = x + \pi$ je lijeva kosa, a pravac $y = x - \pi$ je desna kosa asimptota.

c) $f(x) = x^2 \cdot 2^{-x}$

Domena: $D_f = \mathbb{R}$

Nultočke: $x_0 = 0$

Ekstremi:

$$
\begin{aligned}
f(x) &= x^2 \cdot 2^{-x} \\
f'(x) &= 2x \cdot 2^{-x} + x^2 \cdot 2^{-x} \ln 2 \cdot (-1) \\
&= x \cdot 2^{-x} (2 - x \ln 2) \\
&= (2x - x^2 \ln 2) \cdot 2^{-x}
\end{aligned}
$$

Odredimo stacionarne točke:

$x_1 = 0,\ x_2 = \dfrac{2}{\ln 2}$

	$-\infty$ … x_1	x_1 … x_2	x_2 … $+\infty$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$
		m	M

Funkcija f u $x = 0$ ima lokalni minimum i $f(0) = 0$, te u $x = \dfrac{2}{\ln 2}$ ima lokalni maksimum i $f\left(\dfrac{2}{\ln 2}\right) = \dfrac{4}{\ln^2 2} \cdot 2^{-\frac{2}{\ln 2}}$.

Točke infleksije:

$$\begin{aligned} f''(x) &= 2^{-x}(2-2x\ln 2)+(2x-x^2\ln 2)(-2^{-x}\ln 2)\\ &= 2^{-x}(2-2x\ln 2-2x\ln 2+x^2\ln^2 2)\\ &= 2^{-x}(2-4x\ln 2+x^2\ln^2 2)\\[1em] 0 &= 2^{-x}(2-4x\ln 2+x^2\ln^2 2)\\ x_{1,2} &= \frac{4\ln 2\pm\sqrt{16\ln^2 2-8\ln^2 2}}{2\ln^2 2}\\ &= \frac{4\ln 2\pm 2\sqrt{2}\ln 2}{2\ln^2 2} \end{aligned}$$

	$-\infty$	x_1	x_2	$+\infty$
f''	$+$	$-$	$+$	
f	$\cup$	$\cap$	$\cup$	

Asimptote:

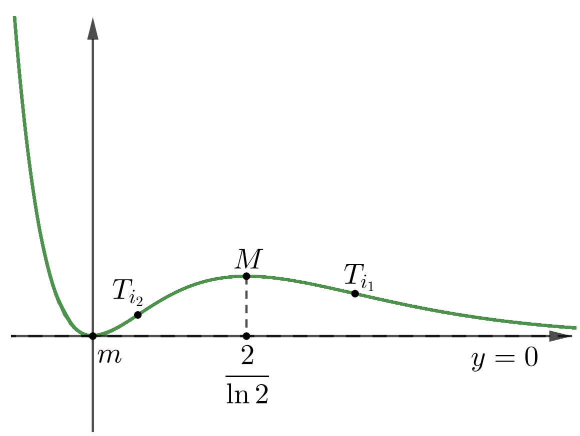
Nema vertikalnih asimptota.

$$\begin{aligned} \lim_{x\to+\infty} x^2\cdot 2^{-x} &= \lim_{x\to+\infty}\frac{x^2}{2^x}\left(=\frac{\infty}{\infty}\right)\\ &\overset{\text{L'H}}{=} \lim_{x\to+\infty}\frac{2x}{2^x\ln 2}\left(=\frac{\infty}{\infty}\right)\\ &\overset{\text{L'H}}{=} \lim_{x\to+\infty}\frac{2}{2^x\ln^2 2}\\ &= 0\\ \lim_{x\to-\infty} x^2\cdot 2^{-x} &= \infty\cdot 2^{+\infty}\\ &= +\infty \end{aligned}$$

Pravac $y=0$ je desna horizontalna asimpotota.

Provjeravamo ima li f lijevu kosu asimptotu:

$$\begin{aligned} k &= \lim_{x\to-\infty}\frac{f(x)}{x}\\ &= \lim_{x\to-\infty}\frac{x^2\cdot 2^{-x}}{x}\\ &= \lim_{x\to-\infty} x\cdot 2^{-x}\\ &= \infty\cdot 2^{+\infty} = -\infty \end{aligned}$$



Nema kosu asimptotu.

d) $f(x)=\arcsin(e^x-1)$

Domena:

$D_{\text{arcsin}} = [-1, 1] \Longrightarrow -1 \leq e^x - 1 \leq 1$

$$\begin{array}{rclcrcl} -1 & \leq & e^x - 1 & \quad & e^x - 1 & \leq & 1 \\ 0 & \leq & e^x & & e^x & \leq & 2 \\ & x \in \mathbb{R} & & & & x \in \langle -\infty, \ln 2] & \end{array}$$

$$D_f = \langle -\infty, \ln 2]$$

Nultočke:

$$\begin{aligned} e^x - 1 &= 0 \\ e^x &= 1 \\ x &= 0 \end{aligned}$$

Ekstremi:

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(e^x-1)^2}} \cdot e^x \\ &= \frac{e^x}{\sqrt{1-(e^x-1)^2}} > 0 \end{aligned}$$

Funkcija f nema ektrema i stalno raste.

Točke infleksije:

$$\begin{aligned} f'(x) &= \frac{e^x}{\sqrt{2e^x - e^{2x}}} \\ f''(x) &= \frac{e^x \cdot \sqrt{2e^x - e^{2x}} - e^x \cdot \frac{1}{2\sqrt{2e^x - e^{2x}}} \cdot (2e^x - 2e^{2x})}{2e^x - e^{2x}} \\ &= \frac{e^x \cdot \sqrt{2e^x - e^{2x}} - \frac{e^x(e^x - e^{2x})}{\sqrt{2e^x - e^{2x}}}}{2e^x - e^{2x}} \\ &= \frac{\frac{e^x(2e^x - e^{2x}) - e^x(e^x - e^{2x})}{\sqrt{2e^x - e^{2x}}}}{2e^x - e^{2x}} \\ &= \frac{e^x(2e^x - e^{2x}) - e^x(e^x - e^{2x})}{(2e^x - e^{2x})^{\frac{3}{2}}} \\ &= \frac{e^{2x}}{(2e^x - e^{2x})^{\frac{3}{2}}} > 0 \end{aligned}$$

Funkcija f nema točku infleksije i konveksna je na cijeloj domeni.

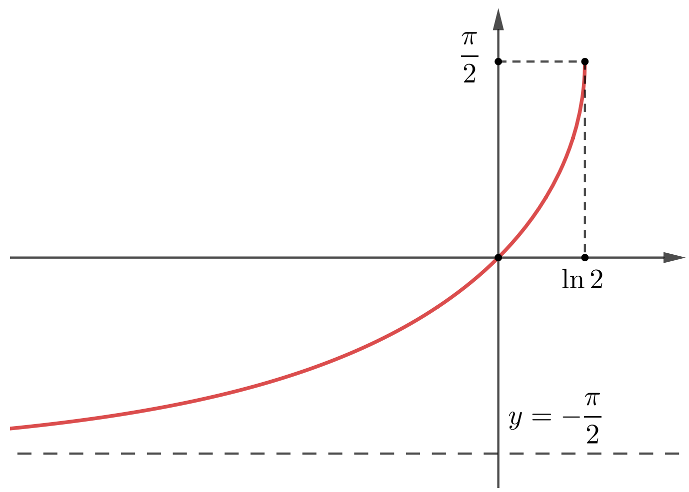
Asimptote:

Nema vertikalnih asimptota.

$$\begin{aligned}\lim_{x\to-\infty} \arcsin(e^x - 1) &= \arcsin(0-1) \\ &= -\frac{\pi}{2}\end{aligned}$$

Pravac $y = -\dfrac{\pi}{2}$ je lijeva horizontalna asimpotota.

Nema kosu asimptotu.



Poglavlje 7

Neodređeni integral

Definicija 7.1. Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$. Za $F : I \to \mathbb{R}$ kažemo da je **primitivna funkcija** funkcije f ako je

$$F'(x) = f(x), \forall x \in I.$$

Uočimo da, ako je F primitivna funkcija funkcije f, onda je skup svih primitivnih funkcija od f dan sa $F + c$, gdje je $c \in \mathbb{R}$ konstanta.

Definicija 7.2. Skup svih primitivnih funkcija od f nazivamo **neodređeni integral** i označavamo sa $\displaystyle\int f(x)\,dx$.

Tablica važnih integrala:

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$x^m, m \in \mathbb{R} \setminus \{-1\}$	$\frac{x^{m+1}}{m+1} + c$	$\sin x$	$-\cos x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$	$\cos x$	$\sin x + c$
$\frac{1}{1+x^2}$	$\operatorname{arctg} x + c$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + c$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + c$
a^x	$\frac{a^x}{\ln a} + c$	$\operatorname{sh} x$	$\operatorname{ch} x$
e^x	$e^x + c$	$\operatorname{ch} x$	$\operatorname{sh} x$

Pravila integriranja:

- $\int (f(x) \pm g(x))\, dx = \int f(x)dx \pm \int g(x)dx$
- $\int (c \cdot f(x))\, dx = c \cdot \int f(x)dx$

Zadatak 7.1. Odredite sljedeće neodređene integrale:

a) $\int (x^2 - x)dx = \frac{x^3}{3} - \frac{x^2}{2} + c$

b) $\int (x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{7}})dx = \frac{4}{5}x^{\frac{5}{4}} - \frac{24}{17}x^{\frac{17}{12}} + \frac{7}{6}x^{\frac{6}{7}} + c$

c) $\int (\mathrm{tg}^2 x)dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \int \left(\frac{1}{\cos^2 x} - 1\right) dx = \mathrm{tg}\, x - x + c$

d) $\int 3^x e^x dx = \int (3e)^x = \frac{(3e)^x}{\ln(3e)} = \frac{3^x e^x}{\ln 3 + 1} + c$

7.1 Metoda supstitucije

$$\int f(x)dx = \left\{ \begin{matrix} t = g(x) & \Rightarrow x = g^{-1}(t) \\ dt = g'(x)dx & \end{matrix} \right\} = \int f\left(g^{-1}(t)\right) \cdot \frac{1}{g'(g^{-1}(t))}dt$$

Zadatak 7.2. Odredite sljedeće neodređene integrale:

$$\begin{aligned} \text{a) } \int \frac{dx}{\sqrt{5x-2}} &= \left\{ \begin{matrix} t = 5x - 2 & \\ dt = 5dx & \Rightarrow \frac{dt}{5} = dx \end{matrix} \right\} \\ &= \int \frac{1}{\sqrt{t}} \cdot \frac{dt}{5} \\ &= \frac{1}{5}\int t^{-\frac{1}{2}}dt \\ &= \frac{1}{5} \cdot 2t^{\frac{1}{2}} + c \\ &= \frac{2}{5}\sqrt{5x-2} + c \end{aligned}$$

$$
\begin{aligned}
\text{b)} \int x\cdot 7^{x^2}dx &= \left\{\begin{matrix} t = x^2 & \\ dt = 2xdx & \Rightarrow \dfrac{dt}{2} = xdx \end{matrix}\right\} \\
&= \frac{1}{2}\int 7^t dt \\
&= \frac{7^t}{2\ln 7} + c \\
&= \frac{7^{x^2}}{2\ln 7} + c
\end{aligned}
$$

$$
\begin{aligned}
\text{c)} \int \cos(kx)dx &= \left\{\begin{matrix} t = kx & \\ dt = kdx & \Rightarrow \dfrac{dt}{k} = dx \end{matrix}\right\} \\
&= \frac{1}{k}\int \cos t dt \\
&= \frac{1}{k}\sin t + c \\
&= \frac{1}{k}\sin(kx) + c
\end{aligned}
$$

$$
\begin{aligned}
\text{d)} \int \frac{3x^2dx}{1+x^6} &= \left\{\begin{matrix} t = x^3 \\ dt = 3x^2dx \end{matrix}\right\} \\
&= \int \frac{dt}{1+t^2} \\
&= \operatorname{arctg} t + c \\
&= \operatorname{arctg} x^3 + c
\end{aligned}
$$

$$
\begin{aligned}
\text{e)} \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx &= \left\{\begin{matrix} t = \sqrt{x} \\ dt = \dfrac{dx}{2\sqrt{x}} \Rightarrow 2dt = \dfrac{dx}{\sqrt{x}} \end{matrix}\right\} \\
&= 2\int \cos t dt \\
&= 2\sin t + c
\end{aligned}
$$

$$= \quad 2\sin\sqrt{x} + c$$

$$\begin{aligned}
\text{f)} \int \operatorname{tg} x\, dx &= \int \frac{\sin x}{\cos x} dx \\
&= \left\{\begin{matrix} t = \cos x \\ dt = -\sin x dx \end{matrix}\right\} \\
&= -\int \frac{dt}{t} \\
&= -\ln|t| + c \\
&= -\ln|\cos x| + c
\end{aligned}$$

$$\begin{aligned}
\text{g)} \int \frac{dx}{\cos x \cdot \sin x} &= \int \frac{\sin x dx}{\cos x \cdot \sin^2 x} \\
&= \int \frac{1}{\operatorname{ctg} x} \cdot \frac{1}{\sin^2 x} dx \\
&= \left\{\begin{matrix} t = \operatorname{ctg} x \\ dt = -\dfrac{1}{\sin^2 x} dx \end{matrix}\right\} \\
&= -\int \frac{dt}{t} \\
&= -\ln|t| + c \\
&= -\ln|\operatorname{ctg} x| + c
\end{aligned}$$

$$\begin{aligned}
\text{h)} \int x(2x+10)^{10} dx &= \left\{\begin{matrix} t = 2x + 10 & \Rightarrow x = \dfrac{t-10}{2} \\ dt = 2dx & \Rightarrow \dfrac{dt}{2} = dx \end{matrix}\right\} \\
&= \frac{1}{2} \int \frac{t-10}{2} \cdot t^{10} dt \\
&= \frac{1}{4} \int (t^{11} - 10t^{10}) dt \\
&= \frac{1}{4} \cdot \frac{t^{12}}{12} - \frac{10}{4} \cdot \frac{t^{11}}{11} + c \\
&= \frac{(2x+10)^{12}}{48} - \frac{5(2x+10)^{11}}{22} + c
\end{aligned}$$

$$\text{i)} \int \frac{dx}{\sqrt{e^x-1}} = \left\{\begin{aligned} t^2 = e^x - 1 \Rightarrow e^x = t^2+1 \\ 2tdt = e^x dx \Rightarrow \frac{2tdt}{t^2+1} = dx \end{aligned}\right\}$$

$$= \int \frac{2tdt}{t(t^2+1)}$$

$$= 2\int \frac{dt}{t^2+1}$$

$$= 2\operatorname{arctg} t + c$$

$$= 2\operatorname{arctg}\sqrt{e^x-1} + c$$

$$\text{j)} \int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1}$$

$$= \left\{\begin{aligned} t = \frac{x}{a} \\ dt = \frac{dx}{a} \end{aligned}\right\}$$

$$= \frac{1}{a^2}\int \frac{adt}{t^2+1}$$

$$= \frac{1}{a}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{a}\operatorname{arctg} t + c$$

$$= \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c$$

$$\text{k)} \int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= \left\{\begin{aligned} t = \frac{x}{a} \\ dt = \frac{dx}{a} \end{aligned}\right\}$$

$$= \frac{1}{a}\int \frac{a\,dt}{\sqrt{1-t^2}}$$

$$= \arcsin t + c$$

$$= \arcsin\frac{x}{a} + c$$

Zadatak 7.3. Odredite funkciju F takvu da je $F'(x) = \dfrac{x}{\sqrt{(x^2+1)^3}}$ te $F(0) = 1$.

Rješenje:

$$\begin{aligned}
F(x) &= \int \frac{x}{\sqrt{(x^2+1)^3}} dx \\
&= \left\{ \begin{aligned} & t = x^2 + 1 \\ & dt = 2x\, dx \Rightarrow \frac{dt}{2} = x\, dx \end{aligned} \right\} \\
&= \frac{1}{2} \int \frac{dt}{t^{\frac{3}{2}}} \\
&= \frac{1}{2} \cdot (-2) \cdot t^{-\frac{1}{2}} + c \\
&= -\frac{1}{\sqrt{x^2+1}} + c \\
F(0) &= -\frac{1}{\sqrt{0^2+1}} + c \\
1 &= -1 + c \\
c &= 2
\end{aligned}$$

$$F(x) = -\frac{1}{\sqrt{x^2+1}} + 2$$

Zadatak 7.4. Odredite sljedeće integrale:

$$\begin{aligned}
\text{a)} \int \frac{x^2+1}{\sqrt{x}} dx &= \int \frac{x^2}{\sqrt{x}} dx + \int \frac{1}{\sqrt{x}} dx \\
&= \int x^{\frac{3}{2}} dx + \int x^{-\frac{1}{2}} dx \\
&= \frac{2}{5} x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + c
\end{aligned}$$

$$\begin{aligned}
\text{b)} \int x(3x+1)^5 dx &= \left\{ \begin{aligned} & t = 3x + 1 && \Rightarrow \quad x = \frac{t-1}{3} \\ & dt = 3dx && \Rightarrow \quad dx = \frac{dt}{3} \end{aligned} \right\} \\
&= \int \frac{t-1}{3} \cdot t^5 \cdot \frac{dt}{3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{9}\int \left((t-1)t^5\right)\, dt \\
&= \frac{1}{9}\int \left(t^6 - t^5\right)\, dt \\
&= \frac{1}{9}\left(\frac{1}{7}t^7 - \frac{1}{6}t^6\right) + c \\
&= \frac{1}{9}\left(\frac{(3x+1)^7}{7} - \frac{(3x+1)^6}{6}\right) + c
\end{aligned}
$$

$$
\begin{aligned}
\text{c) } \int x\sqrt{2x+1}\, dx &= \left\{\begin{array}{lll} t = 2x+1 & \Rightarrow & x = \dfrac{t-1}{2} \\ dt = 2dx & \Rightarrow & dx = \dfrac{dt}{2} \end{array}\right\} \\
&= \int \frac{t-1}{2}\cdot\sqrt{t}\cdot\frac{dt}{2} \\
&= \frac{1}{4}\int \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)\, dt \\
&= \frac{1}{4}\left(\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right) + c \\
&= \frac{(2x+1)^{\frac{5}{2}}}{10} - \frac{(2x+1)^{\frac{3}{2}}}{6} + c
\end{aligned}
$$

$$
\begin{aligned}
\text{d) } \int x^3\sqrt{x^2+1}\, dx &= \left\{\begin{array}{lll} t = x^2+1 & \Rightarrow & x^2 = t-1 \\ dt = 2x\, dx & \Rightarrow & \dfrac{dt}{2} = x\, dx \end{array}\right\} \\
&= \frac{1}{2}\int \sqrt{t}\cdot(t-1)\, dt \\
&= \frac{1}{2}\int (t^{\frac{3}{2}} - t^{\frac{1}{2}})dt \\
&= \frac{1}{2}\left(\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right) + c \\
&= \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{(x^2+1)^{\frac{3}{2}}}{3} + c
\end{aligned}
$$

$$\text{e)} \int \frac{\sin x}{\cos^3 x}\,dx = \begin{Bmatrix} t = \cos x \\ dt = -\sin x\,dx \end{Bmatrix}$$

$$= -\int \frac{dt}{t^3}$$

$$= \frac{1}{2}t^{-2} + c$$

$$= \frac{1}{2\cos^2 c} + c$$

7.2 Metoda parcijalne integracije

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)dx$$

ili

$$\int u\,dv = uv - \int v\,du$$

Zadatak 7.5. Odredite sljedeće integrale:

$$\text{a)} \int xe^x\,dx = \begin{Bmatrix} u = x & du = dx \\ dv = e^x\,dx & v = e^x \end{Bmatrix}$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

$$\text{b)} \int x\sin x\,dx = \begin{Bmatrix} u = x & du = dx \\ dv = \sin x\,dx & v = -\cos x \end{Bmatrix}$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c$$

$$\text{c) } \int x \ln x \, dx = \left\{ \begin{array}{ll} u = \ln x & du = \dfrac{dx}{x} \\ \\ dv = x\, dx & v = \dfrac{x^2}{2} \end{array} \right\}$$

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x}$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

$$\text{d) } \int \ln x \, dx = \left\{ \begin{array}{ll} u = \ln x & du = \dfrac{dx}{x} \\ \\ dv = dx & v = x \end{array} \right\}$$

$$= x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - x + c$$

$$\text{e) } \int \operatorname{arctg} x \, dx = \left\{ \begin{array}{ll} u = \operatorname{arctg} x & du = \dfrac{dx}{1+x^2} \\ \\ dv = dx & v = x \end{array} \right\}$$

$$= x \operatorname{arctg} x - \int \frac{x}{1+x^2} dx$$

$$= \left\{ \begin{array}{l} t = 1 + x^2 \\ dt = 2x \, dx \end{array} \right\}$$

$$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{dt}{t}$$

$$= x \operatorname{arctg} x - \frac{1}{2} \ln |t| + c$$

$$= x \operatorname{arctg} x - \frac{1}{2} \ln(1 + x^2) + c$$

$$
\begin{aligned}
\text{f) } \int e^x \sin x \, dx &= \left\{ \begin{matrix} u = \sin x & du = \cos x \, dx \\ dv = e^x \, dx & v = e^x \end{matrix} \right\} \\
&= e^x \sin x - \int e^x \cos x \, dx \\
&= \left\{ \begin{matrix} u = \cos x & du = -\sin x \, dx \\ dv = e^x \, dx & v = e^x \end{matrix} \right\} \\
&= e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \, dx \right) \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\
\int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\
2 \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x \\
\int e^x \sin x \, dx &= \frac{1}{2} e^x (\sin x - \cos x) + c
\end{aligned}
$$

$$
\begin{aligned}
\text{g) } \int \cos(\ln x) \, dx &= \left\{ \begin{matrix} u = \cos(\ln x) & du = \dfrac{-\sin(\ln x)}{x} \, dx \\ dv = dx & v = x \end{matrix} \right\} \\
&= x \, \cos(\ln x) - \int x \cdot \frac{-\sin(\ln x)}{x} \, dx \\
&= x \, \cos(\ln x) + \int \sin(\ln x) \, dx \\
&= \left\{ \begin{matrix} u = \sin(\ln x) & du = \dfrac{\cos(\ln x)}{x} \, dx \\ dv = dx & v = x \end{matrix} \right\} \\
&= x \, \cos(\ln x) + \left(x \, \sin(\ln x) - \int x \cdot \frac{\cos(\ln x)}{x} \, dx \right) \\
&= x \, \cos(\ln x) + x \, \sin(\ln x) - \int \cos(\ln x) \, dx \\
\int \cos(\ln x) \, dx &= x \, \cos(\ln x) + x \, \sin(\ln x) - \int \cos(\ln x) \, dx \\
2 \int \cos(\ln x) \, dx &= x \, \cos(\ln x) + x \, \sin(\ln x)
\end{aligned}
$$

$$\int \cos(\ln x)\,dx \quad = \quad \frac{1}{2}x(\cos(\ln x) + \sin(\ln x))$$

$$\text{h)} \int \sqrt{x}\ln x\,dx = \left\{\begin{matrix} u = \ln x & du = \dfrac{dx}{x} \\ dv = \sqrt{x}\,dx & v = \dfrac{2}{3}x^{\frac{3}{2}} \end{matrix}\right\}$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{3}{2}} \cdot \frac{1}{x}\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + c$$

$$\text{i)} \int \operatorname{arctg}\sqrt{x}\,dx = \left\{\begin{matrix} t^2 = x \\ 2t\,dt = dx \end{matrix}\right\}$$

$$= 2\int \operatorname{arctg} t \cdot t\,dt$$

$$= \left\{\begin{matrix} u = \operatorname{arctg} t & du = \dfrac{1}{1+t^2}dt \\ dv = t\,dx & v = \dfrac{1}{2}t^2 \end{matrix}\right\}$$

$$= t^2 \cdot \operatorname{arctg} t - 2\int \frac{t^2}{2} \cdot \frac{1}{1+t^2}dt$$

$$= t^2 \cdot \operatorname{arctg} t - \int \frac{1+t^2-1}{1+t^2}dt$$

$$= t^2 \cdot \operatorname{arctg} t - \left(\int \frac{1+t^2}{1+t^2}dt - \int \frac{1}{1+t^2}dt\right)$$

$$= t^2 \cdot \operatorname{arctg} t - (t - \operatorname{arctg} t + c)$$

$$= t^2 \cdot \operatorname{arctg} t - t + \operatorname{arctg} t + c$$

$$= x\operatorname{arctg}\sqrt{x} - \sqrt{x} + \operatorname{arctg}\sqrt{x} + c$$

$$
\text{j)} \int \ln(1+x^2)\,dx = \begin{Bmatrix} u = \ln(1+x^2) & du = \dfrac{2x}{1+x^2}dx \\ & \\ dv = dx & v = x \end{Bmatrix}
$$

$$
= x\ln(1+x^2) - 2\int \frac{x^2}{1+x^2}dx
$$

$$
= x\ln(1+x^2) - 2\left(\int \frac{1+x^2}{1+x^2}dx - \int \frac{1}{1+x^2}dx\right)
$$

$$
= x\ln(1+x^2) - 2x + 2\operatorname{arctg} x + c
$$

$$
\text{k)} \int (x^2+3x)\sin x\,dx = \begin{Bmatrix} u = x^2+3x & du = (2x+3)\,dx \\ dv = \sin x\,dx & v = -\cos x \end{Bmatrix}
$$

$$
= -(x^2+3x)\cos x + \int (2x+3)\cos x\,dx
$$

$$
= \begin{Bmatrix} u = 2x+3 & du = 2\,dx \\ dv = \cos x\,dx & v = \sin x \end{Bmatrix}
$$

$$
= -(x^2+3x)\cos x + (2x+3)\sin x - 2\int \sin x\,dx
$$

$$
= -(x^2+3x)\cos x + (2x+3)\sin x + 2\cos x + c
$$

$$
\text{l)} \int e^{\sqrt{x}}\,dx = \begin{Bmatrix} t^2 = x \\ 2t\,dt = dx \end{Bmatrix}
$$

$$
= 2\int t\cdot e^t\,dt
$$

$$
= \begin{Bmatrix} u = t & du = dt \\ dv = e^t\,dt & v = e^t \end{Bmatrix}
$$

$$
= 2t\cdot e^t - 2\int e^t\,dt
$$

$$
= 2t\cdot e^t - 2e^t + c
$$

$$
= 2\sqrt{x}\cdot e^{\sqrt{x}} - 2e^{\sqrt{x}} + c
$$

7.3 Integriranje racionalnih funkcija

Zadatak 7.6. Odredite sljedeće integrale:

$$\text{a)} \int \frac{1}{2x+1}\,dx = \begin{Bmatrix} t = 2x+1 \\ dt = 2\,dx \end{Bmatrix}$$

$$= \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}\ln|t| + c$$

$$= \frac{1}{2}\ln|2x+1| + c$$

$$\text{b)} \int \frac{3x+1}{x^2+1}\,dx = \int \frac{3x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \begin{Bmatrix} t = x^2+1 \\ dt = 2x\,dx \end{Bmatrix}$$

$$= \frac{3}{2}\int \frac{dt}{t} + \operatorname{arctg} x + c$$

$$= \frac{3}{2}\ln|t| + \operatorname{arctg} x + c$$

$$= \frac{3}{2}\ln(x^2+1) + \operatorname{arctg} x + c$$

$$\text{c)} \int \frac{dx}{x^2+x+1} = \int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$= \int \frac{dx}{(x+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \begin{Bmatrix} t = x+\dfrac{1}{2} \\ dt = dx \end{Bmatrix}$$

$$= \int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t}{\sqrt{3}} + c \\
&= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + c
\end{aligned}
$$

d)
$$
\begin{aligned}
\int \frac{x\,dx}{2x^2-3} &= \left\{ \begin{matrix} t = 2x^2 - 3 \\ dt = 4x\,dx \end{matrix} \right\} \\
&= \frac{1}{4} \int \frac{dt}{t} \\
&= \frac{1}{4} \ln |t| + c \\
&= \frac{1}{4} \ln |2x^2 - 3| + c
\end{aligned}
$$

e)
$$
\begin{aligned}
\int \frac{dx}{x^2-4} &= \int \frac{dx}{(x-2)(x+2)} = (*) \\
\frac{1}{x^2-4} &= \frac{A}{x-2} + \frac{B}{x+2} \quad \Big/ \cdot (x^2-4) \\
1 &= A(x+2) + B(x-2) \\
1 &= x(A+B) + 2A - 2B \\
0 &= A + B \\
1 &= 2A - 2B \\
1 &= 4A \\
A &= \frac{1}{4} \\
B &= -\frac{1}{4} \\
(*) &= \frac{1}{4} \int \frac{dx}{x-2} - \frac{1}{4} \int \frac{dx}{x+2} \\
&= \frac{1}{4} \ln |x-2| - \frac{1}{4} \ln |x+2| + c
\end{aligned}
$$

$$\text{f)} \int \frac{dx}{(x-a)^p} = \begin{Bmatrix} t = x - a \\ dt = dx \end{Bmatrix}$$

$$= \frac{1}{4}\int \frac{dt}{t^p}$$

$$= \frac{t^{-p+1}}{-p+1} + c$$

$$= \frac{1}{-p+1} \cdot \frac{1}{(x-a)^{(p-1)}} + c$$

Napomena: Integral $\int \frac{dx}{ax^2+bx+c}$ rješavamo rastavom na parcijalne razlomke, ako je $D = b^2 - 4ac \geq 0$. Inače, ako je $D < 0$, zapisujemo ga u obliku $\int \frac{dx}{t^2+a^2}$.

Zadatak 7.7. Odredite sljedeće integrale:

$$\text{a)} \int \frac{-x^2+x+1}{(x-1)^3}dx = (*)$$

$$\frac{-x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \quad \Big/ \cdot (x-1)^3$$

$$-x^2+x+1 = A(x^2-2x+1) + B(x-1) + C$$

$$-x^2+x+1 = Ax^2 + x(-2A+B) + A - B + C$$

$$\begin{aligned} A &= -1 \\ -2A+B &= 1 \Longrightarrow B = -1 \\ A-B+C &= 1 \Longrightarrow C = 1 \end{aligned}$$

$$(*) = -\int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x-1)^3}$$

$$= -\ln|x-1| + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + c$$

$$\text{b)} \int \frac{x^2+1}{x^4-x^3}dx = \int \frac{x^2+1}{x^3(x-1)}dx = (*)$$

$$\frac{x^2+1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} \quad \Big/ \cdot x^3(x-1)$$

$$
\begin{aligned}
x^2+1 &= A(x^3-x^2)+B(x^2-x)+C(x-1)+Dx^3\\
x^2+1 &= x^3(A+D)+x^2(-A+B)+x(-B+C)-C
\end{aligned}
$$

$$
\begin{aligned}
-C &= 1 \quad \Longrightarrow C=-1\\
-B+C &= 0 \quad \Longrightarrow B=-1\\
-A+B &= 1 \quad \Longrightarrow A=-2\\
A+D &= 0 \quad \Longrightarrow D=2
\end{aligned}
$$

$$
\begin{aligned}
(*) &= -2\int\frac{dx}{x}-\int\frac{dx}{x^2}-\int\frac{dx}{x^3}+2\int\frac{dx}{x-1}\\
&= -2\ln|x|+\frac{1}{x}+\frac{1}{2x^2}+2\ln|x-1|+c
\end{aligned}
$$

c) $$
\begin{aligned}
\int\frac{x^2-12}{x(x^2-4)}dx &= \int\frac{x^2-12}{x(x-2)(x+2)}dx=(*)\\
\frac{x^2-12}{x(x-2)(x+2)} &= \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2} \quad \Big/\cdot x(x-2)(x+2)\\
x^2-12 &= A(x^2-4)+B(x^2+2x)+C(x^2-2x)\\
x^2-12 &= x^2(A+B+C)+x(2B-2C)-4A
\end{aligned}
$$

$$
\begin{aligned}
-4A &= -12 \quad \Longrightarrow A=3\\
2B-2C &= 0\\
A+B+C &= 1
\end{aligned}
$$

$$
\begin{aligned}
B-C &= 0\\
B+C &= -2
\end{aligned}
$$

$$
B = -1 \quad \Longrightarrow C=-1
$$

$$
\begin{aligned}
(*) &= 3\int\frac{dx}{x}-\int\frac{dx}{x-2}-\int\frac{dx}{x+2}\\
&= 3\ln|x|-\ln|x-2|-\ln|x+2|+c\\
&= \ln\left|\frac{x^3}{x^2-4}\right|+c
\end{aligned}
$$

d)

$$\int \frac{dx}{x^4-16} = \int \frac{dx}{(x-2)(x+2)(x^2+4)} = (*)$$

$$\frac{1}{x^4-16} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4} \quad \Big/ \cdot (x-2)(x+2)(x^2+4)$$

$$\begin{aligned}
1 &= A(x^3+2x^2+4x+8) + B(x^3-2x^2+4x-8) + (Cx+D)(x^2-4) \\
1 &= A(x^3+2x^2+4x+8) + B(x^3-2x^2+4x-8) + Cx^3 - 4Cx + Dx^2 - 4D \\
1 &= x^3(A+B+C) + x^2(2A-2B+D) + x(4A+4B-4C) + 8A-8B-4D
\end{aligned}$$

$$\begin{aligned}
A+B+C &= 0 \\
2A-2B+D &= 0 \\
4A+4B-4C &= 0 \\
8A-8B-4D &= 1
\end{aligned}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ 4 & 4 & -4 & 0 & 0 \\ 8 & -8 & 0 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \\ / : 4 \\ \\ \end{array} \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 8 & -8 & 0 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \text{II}-2\text{I} \\ \text{III}-\text{I} \\ \text{IV}-8\text{I} \end{array} \sim$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -16 & -8 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \text{IV}-4\text{II} \end{array} \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -8 & 1 \end{array}\right]$$

$$D = -\frac{1}{8} \qquad C = 0 \quad -4B - 2C + D = 0 \quad A + B + C = 0$$

$$-4B = \frac{1}{8} \qquad A = \frac{1}{32}$$

$$B = -\frac{1}{32}$$

$$\begin{aligned}
(*) &= \frac{1}{32}\int \frac{dx}{x-2} - \frac{1}{32}\int \frac{dx}{x+2} - \frac{1}{8}\int \frac{dx}{x^2+4} \\
&= \frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{16}\operatorname{arctg}\frac{x}{2} + c
\end{aligned}$$

e) $\int \frac{x^3}{x^2+x+1}dx \quad = \quad (*)$

$$\begin{array}{ll} x^3 & : (x^2+x+1) = x-1 \\ -(x^3+x^2+x) & \\ \hline \quad -x^2-x & \\ \quad -(-x^2-x-1) & \\ \hline \qquad 1 & \end{array}$$

$$\begin{aligned} (*) &= \int \left(x-1+\frac{1}{x^2+x+1}\right)dx \\ &= \int (x-1)\,dx + \int \frac{1}{x^2+x+1}dx \\ &= \frac{x^2}{2} - x + \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx \\ &= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)+c \end{aligned}$$

f) $\int \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}dx \quad = \quad (*)$

$$\begin{aligned} \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2} &= \frac{Ax+B}{x^2+2x+2}+\frac{Cx+D}{(x^2+2x+2)^2} \quad \Big/ \cdot (x^2+2x+2)^2 \\ 2x^3+3x^2+x-1 &= (Ax+B)(x^2+2x+2)+Cx+D \\ 2x^3+3x^2+x-1 &= Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B+Cx+D \\ 2x^3+3x^2+x-1 &= Ax^3+x^2(2A+B)+x(2A+2B+C)+2B+D \end{aligned}$$

$$\begin{aligned} A &= 2 \\ 2A+B &= 3 \implies B=-1 \\ 2A+2B+C &= 1 \implies C=-1 \\ 2B+D &= -1 \implies D=1 \end{aligned}$$

$$
\begin{aligned}
(*) &= \int \frac{2x-1}{x^2+2x+2}dx + \int \frac{-x+1}{(x^2+2x+2)^2}dx \\
&= \int \frac{2x-1}{(x+1)^2+1}dx + \int \frac{-x+1}{\left((x+1)^2+1\right)^2}dx \\
&= \left\{\begin{matrix} t = x+1 \quad \Rightarrow x = t-1 \\ dx = dt \end{matrix}\right\} \\
&= \int \frac{2t-3}{t^2+1}dt + \int \frac{-t+2}{\left(t^2+1\right)^2}dt \\
&= \underbrace{\int \frac{2t}{t^2+1}dt}_{\Delta} - 3\int \frac{1}{t^2+1}dt \underbrace{- \int \frac{t}{\left(t^2+1\right)^2}dt}_{\square} \underbrace{+2\int \frac{1}{\left(t^2+1\right)^2}dt}_{\blacksquare} = (*)
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \int \frac{2t}{t^2+1}dt \\
&= \left\{\begin{matrix} w = t^2+1 \\ dw = 2t\,dt \end{matrix}\right\} \\
&= \int \frac{dw}{w} \\
&= \ln|w| + c \\
&= \ln(t^2+1)
\end{aligned}
$$

$$
\begin{aligned}
\square &= -\int \frac{t}{\left(t^2+1\right)^2}dt \\
&= \left\{\begin{matrix} w = t^2+1 \\ dw = 2t\,dt \Rightarrow \dfrac{dw}{2} = t\,dt \end{matrix}\right\} \\
&= -\frac{1}{2}\int \frac{dw}{w^2} \\
&= \frac{1}{2}\cdot\frac{1}{w} + c \\
&= \frac{1}{2(t^2+1)} + c
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare \quad &= \int \frac{1}{(t^2+1)^2} dt \\
&= \int \frac{t^2+1-t^2}{(t^2+1)^2} dt \\
&= \int \frac{t^2+1}{(t^2+1)^2} dt - \int \frac{t^2}{(t^2+1)^2} dt \\
&= \int \frac{1}{t^2+1} dt - \int \frac{t \cdot t}{(t^2+1)^2} dt \\
&= \left\{ \begin{array}{ll} u = t & \Rightarrow \;\; du = dt \\ dv = \underbrace{\dfrac{t}{(t^2+1)^2}\, dt}_{-\square} & \Rightarrow \;\; v = -\dfrac{1}{2(t^2+1)} \end{array} \right\} \\
&= \operatorname{arctg} t + \frac{t}{2(t^2+1)} - \frac{1}{2} \operatorname{arctg} t \\
&= \frac{1}{2} \operatorname{arctg} t + \frac{t}{2(t^2+1)}
\end{aligned}
$$

$$
\begin{aligned}
(*) \quad &= \ln(t^2+1) - 3\operatorname{arctg} t + \frac{1}{2(t^2+1)} + 2\left(\frac{1}{2}\operatorname{arctg} t + \frac{t}{2(t^2+1)}\right) + c \\
&= \ln(t^2+1) - 3\operatorname{arctg} t + \frac{1}{2(t^2+1)} + \operatorname{arctg} t + \frac{t}{(t^2+1)} + c \\
&= \ln(t^2+1) - 2\operatorname{arctg} t + \frac{1+2t}{2(t^2+1)} + c \\
&= \ln(x^2+2x+2) - 2\operatorname{arctg}(x+1) + \frac{2x+3}{2(x^2+2x+2)} + c
\end{aligned}
$$

$$
\begin{aligned}
\text{g)} \int \frac{e^x}{e^{2x} - 2e^x + 10} dx \quad &= \left\{ \begin{array}{l} t = e^x \\ dt = e^x \, dx \end{array} \right\} \\
&= \int \frac{dt}{t^2 - 2t + 10} \\
&= \int \frac{dt}{(t-1)^2 + 3^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{Bmatrix} w = t - 1 \\ dw = dt \end{Bmatrix} \\
&= \int \frac{dw}{w^2 + 3^2} \\
&= \frac{1}{3} \operatorname{arctg} \frac{w}{3} + c \\
&= \frac{1}{3} \operatorname{arctg} \frac{t-1}{3} + c \\
&= \frac{1}{3} \operatorname{arctg} \frac{e^x - 1}{3} + c
\end{aligned}
$$

7.4 Integriranje iracionalnih funkcija

Zadatak 7.8. Odredite sljedeće integrale:

$$
\begin{aligned}
\text{a)} \int \frac{dx}{2\sqrt{x} + \sqrt[3]{x}} &= \begin{Bmatrix} x = t^6 \\ dx = 6t^5\, dt \end{Bmatrix} \\
&= \int \frac{6t^5\, dt}{2t^3 + t^2} \\
&= \int \frac{6t^{\cancel{5}3}\, dt}{\cancel{t^2}(2t+1)} \\
&= \int \frac{6t^3\, dt}{2t+1} = (*)
\end{aligned}
$$

$$
\begin{array}{rl}
6t^3 & : (2t+1) = 3t^2 - \dfrac{3}{2}t + \dfrac{3}{4} \\
-(6t^3 + 3t^2) & \\
\hline
-3t^2 & \\
-\left(-3t^2 - \frac{3}{2}t\right) & \\
\hline
\frac{3}{2}t & \\
-\left(\frac{3}{2}t + \frac{3}{4}\right) & \\
\hline
-\frac{3}{4} &
\end{array}
$$

$$\begin{aligned}
(*) &= \int \left(3t^2 - \frac{3}{2}t + \frac{3}{4} - \frac{\frac{3}{4}}{2t+1}\right) dt \\
&= t^3 - \frac{3}{4}t^2 + \frac{3}{4}t - \frac{3}{4}\int \frac{1}{2t+1} dt \\
&= t^3 - \frac{3}{4}t^2 + \frac{3}{4}t - \frac{3}{8}\ln|2t+1| + c \\
&= \sqrt{x} - \frac{3}{4}\sqrt[3]{x} + \frac{3}{4}\sqrt[6]{x} - \frac{3}{8}\ln|2\sqrt[6]{x}+1| + c
\end{aligned}$$

$$\begin{aligned}
\text{b)} \int \frac{x+2}{\sqrt{4-x^2}}\, dx &= \int \frac{x}{\sqrt{4-x^2}}\, dx + 2\int \frac{1}{\sqrt{4-x^2}}\, dx \\
&= \left\{\begin{array}{l} t = 4-x^2 \\ dt = -2x\, dx \Rightarrow -\dfrac{dt}{2} = x\, dx \end{array}\right\} \\
&= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} + 2\int \frac{1}{\sqrt{4-x^2}}\, dx \\
&= -\frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{t} + 2\text{arcsin}\, \frac{x}{2} + c \\
&= -\sqrt{4-x^2} + 2\text{arcsin}\, \frac{x}{2} + c
\end{aligned}$$

$$\begin{aligned}
\text{c)} \int \sqrt{e^x - 1}\, dx &= \left\{\begin{array}{lcl} t^2 = e^x - 1 & \Rightarrow & e^x = t^2 + 1 \\ 2t\, dt = e^x\, dx & \Rightarrow & \dfrac{2t\, dt}{t^2+1} = dx \end{array}\right\} \\
&= \int t \cdot \frac{2t\, dt}{t^2+1} \\
&= 2\int \frac{t^2}{t^2+1}\, dt \\
&= 2\int \frac{t^2+1-1}{t^2+1}\, dt \\
&= 2\int dt - 2\int \frac{1}{t^2+1}\, dt \\
&= 2t - 2\,\text{arctg}\, t + c \\
&= 2\sqrt{e^x-1} - 2\,\text{arctg}\, \sqrt{e^x-1} + c
\end{aligned}$$

d)

$$\int \ln(x+\sqrt{x^2+1})\,dx = \left\{\begin{array}{ll} u = \ln(x+\sqrt{x^2+1}) & dv = dx \\ du = \dfrac{1}{x+\sqrt{x^2+1}}\left(1+\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right)dx & v = x \\ \quad = \dfrac{1}{\cancel{x+\sqrt{x^2+1}}}\left(\dfrac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right)dx & \\ \quad = \dfrac{1}{\sqrt{x^2+1}}dx & \end{array}\right\}$$

$$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}\,dx$$

$$= \left\{\begin{array}{l} t = x^2+1 \\ dt = 2x\,dx \end{array}\right\}$$

$$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{2}\int \frac{dt}{\sqrt{t}}\,dx$$

$$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{\cancel{2}}\cdot \cancel{2}\sqrt{t} + c$$

$$= x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + c$$

e) $$\int e^{\sqrt{2x+1}}\,dx = \left\{\begin{array}{l} t^2 = 2x+1 \\ \cancel{2}t\,dt = \cancel{2}\,dx \end{array}\right\}$$

$$= \int e^t \cdot t\,dt$$

$$= \left\{\begin{array}{lll} u = t & \Rightarrow & du = dt \\ dv = e^t\,dt & \Rightarrow & v = e^t \end{array}\right\}$$

$$= te^t - \int e^t\,dt$$

$$= te^t - e^t + c$$

$$= \sqrt{2x+1}\,e^{\sqrt{2x+1}} - e^{\sqrt{2x+1}} + c$$

$$
\begin{aligned}
\text{f) } \int \frac{\ln\sqrt{x+3}}{\sqrt{x+3}}\,dx &= \left\{\begin{matrix} t^2 = x+3 \\ 2t\,dt = dx \end{matrix}\right\} \\
&= \int \frac{\ln t}{\cancel{t}} \cdot 2\cancel{t}\,dt \\
&= 2\int \ln t\,dt \\
&= \left\{\begin{matrix} u = \ln t & \Rightarrow & du = \dfrac{dt}{t} \\ dv = dt & \Rightarrow & v = t \end{matrix}\right\} \\
&= 2t\ln t - 2\int \cancel{t} \cdot \frac{dt}{\cancel{t}} \\
&= 2t\ln t - 2t + c \\
&= 2\sqrt{x+3}\cdot\ln\sqrt{x+3} - 2\sqrt{x+3} + c
\end{aligned}
$$

$$
\begin{aligned}
\text{g) } \int \frac{(\sqrt{x}+1)\sin\sqrt{x}}{\sqrt{x}}\,dx &= \left\{\begin{matrix} t = \sqrt{x} \\ dt = \dfrac{dx}{2\sqrt{x}} \Rightarrow 2dt = \dfrac{dx}{\sqrt{x}} \end{matrix}\right\} \\
&= 2\int (t+1)\sin t\,dt \\
&= \left\{\begin{matrix} u = t+1 & \Rightarrow & du = dt \\ dv = \sin t\,dt & \Rightarrow & v = -\cos t \end{matrix}\right\} \\
&= -2(t+1)\cos t + 2\int \cos t\,dt \\
&= -2(t+1)\cos t + 2\sin t + c \\
&= -2(\sqrt{x}+1)\cos\sqrt{x} + 2\sin\sqrt{x} + c
\end{aligned}
$$

$$
\begin{aligned}
\text{h) } \int \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx &= \left\{\begin{matrix} t^2 = e^x - 1 & \Rightarrow & e^x = t^2+1 \\ 2t\,dt = e^x\,dx & \Rightarrow & \dfrac{2t}{t^2+1}\,dt = dx \end{matrix}\right\} \\
&= \int \frac{\cancel{(t^2+1)}\cdot t}{t^2+4} \cdot \frac{2t}{\cancel{t^2+1}}\,dt \\
&= 2\int \frac{t^2}{t^2+4}\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int \frac{t^2+4-4}{t^2+4}\,dt \\
&= 2\int \left(1-\frac{4}{t^2+4}\right)\,dt \\
&= 2\left(t-\cancel{4}^{2}\cdot\frac{1}{\cancel{2}}\operatorname{arctg}\frac{t}{2}\right)+c \\
&= 2\left(\sqrt{e^x-1}-2\operatorname{arctg}\frac{\sqrt{e^x-1}}{2}\right)+c
\end{aligned}
$$

7.5 Integriranje trigonometrijskih funkcija

Prisjetimo se nekih jednakosti koje vrijede za trigonometrijske funkcije:

i) $\sin^2 x+\cos^2 x=1$

ii) $\sin(2x)=2\sin x\cos x$

iii) $\cos(2x)=\cos^2 x-\sin^2 x$

iv) $\sin x\cos y=\frac{1}{2}\left[\sin(x-y)+\sin(x+y)\right]$

v) $\sin x\sin y=\frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$

vi) $\cos x\cos y=\frac{1}{2}\left[\cos(x-y)+\cos(x+y)\right]$

Ako zbrojimo jednakosti i) i iii) dobijemo:

vii) $\cos^2 x=\dfrac{1+\cos(2x)}{2}$,

Ako ih oduzmemo dobijemo:

viii) $\sin^2 x=\dfrac{1-\cos(2x)}{2}$

Zadatak 7.9. Odredite sljedeće integrale:

a)
$$
\begin{aligned}
\int \sin^2 x\,dx &= \int \frac{1-\cos(2x)}{2}\,dx \\
&= \frac{1}{2}\int dx-\frac{1}{2}\int \cos(2x)\,dx
\end{aligned}
$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$$

$$\text{b) } \int \cos^2 x \, dx = \int \frac{1 + \cos(2x)}{2} \, dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x) \, dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$$

$$\text{c) } \int \sin(kx) \, dx = \left\{ \begin{array}{l} t = kx \\ dt = k \, dx \quad \Rightarrow \quad \dfrac{dt}{k} = dx \end{array} \right\}$$

$$= \frac{1}{k}\int \sin t \, dt$$

$$= -\frac{1}{k}\cos t + c$$

$$= -\frac{1}{k}\cos(kx) + c$$

Napomena: Slično se dobije i $\int \cos(kx) \, dx = \frac{1}{k}\sin(kx) + c$

$$\text{d) } \int \sin(3x)\cos(2x) \, dx = \frac{1}{2}\int \left[\sin(3x - 2x) + \sin(3x + 2x)\right] \, dx$$

$$= \frac{1}{2}\int \left[\sin x + \sin(5x)\right] \, dx$$

$$= -\frac{1}{2}\cos x - \frac{1}{10}\cos(5x) + c$$

$$\text{e) } \int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int (2\sin x \cos x)^2 \, dx$$

$$= \frac{1}{4}\int \sin^2(2x) \, dx$$

$$
\begin{aligned}
&= \left\{ \begin{array}{l} t = 2x \\[2ex] dt = 2\,dx \quad \Rightarrow \quad \dfrac{dt}{2} = dx \end{array} \right\} \\
&= \frac{1}{8}\int \sin^2(t)\,dt \\
&= \frac{1}{8}\left(\frac{1}{2}t - \frac{1}{4}\sin(2t)\right) + c \\
&= \frac{\cancel{2}x}{\cancel{16}^{8}} - \frac{1}{32}\sin(4x) + c \\
&= \frac{x}{8} - \frac{1}{32}\sin(4x) + c
\end{aligned}
$$

$$
\begin{aligned}
\text{f)} \int \cos^4 x\,dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int \big(1 + 2\cos(2x) + \cos^2(2x)\big)\,dx \\
&= \frac{1}{4}x + \frac{1}{\cancel{4}^{2}} \cdot \cancel{2} \cdot \frac{1}{2}\sin(2x) + \frac{1}{4}\int \cos^2(2x)\,dx \\
&= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx \\
&= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + c \\
&= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c
\end{aligned}
$$

$$
\begin{aligned}
\text{g)} \int \sin^5 x\,dx &= \int \sin x \sin^4 x\,dx \\
&= \int \sin x(1-\cos^2 x)^2\,dx \\
&= \left\{ \begin{array}{l} t = \cos x \\[2ex] dt = -\sin x\,dx \end{array} \right\} \\
&= -\int (1-t^2)^2\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\int (1-2t^2+t^4)\,dt \\
&= -t+\frac{2}{3}t^3-\frac{1}{5}t^5+c \\
&= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+c
\end{aligned}
$$

h) $$
\begin{aligned}
\int \sqrt{\sin^3 x}\cos^3 x\,dx &= \int \sqrt{\sin^3 x}\cos^2 x\cos x\,dx \\
&= \int \sqrt{\sin^3 x}(1-\sin^2 x)\cos x\,dx \\
&= \begin{Bmatrix} t=\sin x \\ dt=\cos x\,dx \end{Bmatrix} \\
&= \int \sqrt{t^3}(1-t^2)\,dt \\
&= \int \left(t^{\frac{3}{2}}-t^{\frac{7}{2}}\right)\,dt \\
&= \frac{2}{5}t^{\frac{5}{2}}-\frac{2}{9}t^{\frac{9}{2}}+c \\
&= \frac{2}{5}\sqrt{\sin^5 x}-\frac{2}{9}\sqrt{\sin^9 x}+c
\end{aligned}
$$

i) $$
\begin{aligned}
\int \frac{\sin(\ln x)}{x}\,dx &= \begin{Bmatrix} t=\ln x \\ dt=\dfrac{dx}{x} \end{Bmatrix} \\
&= \int \sin t\,dt \\
&= -\cos t+c \\
&= -\cos(\ln x)+c
\end{aligned}
$$

Napomena: Integrale oblika $\int \sqrt{a^2-x^2}\,dx$ rješavamo supstitucijom $x = a\cdot\sin t$, a integrale oblika $\int \sqrt{a^2+x^2}\,dx$ supstitucijom $x = a\cdot \operatorname{sh} t$.

j)
$$\begin{aligned}
\int \sqrt{a^2-x^2}\,dx &= \left\{\begin{matrix} x = a\cdot\sin t \\ dx = a\cdot\cos t\,dt \end{matrix}\right\} \\
&= \int \sqrt{a^2-a^2\sin^2 t}\cdot a\cdot\cos t\,dt \\
&= a^2\int \sqrt{1-\sin^2 t}\cdot\cos t\,dt \\
&= a^2\int \sqrt{\cos^2 t}\cdot\cos t\,dt \\
&= a^2\int \cos^2 t\,dt \\
&= a^2\int \left(\frac{1+\cos(2t)}{2}\right)\,dt \\
&= \frac{a^2}{2}\int (1+\cos(2t))\,dt \\
&= \frac{a^2}{2}\left(t-\frac{1}{2}\sin(2t)\right)+c
\end{aligned}$$

k)
$$\begin{aligned}
\int \sqrt{a^2+x^2}\,dx &= \left\{\begin{matrix} x = a\cdot\operatorname{sh} t \\ dx = a\cdot\operatorname{ch} t\,dt \end{matrix}\right\} \\
&= \int \sqrt{a^2+a^2\operatorname{sh}^2 t}\cdot a\cdot\operatorname{ch} t\,dt \\
&= a^2\int \sqrt{1+\operatorname{sh}^2 t}\cdot\operatorname{ch} t\,dt \\
&= a^2\int \sqrt{\operatorname{ch}^2 t}\cdot\operatorname{ch} t\,dt \\
&= a^2\int \operatorname{ch}^2 t\,dt \\
&= a^2\int \left(\frac{1+\operatorname{ch}(2t)}{2}\right)\,dt
\end{aligned}$$

$$\begin{aligned} &= \frac{a^2}{2}\int (1+\mathrm{ch}(2t))\ dt \\ &= \frac{a^2}{2}\left(t+\frac{1}{2}\mathrm{sh}\,(2t)\right)+c \end{aligned}$$

Promotrimo supstituciju $t = \mathrm{tg}\dfrac{x}{2}$. Tada je:

$$\begin{aligned} dt &= \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\,dx \\ &= \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx \quad : \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} \\ &= \frac{\mathrm{tg}^2\frac{x}{2}+1}{2}\,dx \\ &= \frac{t^2+1}{2}\,dx \\ dx &= \frac{2\,dt}{t^2+1} \end{aligned}$$

$$\begin{aligned} \sin x &= \sin\left(2\cdot\frac{x}{2}\right) \\ &= 2\cdot\sin\frac{x}{2}\cos\frac{x}{2} \\ &= 2\cdot\frac{\sin\frac{x}{2}\cos\frac{x}{2}}{1} \\ &= 2\cdot\frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \quad : \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} \\ &= 2\cdot\frac{\mathrm{tg}\frac{x}{2}}{\mathrm{tg}^2\frac{x}{2}+1} \\ &= \frac{2t}{t^2+1} \end{aligned}$$

$$\begin{aligned} \cos x &= \cos\left(2\cdot\frac{x}{2}\right) \\ &= \cos^2\frac{x}{2}-\sin^2\frac{x}{2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2}}{1} \\
&= \frac{\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \quad : \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} \\
&= \frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1} \\
&= \frac{1-t^2}{t^2+1}
\end{aligned}
$$

Dakle, kod supstitucije $t=\operatorname{tg}\dfrac{x}{2}$ vrijedi:

$$
\boxed{dx=\frac{2\,dt}{t^2+1} \quad \sin x=\frac{2t}{t^2+1} \quad \cos x=\frac{1-t^2}{t^2+1}}
$$

Zadatak 7.10. Odredite sljedeće integrale:

a)

$$
\begin{aligned}
\int \frac{\cos x}{1+\cos x}\,dx &= \left\{\begin{array}{l} t=\operatorname{tg}\dfrac{x}{2} \\ dx=\dfrac{2\,dt}{1+t^2} \\ \cos x=\dfrac{1-t^2}{1+t^2} \end{array}\right\} \\
&= \int \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2} \\
&= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2} \\
&= \int \frac{\frac{1-t^2}{\cancel{1+t^2}}}{\frac{\cancel{2}}{\cancel{1+t^2}}}\cdot\frac{\cancel{2}\,dt}{1+t^2} \\
&= \int \frac{1-t^2}{1+t^2}\,dt \\
&= \int \frac{2-1-t^2}{1+t^2}\,dt
\end{aligned}
$$

$$\begin{aligned}
&= \int \left(-1 + \frac{2}{1+t^2}\right)\, dt \\
&= -t + 2\mathrm{arctg}\, t + c \\
&= -\mathrm{tg}\frac{x}{2} + 2\mathrm{arctg}\left(\mathrm{tg}\frac{x}{2}\right) + c \\
&= -\mathrm{tg}\frac{x}{2} + x + c
\end{aligned}$$

b)

$$\begin{aligned}
\int \frac{dx}{8 - 4\sin x + 7\cos x} &= \left\{t = \mathrm{tg}\frac{x}{2}\right\} \\
&= \int \frac{\frac{2\,dt}{1+t^2}}{8 - \frac{8t}{1+t^2} + \frac{7-7t^2}{1+t^2}}. \\
&= \int \frac{\frac{2\,dt}{\cancel{1+t^2}}}{\frac{8+8t^2-8t+7-7t^2}{\cancel{1+t^2}}} \\
&= 2\int \frac{dt}{t^2 - 8t + 15} \\
&= 2\int \frac{dt}{(t-3)(t-5)} = (*) \\
\frac{1}{(t-3)(t-5)} &= \frac{A}{t-3} + \frac{B}{t-5} \quad \Big/ \cdot (t-3)(t-5) \\
1 &= A(t-5) + B(t-3) \\
1 &= t(A+B) - 5A - 3B \\
0 &= A + B \\
1 &= -5A - 3B \\
A &= -\frac{1}{2} \\
B &= \frac{1}{2} \\
(*) &= -\frac{1}{2}\int \frac{dt}{t-3} + \frac{1}{2}\int \frac{dt}{t-5}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\ln|t-3| + \frac{1}{2}\ln|t-5| + c \\
&= \frac{1}{2}\ln\left|\frac{t-5}{t-3}\right| + c \\
&= \frac{1}{2}\ln\left|\frac{\mathrm{tg}\frac{x}{2}-5}{\mathrm{tg}\frac{x}{2}-3}\right| + c
\end{aligned}
$$

Poglavlje 8

Određeni integral

Teorem 8.1. Neka je $f : [a, b] \to \mathbb{R}$ neprekidna funkcija na segmentu $[a, b]$. Tada vrijedi **Newton-Leibnizova formula**:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

gdje je F bilo koja primitivna funkcija od f.

Svojstva određenog integrala:

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx, \forall b \in \langle a, c\rangle$
- $\int_a^b \left(f(x) \pm g(x)\right)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b \left(c \cdot f(x)\right)\,dx = c \cdot \int_a^b f(x)\,dx$

- $\int\limits_{-a}^{a} f(x)\,dx = 0$, ako je f neparna na $[-a,a]$
- $\int\limits_{-a}^{a} f(x)\,dx = 2\int\limits_{0}^{a} f(x)\,dx$, ako je f parna na $[-a,a]$

8.1 Metoda supstitucije u određenom integralu

Zadatak 8.1. Odredite sljedeće integrale:

a)
$$
\begin{aligned}
\int\limits_{0}^{\frac{\pi}{2}} \frac{\cos x\,dx}{1+\sin^2 x} &= \left\{\begin{matrix} t=\sin x & x=0 & \mapsto & t=0 \\ dt=\cos x\,dx & x=\dfrac{\pi}{2} & \mapsto & t=1 \end{matrix}\right\} \\
&= \int\limits_{0}^{1} \frac{dt}{1+t^2} \\
&= \operatorname{arctg} t\Big|_{0}^{1} \\
&= \operatorname{arctg} 1 - \operatorname{arctg} 0 \\
&= \frac{\pi}{4}
\end{aligned}
$$

b)
$$
\begin{aligned}
\int\limits_{3}^{29} \frac{\sqrt[3]{(x-2)^2}}{\sqrt[3]{(x-2)^2}+3}\,dx &= \left\{\begin{matrix} t^3=x-2 & \Rightarrow & t=\sqrt[3]{x-2} & x=3 & \mapsto & t=1 \\ 3t^2\,dt=dx & & & x=29 & \mapsto & t=3 \end{matrix}\right\} \\
&= \int\limits_{1}^{3} \frac{t^2}{t^2+3}\cdot 3t^2\,dt \\
&= 3\int\limits_{1}^{3} \frac{t^4}{t^2+3}\,dt = (*)
\end{aligned}
$$

$$
\begin{array}{rl}
t^4 & :(t^2+3) = t^2-3 \\
-(t^4+3t^2) & \\
\hline
-3t^2 & \\
-(-3t^2-9) & \\
\hline
9 &
\end{array}
$$

$$
\begin{aligned}
(*) &= 3\int_1^3 \left(t^2 - 3 + \frac{9}{t^2+3}\right)\, dt \\
&= 3\left(\frac{1}{3}t^3 - 3t + \frac{9}{\sqrt{3}}\,\mathrm{arctg}\frac{t}{\sqrt{3}}\right)\Bigg|_1^3 \\
&= 3\left(\frac{1}{3}\cdot 3^3 - 3\cdot 3 + \frac{9}{\sqrt{3}}\,\mathrm{arctg}\frac{3}{\sqrt{3}}\right) - 3\left(\frac{1}{3}\cdot 1^3 - 3\cdot 1 + \frac{9}{\sqrt{3}}\,\mathrm{arctg}\frac{1}{\sqrt{3}}\right) \\
&= 3\left(3\sqrt{3}\cdot\frac{\pi}{3}\right) - 3\left(\frac{1}{3} - 3 + 3\sqrt{3}\cdot\frac{\pi}{6}\right) \\
&= 3\sqrt{3}\pi + 8 - \frac{3\sqrt{3}\pi}{2} \\
&= 8 + \frac{3\sqrt{3}\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{c) } \int_0^{\ln 2} \sqrt{e^x - 1}\, dx &= \left\{\begin{matrix} t^2 = e^x - 1 & \Rightarrow & e^x = t^2 + 1 & x = 0 & \mapsto & t = 0 \\ 2t\,dt = e^x\,dx & \Rightarrow & \dfrac{2t\,dt}{t^2+1} = dx & x = \ln 2 & \mapsto & t = 1 \end{matrix}\right\} \\
&= \int_0^1 t\cdot\frac{2t\,dt}{t^2+1} \\
&= 2\int_0^1 \frac{t^2\,dt}{t^2+1} \\
&= 2\int_0^1 \frac{t^2+1-1}{t^2+1}\,dt \\
&= 2\int_0^1 \left(1 - \frac{1}{t^2+1}\right)\, dt \\
&= 2\left(t - \mathrm{arctg}\, t\right)\Big|_0^1 \\
&= 2\left(1 - \mathrm{arctg}\, 1\right) - 2\left(0 - \mathrm{arctg}\, 0\right) \\
&= 2 - \frac{\pi}{2}
\end{aligned}
$$

$$\text{d)} \int_1^{e^4} \frac{dx}{x\sqrt{9-2\ln x}} = \left\{ \begin{matrix} t^2 = 9 - 2\ln x & \Rightarrow & t = \sqrt{9-2\ln x} & x = 1 & \mapsto & t = 3 \\ \cancel{2}t\,dt = -\frac{\cancel{2}^1}{x}\,dx & & & x = e^4 & \mapsto & t = 1 \end{matrix} \right\}$$

$$= -\int_3^1 \frac{\cancel{t}\,dt}{\cancel{t}}$$

$$= \int_1^3 dt$$

$$= t\Big|_1^3$$

$$= 2$$

8.2 Metoda parcijalne integracije u određenom integralu

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Zadatak 8.2. Odredite sljedeće integrale:

$$\text{a)} \int_0^1 xe^x\,dx = \left\{ \begin{matrix} u = x & \Rightarrow & du = dx \\ dv = e^x\,dx & \Rightarrow & v = e^x \end{matrix} \right\}$$

$$= xe^x\Big|_0^1 - \int_0^1 e^x\,dx$$

$$= (1 \cdot e^1 - 0 \cdot e^0) - e^x\Big|_0^1$$

$$= e - (e^1 - e^0)$$

$$= 1$$

$$\text{b)} \int_0^1 x^2 e^{2x}\,dx = \left\{\begin{matrix} u = x^2 & \Rightarrow & du = 2x\,dx \\ dv = e^{2x}\,dx & \Rightarrow & v = \frac{1}{2}e^{2x} \end{matrix}\right\}$$

$$= \frac{1}{2}x^2e^{2x}\Big|_0^1 - \int_0^1 \frac{1}{\cancel{2}} \cdot \cancel{2}xe^{2x}\,dx$$

$$= \left(\frac{1}{2}\cdot 1^2 \cdot e^2 - \frac{1}{2}\cdot 0^2 \cdot e^0\right) - \int_0^1 xe^{2x}\,dx$$

$$= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = e^{2x}\,dx & \Rightarrow & v = \frac{1}{2}e^{2x} \end{matrix}\right\}$$

$$= \frac{1}{2}\cdot e^2 - \left(\frac{1}{2}xe^{2x}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx\right)$$

$$= \cancel{\frac{1}{2}\cdot e^2} - \cancel{\frac{1}{2}\cdot e^2} + \frac{1}{4}\left(e^2 - e^0\right)$$

$$= \frac{1}{4}\left(e^2 - 1\right)$$

$$\text{c)} \int_0^1 x^3 e^{2x}\,dx = \left\{\begin{matrix} u = x^3 & \Rightarrow & du = 3x^2\,dx \\ dv = e^{2x}\,dx & \Rightarrow & v = \frac{1}{2}e^{2x} \end{matrix}\right\}$$

$$= \frac{1}{2}x^3e^{2x}\Big|_0^1 - \frac{3}{2}\int_0^1 x^2e^{2x}\,dx$$

$$= \left(\frac{1}{2}\cdot 1^3 \cdot e^2 - \frac{1}{2}\cdot 0^3 \cdot e^0\right) - \frac{3}{2}\cdot\frac{1}{4}(e^2 - 1)$$

$$= \frac{1}{8}\left(e^2 + 3\right)$$

$$\text{d)} \int_1^e \ln x\,dx = \left\{\begin{matrix} u = \ln x & \Rightarrow & du = \frac{dx}{x} \\ dv = dx & \Rightarrow & v = x \end{matrix}\right\}$$

$$= x\ln x\Big|_1^e - \int_1^e \cancel{x}\cdot\frac{dx}{\cancel{x}}$$

$$
\begin{aligned}
&= (e\cdot\ln e - 1\cdot\ln 1) - x\Big|_1^e \\
&= e-(e-1) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\text{e) } \int\limits_0^1 \ln(1+x^2)\,dx &= \left\{\begin{matrix} u=\ln(1+x^2) & \Rightarrow & du=\dfrac{2x\,dx}{1+x^2} \\ dv=dx & \Rightarrow & v=x \end{matrix}\right\} \\
&= x\ln(1+x^2)\Big|_0^1 - \int\limits_0^1 x\cdot\frac{2x\,dx}{1+x^2} \\
&= \ln 2 - 2\int\limits_0^1 \frac{x^2}{1+x^2}\,dx \\
&= \ln 2 - 2\int\limits_0^1 \frac{1+x^2-1}{1+x^2}\,dx \\
&= \ln 2 - 2\int\limits_0^1 \left(1-\frac{1}{1+x^2}\right)\,dx \\
&= \ln 2 - 2\,(x-\operatorname{arctg} x)\Big|_0^1 \\
&= \ln 2 - 2\left(1-\frac{\pi}{4}\right) \\
&= \ln 2 - 2 + \frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{f) } \int\limits_0^{\frac{\pi}{2}} x\cos x\,dx &= \left\{\begin{matrix} u=x & \Rightarrow & du=dx \\ dv=\cos x\,dx & \Rightarrow & v=\sin x \end{matrix}\right\} \\
&= x\sin x\Big|_0^{\frac{\pi}{2}} - \int\limits_0^{\frac{\pi}{2}} \sin x\,dx \\
&= \frac{\pi}{2}\cdot\sin\frac{\pi}{2} + \cos x\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} - 1
\end{aligned}
$$

$$
\begin{aligned}
\text{g)}\ \int_0^\pi e^x \sin x\, dx &= \left\{\begin{matrix} u = \sin x & \Rightarrow & du = \cos x\, dx \\ dv = e^x\, dx & \Rightarrow & v = e^x \end{matrix}\right\} \\
&= \underbrace{e^x \sin x \Big|_0^\pi}_{0} - \int_0^\pi e^x \cos x\, dx \\
&= \left\{\begin{matrix} u = \cos x & \Rightarrow & du = -\sin x\, dx \\ dv = e^x\, dx & \Rightarrow & v = e^x \end{matrix}\right\} \\
&= -\left(e^x \cos x \Big|_0^\pi - \int_0^\pi e^x(-\sin x)\, dx \right) \\
&= -e^\pi \underbrace{\cos \pi}_{-1} + \underbrace{e^0 \cos 0}_{1} - \int_0^\pi e^x \sin x\, dx \\
\int_0^\pi e^x \sin x\, dx &= 1 + e^\pi - \int_0^\pi e^x \sin x\, dx \\
2\int_0^\pi e^x \sin x\, dx &= 1 + e^\pi \\
\int_0^\pi e^x \sin x\, dx &= \frac{1 + e^\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{h)}\ \int_0^1 x e^{-x}\, dx &= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = e^{-x}\, dx & \Rightarrow & v = -e^{-x} \end{matrix}\right\} \\
&= -xe^{-x} \Big|_0^1 - \int_0^1 -e^{-x}\, dx \\
&= -e^{-1} + \int_0^1 e^{-x}\, dx \\
&= -e^{-1} + (-e^{-x}) \Big|_0^1 \\
&= -e^{-1} - e^{-1} + e^0 \\
&= 1 - \frac{2}{e}
\end{aligned}
$$

8.3 Nepravi integral

1. tip: Integrand f nije ograničen na području integracije. Neka je $f : [a, b\rangle \to \mathbb{R}$ neprekidna i $\lim\limits_{x\to b^-} f(x) = +\infty$. Ako postoji $L = \lim\limits_{c\to b^-} \int\limits_a^c f(x)\,dx$ kažemo da nepravi integral **konvergira** i pišemo:

$$\int\limits_a^b f(x)\,dx = \lim_{c\to b^-} \int\limits_a^c f(x)\,dx$$

Inače, kažemo da divergira. Slično postupamo i u slučajevima kada $f(x) \to -\infty$ i za slučajeve kada f nije ograničena u a.

Zadatak 8.3. Odredite sljedeće integrale:

a) $$\begin{aligned} \int\limits_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{b\to 1^-} \int\limits_0^b \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{b\to 1^-} \left(\arcsin x \Big|_0^b \right) \\ &= \lim_{b\to 1^-} (\arcsin b - \arcsin 0) \\ &= \frac{\pi}{2} \end{aligned}$$

Integral je nepravi jer $\dfrac{1}{\sqrt{1-x^2}} \to +\infty$ kad $x \to 1^-$.

b) $$\begin{aligned} \int\limits_0^1 \frac{dx}{\sqrt[3]{x}} &= \lim_{a\to 0^+} \int\limits_a^1 \frac{dx}{\sqrt[3]{x}} \\ &= \lim_{a\to 0^+} \left(\frac{3}{2} \cdot x^{\frac{2}{3}} \Big|_a^1 \right) \\ &= \lim_{a\to 0^+} \left(\frac{3}{2} \cdot 1^{\frac{2}{3}} - \frac{3}{2} \cdot a^{\frac{2}{3}} \right) \\ &= \frac{3}{2} \end{aligned}$$

c) $$\int\limits_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int\limits_a^1 \frac{dx}{x}$$

$$
\begin{aligned}
&= \lim_{a\to 0^+}\left(\ln|x|\Big|_a^1\right)\\
&= \lim_{a\to 0^+}(\ln 1-\ln a)\\
&= 0-(-\infty)\\
&= +\infty \Longrightarrow \text{integral divergira}
\end{aligned}
$$

Napomena: Ako se točka u kojoj funkcija nije definirana ne nalazi na rubu nego unutar segmenta integracije, onda se interval rastavi na dva intervala s tom točkom kao granicom.

d)
$$
\begin{aligned}
\int_{-1}^{1}\frac{dx}{x^2} &= \int_{-1}^{0}\frac{dx}{x^2}+\int_{0}^{1}\frac{dx}{x^2}\\
&= \lim_{b\to 0^-}\int_{-1}^{b}\frac{dx}{x^2}+\lim_{a\to 0^+}\int_{a}^{1}\frac{dx}{x^2}\\
&= \lim_{b\to 0^-}\left(-\frac{1}{x}\Big|_{-1}^{b}\right)+\lim_{a\to 0^+}\left(-\frac{1}{x}\Big|_{a}^{1}\right)\\
&= \underbrace{\lim_{b\to 0^-}\frac{-1}{b}}_{+\infty}-1+\underbrace{\lim_{a\to 0^+}\frac{1}{a}}_{+\infty}-1\\
&= +\infty \Longrightarrow \text{integral divergira}
\end{aligned}
$$

2. tip: Područje integracije je neograničeno. Neka je $f:[a,+\infty\rangle\to\mathbb{R}$ neprekidna. Ako postoji $L=\lim\limits_{b\to+\infty}\int\limits_a^b f(x)\,dx$ kažemo da nepravi integral **konvergira** i pišemo:

$$
\int_a^{+\infty} f(x)\,dx=\lim_{b\to+\infty}\int_a^b f(x)\,dx
$$

Inače, kažemo da divergira. Slično postupamo i u slučajevima kada $f:\langle-\infty,b]\to\mathbb{R}$ i $f:\langle-\infty,+\infty\rangle\to\mathbb{R}$.

Zadatak 8.4. Odredite sljedeće integrale:

a) $$\int_0^{+\infty} e^{-x}\,dx = \lim_{b\to+\infty}\int_0^b e^{-x}\,dx = \lim_{b\to+\infty}\left(-e^{-x}\Big|_0^b\right) = \lim_{b\to+\infty}\left(-e^{-b}+e^{0}\right) = 1$$

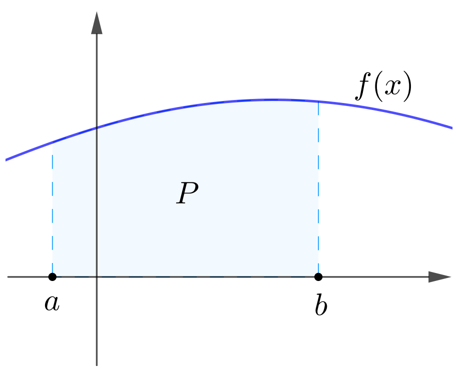
b) $$\int_1^{+\infty}\frac{dx}{x} = \lim_{b\to+\infty}\int_1^b\frac{dx}{x} = \lim_{b\to+\infty}\left(\ln x\Big|_1^b\right) = \lim_{b\to+\infty}\ln b = +\infty$$

c) $$\begin{aligned}\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} &= \int_{-\infty}^{0}\frac{dx}{1+x^2}+\int_{0}^{+\infty}\frac{dx}{1+x^2}\\ &= \lim_{a\to-\infty}\int_a^0\frac{dx}{1+x^2}+\lim_{b\to+\infty}\int_0^b\frac{dx}{1+x^2}\\ &= \lim_{a\to-\infty}\left(\operatorname{arctg} x\Big|_a^0\right)+\lim_{b\to+\infty}\left(\operatorname{arctg} x\Big|_0^b\right)\\ &= \lim_{a\to-\infty}(\underbrace{\operatorname{arctg} 0}_{0}-\underbrace{\operatorname{arctg} a}_{-\frac{\pi}{2}})+\lim_{b\to+\infty}(\underbrace{\operatorname{arctg} b}_{\frac{\pi}{2}}-\underbrace{\operatorname{arctg} 0}_{0})\\ &= \pi\end{aligned}$$

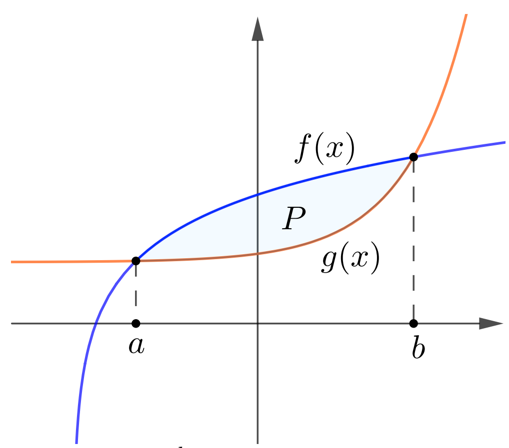
Poglavlje 9

Primjena integrala

9.1 Površine ravninskih likova



$$P = \int_a^b f(x)\,dx$$



$$P = \int_a^b \left(f(x) - g(x)\right)\,dx$$

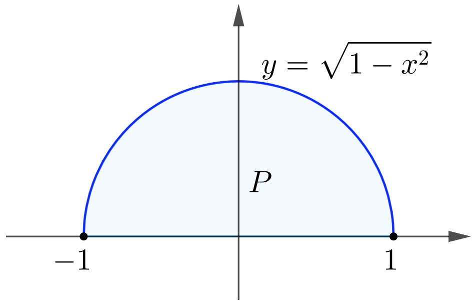
Zadatak 9.1. Odredite površinu lika omeđenog krivuljom:

a) $y = \sqrt{1-x^2}$ i osi x.

Odredimo sjecišta krivulja:

$$\begin{aligned}\sqrt{1-x^2} &= 0\\ 1-x^2 &= 0\\ x^2 &= 1\end{aligned}$$

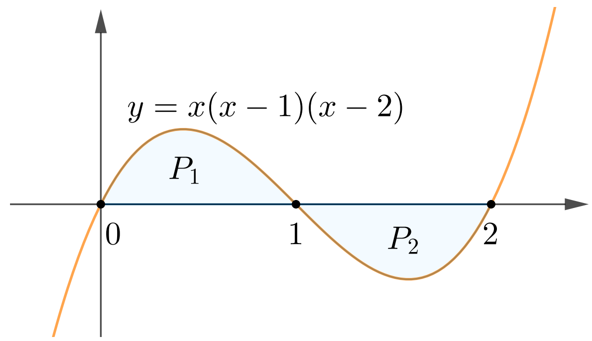
$x_1 = -1$ i $x_2 = 1$



$$\begin{aligned}
P &= \int_{-1}^{1} \sqrt{1-x^2}\,dx\\
&= \left\{\begin{matrix} x=\sin t & x=-1 & \mapsto & t=-\dfrac{\pi}{2}\\ dx=\cos t\,dt & x=1 & \mapsto & t=\dfrac{\pi}{2}\end{matrix}\right\}\\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t\,dt\\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt\\
&= \left(\frac{1}{2}t-\frac{1}{4}\sin(2t)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= \left(\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{4}\sin\pi\right)-\left(\frac{1}{2}\cdot\frac{-\pi}{2}-\frac{1}{4}\sin(-\pi)\right)\\
&= \frac{\pi}{4}-0+\frac{\pi}{4}+0\\
&= \frac{\pi}{2}
\end{aligned}$$

b) $y = x(x-1)(x-2)$ i osi x.

$$\begin{aligned} y &= x(x-1)(x-2) \\ &= (x^2-x)(x-2) \\ &= x^3-2x^2-x^2+2x \\ &= x^3-3x^2+2x \end{aligned}$$



$$\begin{aligned} P_1 &= \int_0^1 \left(x^3-3x^2+2x\right)\, dx \\ &= \left(\frac{1}{4}x^4 - x^3 + x^2\right)\Big|_0^1 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P_2 &= -\int_1^2 \left(x^3-3x^2+2x\right)\, dx \\ &= -\left(\frac{1}{4}x^4 - x^3 + x^2\right)\Big|_1^2 \\ &= -(4-8+4)+\left(\frac{1}{4}-1+1\right) \\ &= \frac{1}{4} \end{aligned}$$

$$P = P_1 + P_2 = \frac{1}{2}$$

Zadatak 9.2. Odredite površinu lika omeđenog krivuljama:

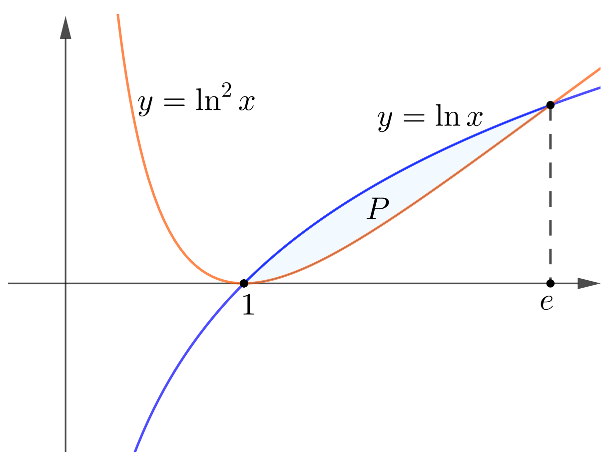
a) $y = \ln x$ i $y = \ln^2 x$

Odredimo sjecišta krivulja:

$$\begin{aligned} \ln^2 x &= \ln x \\ \ln^2 x - \ln x &= 0 \\ \ln x(\ln x - 1) &= 0 \end{aligned}$$

$\ln x = 0$ i $\ln x = 1$

$x_1 = 1$ i $x_2 = e$



$$\begin{aligned} P &= \int_1^e \left(\ln x - \ln^2 x\right)\, dx \\ &= \underbrace{\int_1^e \ln x\, dx}_{\Delta} - \underbrace{\int_1^e \ln^2 x\, dx}_{\square} \end{aligned}$$

$$\begin{aligned}
\Delta &= \int_1^e \ln x \, dx \\
&= \left\{ \begin{array}{lll} u = \ln x & \Rightarrow & du = \dfrac{dx}{x} \\ dv = dx & \Rightarrow & v = x \end{array} \right\} \\
&= x \ln x \Big|_1^e - \int_1^e \cancel{x} \cdot \frac{dx}{\cancel{x}} \\
&= e - (e-1) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\square &= \int_1^e \ln^2 x \, dx \\
&= \left\{ \begin{array}{lll} u = \ln^2 x & \Rightarrow & du = \dfrac{2 \ln x \, dx}{x} \\ dv = dx & \Rightarrow & v = x \end{array} \right\} \\
&= x \ln^2 x \Big|_1^e - \int_1^e \cancel{x} \cdot \frac{2 \ln x \, dx}{\cancel{x}} \\
&= e - 2 \underbrace{\int_1^e \ln x \, dx}_{\Delta} \\
&= e - 2
\end{aligned}$$

$$\begin{aligned}
P &= 1 - (e-2) \\
P &= 3 - e
\end{aligned}$$

b) $y = -x^2 - 2x + 1$ i $y = -\dfrac{1}{2}x$

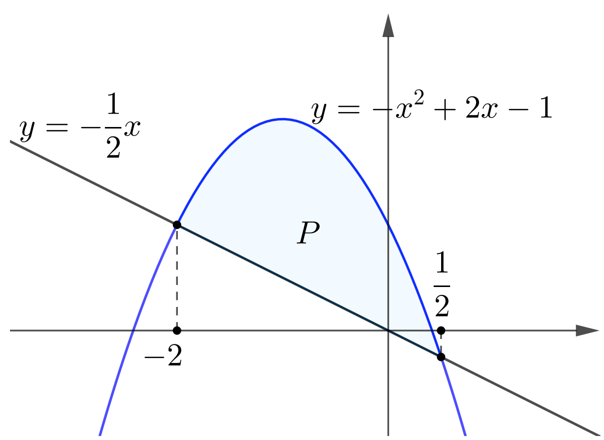
Odredimo sjecišta krivulja:

$$\begin{aligned}
-x^2 - 2x + 1 &= -\frac{1}{2}x \\
-x^2 - \frac{3}{2}x + 1 &= 0 \quad / \cdot (-2) \\
2x^2 + 3x - 2 &= 0
\end{aligned}$$

$$x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot (-2)}}{4}$$

$$x_{1,2} = \frac{-3 \pm \sqrt{25}}{4}$$

$x_1 = -2$ i $x_2 = \dfrac{1}{2}$



$$\begin{aligned}
P &= \int_{-2}^{\frac{1}{2}} \left(-x^2 - 2x + 1 + \frac{1}{2}x \right) \, dx \\
&= \int_{-2}^{\frac{1}{2}} \left(-x^2 - \frac{3}{2}x + 1 \right) \, dx
\end{aligned}$$

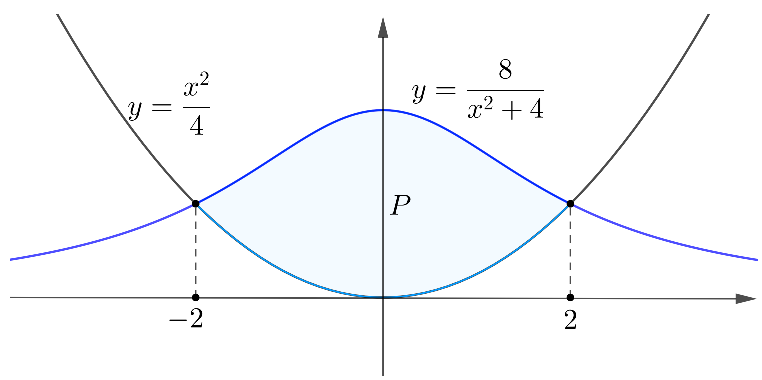
$$
\begin{aligned}
&= \left(-\frac{1}{3}x^3 - \frac{3}{4}x^2 + x\right)\Big|_{-2}^{\frac{1}{2}}\\
&= \left(-\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{3}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\right) - \left(-\frac{1}{3}\left(-2\right)^3 - \frac{3}{4}\left(-2\right)^2 - 2\right)\\
&= \left(-\frac{1}{24} - \frac{3}{16} + \frac{1}{2}\right) - \left(\frac{8}{3} - 5\right)\\
&= \frac{-2-9+24}{48} + \frac{7}{3}\\
&= \frac{13}{48} + \frac{7}{3}\\
&= \frac{13+112}{48}\\
&= \frac{125}{48}
\end{aligned}
$$

c) $y = \dfrac{8}{x^2+4}$ i $y = \dfrac{x^2}{4}$

Odredimo sjecišta krivulja:

$$
\begin{aligned}
\frac{8}{x^2+4} &= \frac{x^2}{4}\\
x^4 + 4x^2 - 32 &= 0\\
x^4 + 8x^2 - 4x^2 - 32 &= 0\\
x^2(x^2+8) - 4(x^2+8) &= 0\\
(x^2-4)(x^2+8) &= 0\\
(x-2)(x+2)(x^2+8) &= 0
\end{aligned}
$$

$x_1 = -2$ i $x_2 = 2$

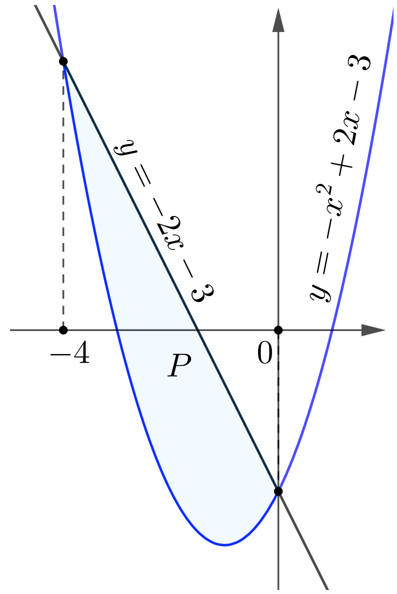


$$
\begin{aligned}
P &= \int_{-2}^{2}\left(\frac{8}{x^2+4} - \frac{x^2}{4}\right)dx\\
&= \left(\cancel{8}^4 \cdot \frac{1}{\cancel{2}}\text{arctg}\,\frac{x}{2} - \frac{1}{4}\cdot\frac{1}{3}x^3\right)\Big|_{-2}^{2}\\
&= \left(4\text{arctg}\,1 - \frac{1}{12}\cdot 2^3\right) - \left(4\text{arctg}\,(-1) - \frac{1}{12}\cdot(-2)^3\right)
\end{aligned}
$$

$$= \left(\cancel{4}\cdot\frac{\pi}{\cancel{4}} - \frac{\cancel{8}^{2}}{\cancel{12}^{3}}\right) - \left(\cancel{4}\cdot\frac{-\pi}{\cancel{4}} - \frac{\cancel{-8}^{-2}}{\cancel{12}^{3}}\right)$$

$$= 2\pi - \frac{4}{3}$$

d) $y = x^2 + 2x - 3$ i $y = -2x - 3$



Odredimo sjecišta krivulja:

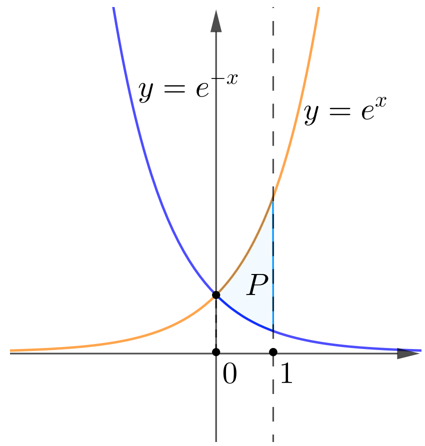
$$\begin{aligned} x^2 + 2x - 3 &= -2x - 3 \\ x^2 + 4x &= 0 \\ x(x+4) &= 0 \end{aligned}$$

$x_1 = -4$ i $x_2 = 0$

$$\begin{aligned} P &= \int_{-4}^{0} \left(-2x - 3 - x^2 - 2x + 3\right)\, dx \\ &= \int_{-4}^{0} \left(-x^2 - 4x\right)\, dx \\ &= \left(-\frac{1}{3}x^3 - 2x^2\right)\Big|_{-4}^{0} \\ &= \left(-\frac{1}{3}\cdot 0^3 - 2\cdot 0^2\right) - \left(-\frac{1}{3}\cdot(-4)^3 - 2\cdot(-4)^2\right) \\ &= -\frac{64}{3} + 32 \\ &= \frac{32}{3} \end{aligned}$$

e) $y = e^x$, $y = e^{-x}$ i $x = 1$

Odredimo sjecišta krivulja:

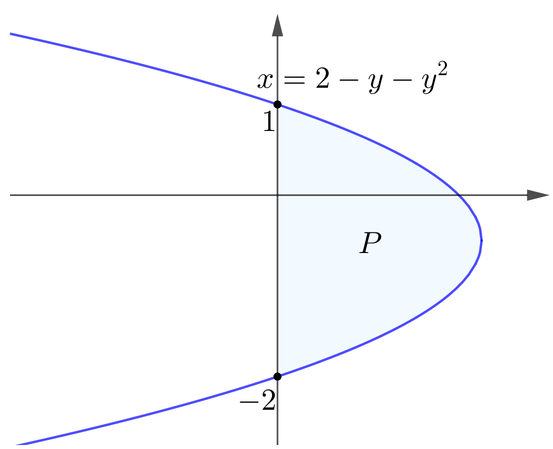


$$\begin{aligned} e^x &= e^{-x} \\ x &= 0 \end{aligned}$$

$$\begin{aligned} P &= \int_0^1 \left(e^x - e^{-x}\right)\, dx \\ &= \left(e^x + e^{-x}\right)\Big|_0^1 \\ &= (e^1 + e^{-1}) - (e^0 + e^0) \\ &= e + \frac{1}{e} - 2 \end{aligned}$$

f) $x = 2 - y - y^2$ i osi y

Odredimo sjecišta krivulje s osi y:



$$\begin{aligned} 2 - y - y^2 &= 0 \\ y^2 + y - 2 &= 0 \\ (y+2)(y-1) &= 0 \end{aligned}$$

$y_1 = -2$ i $y_2 = 1$

$$\begin{aligned} P &= \int_{-2}^1 \left(2 - y - y^2\right)\, dy \\ &= \left(2y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_{-2}^1 \\ &= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) \end{aligned}$$

$$= \frac{9}{2}$$

Zadatak 9.3. U presječnim točkama pravca $y = 0$ i parabole $y = x^2 - 4$ povučene su normale na parabolu. Odredite površinu određenu normalama i parabolom.

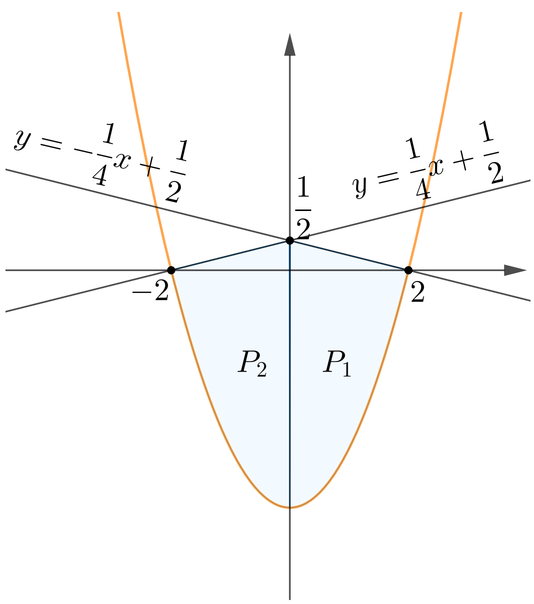
$y' = 2x$, pa je $y'(2) = 4$, a $y'(-2) = -4$.

Normala na parabolu u točki $T_1(-2, 0)$ je:

$$\begin{aligned} y - 0 &= \frac{1}{4}(x+2) \\ n_1 ... y &= \frac{1}{4}x + \frac{1}{2} \end{aligned}$$

Normala na parabolu u točki $T_2(2, 0)$ je:

$$\begin{aligned} y - 0 &= -\frac{1}{4}(x-2) \\ n_2 ... y &= -\frac{1}{4}x + \frac{1}{2} \end{aligned}$$



Kako je $P_1 = P_2$, dovoljno je računati samo P_1.

$$\begin{aligned} P &= 2 \cdot P_1 \\ &= 2 \int_0^2 \left(-\frac{1}{4}x + \frac{1}{2} - x^2 + 4 \right) dx \\ &= 2 \int_0^2 \left(-x^2 - \frac{1}{4}x + \frac{9}{2} \right) dx \\ &= 2 \left(-\frac{1}{3}x^3 - \frac{1}{8}x^2 + \frac{9}{2}x \right) \Big|_0^2 \\ &= 2 \left(-\frac{1}{3} \cdot 2^3 - \frac{1}{8} \cdot 2^2 + \frac{9}{2} \cdot 2 \right) - 0 \\ &= \frac{35}{3} \end{aligned}$$

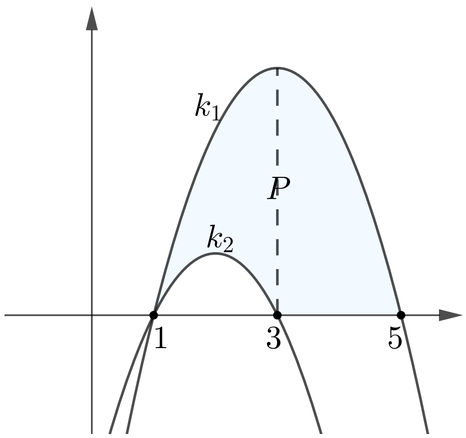
Zadatak 9.4. Odredite površinu omeđenu krivuljama $k_1 ... y = -x^2 + 6x - 5$, $k_2 ... y = -x^2 + 4x - 3$ i osi x.

Odredimo nultočke krivulja:

$$\begin{aligned} -x^2 + 6x - 5 &= 0 \\ -x^2 + 5x + x - 5 &= 0 \\ -x(x-5) + (x-5) &= 0 \\ (-x+1)(x-5) &= 0 \end{aligned}$$

$$\begin{aligned} -x^2 + 4x - 3 &= 0 \\ -x^2 + 3x + x - 3 &= 0 \\ -x(x-3) + (x-3) &= 0 \\ (-x+1)(x-3) &= 0 \end{aligned}$$

$x_1 = 1$ i $x_2 = 5$ $\qquad\qquad$ $x_1 = 1$ i $x_2 = 3$



$$\begin{aligned}
P &= \int_1^3 \left(-x^2+6x-5-(-x^2+4x-3)\right)\,dx + \int_3^5 \left(-x^2+6x-5\right)\,dx \\
&= \int_1^3 (2x-2)\,dx + \int_3^5 \left(-x^2+6x-5\right)\,dx \\
&= (x^2-2x)\Big|_1^3 + \left(-\frac{1}{3}x^3+3x^2-5x\right)\Big|_3^5 \\
&= (9-6-1+2) + \left(-\frac{125}{3}+75-25+\frac{27}{3}-27+15\right) \\
&= 4 + \left(-\frac{98}{3}+38\right) \\
&= 4 + \frac{16}{3} \\
&= \frac{28}{3}
\end{aligned}$$

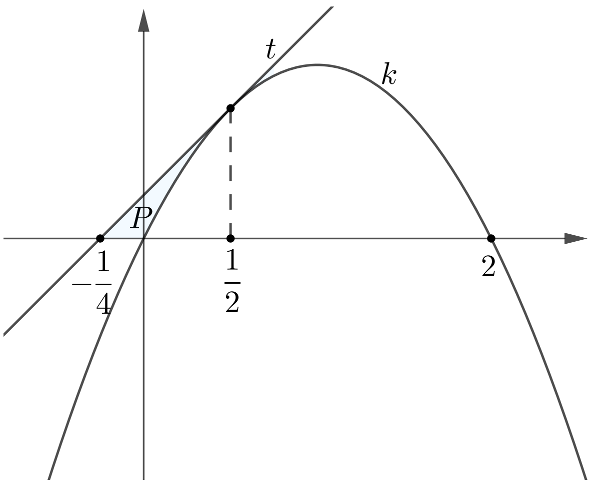
Zadatak 9.5. Odredite površinu omeđenu krivuljom $k ... y = 2x - x^2$, tangentom na krivulju u točki $T\left(\frac{1}{2}, \frac{3}{4}\right)$ i osi x.

Odredimo nultočke krivulje i jednadžbu tangente:

$$\begin{aligned}
2x - x^2 &= 0 \qquad\qquad & y' &= 2-2x \\
x(2-x) &= 0 \qquad\qquad & y'\left(\frac{1}{2}\right) &= 1 = k
\end{aligned}$$

$$x_1 = 0 \text{ i } x_2 = 2 \qquad\qquad y - \frac{3}{4} = 1\left(x - \frac{1}{2}\right)$$

$$t ... y = x + \frac{1}{4}$$



$$\begin{aligned}
P &= \int_{-\frac{1}{4}}^{0}\left(x+\frac{1}{4}\right)\,dx + \int_{0}^{\frac{1}{2}}\left(x+\frac{1}{4}-2x+x^2\right)\,dx \\
&= \int_{-\frac{1}{4}}^{0}\left(x+\frac{1}{4}\right)\,dx + \int_{0}^{\frac{1}{2}}\left(x^2-x+\frac{1}{4}\right)\,dx \\
&= \left(\frac{1}{2}x^2+\frac{1}{4}x\right)\Big|_{-\frac{1}{4}}^{0} + \left(\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{4}x\right)\Big|_{0}^{\frac{1}{2}} \\
&= 0-\left(\frac{1}{32}-\frac{1}{16}\right)+\left(\frac{1}{24}-\frac{1}{8}+\frac{1}{8}\right)-0 \\
&= \frac{1}{32}+\frac{1}{24} \\
&= \frac{7}{96}
\end{aligned}$$

Zadatak 9.6. Odredite površinu omeđenu krivuljama $k_1 ... y = 6x^2 - 5x - 1$, $k_2 ... y = \cos(\pi\, x)$, $x = 0$ i $x = \dfrac{1}{2}$.

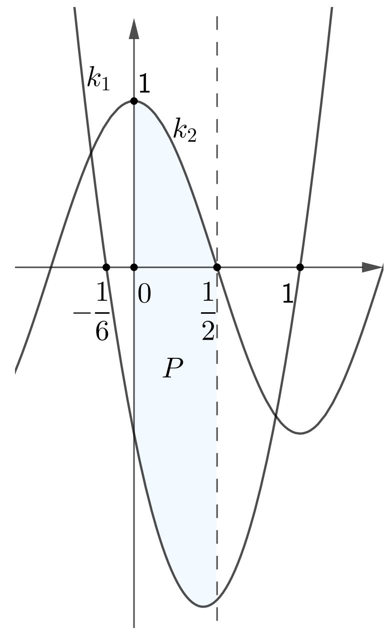
Za $x = 0$ i $x = \dfrac{1}{2}$ imamo:

$$\begin{aligned} \cos 0 &= 1 \\ \cos\frac{\pi}{2} &= 0 \end{aligned}$$

Odredimo nultočke parabole:

$$\begin{aligned} 6x^2 - 5x - 1 &= 0 \\ 6x^2 - 6x + x - 1 &= 0 \\ 6x(x-1) + (x-1) &= 0 \\ (6x+1)(x-1) &= 0 \end{aligned}$$

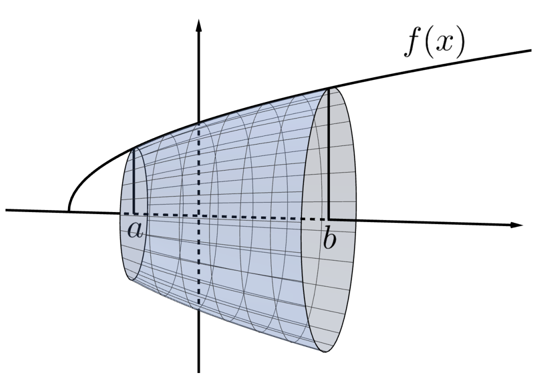
$$x_1 = -\frac{1}{6} \text{ i } x_2 = 1$$

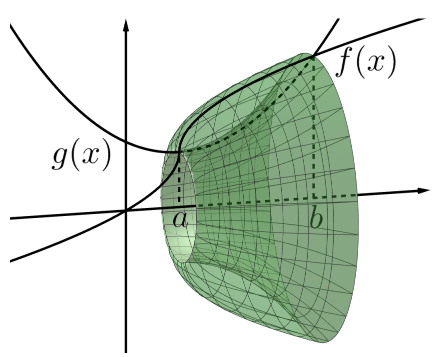


$$\begin{aligned} P &= \int_0^{\frac{1}{2}} \left(\cos(\pi\, x) - 6x^2 + 5x + 1\right)\, dx \\ &= \left(\frac{1}{\pi}\sin(\pi\, x) - 2x^3 + \frac{5}{2}x^2 + x\right)\Big|_0^{\frac{1}{2}} \\ &= \left(\frac{1}{\pi}\sin\frac{\pi}{2} - \frac{1}{4} + \frac{5}{8} + \frac{1}{2}\right) - 0 \\ &= \frac{1}{\pi} + \frac{7}{8} \end{aligned}$$

9.2 Volumen rotacijskih tijela

Volumen tijela dobivenih rotacijom oko osi x:

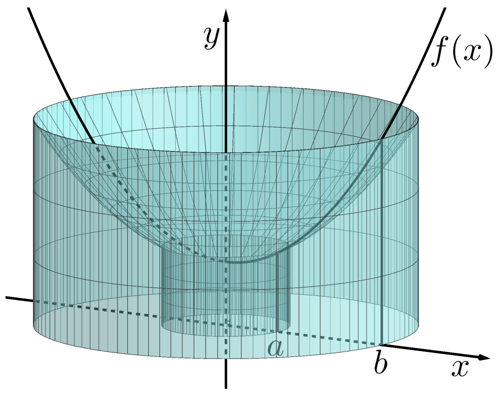




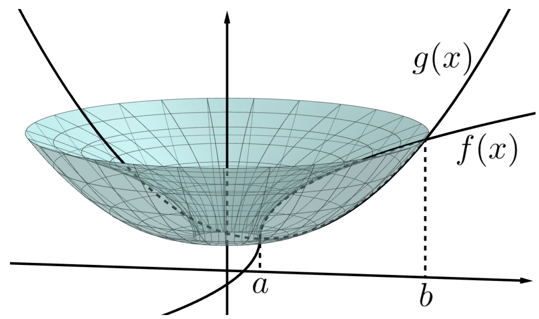
$$V_x = \pi \int_a^b f(x)^2 \, dx$$

$$V_x = \pi \int_a^b \left(f(x)^2 - g(x)^2\right) \, dx$$

Volumen tijela dobivenih rotacijom oko osi y:



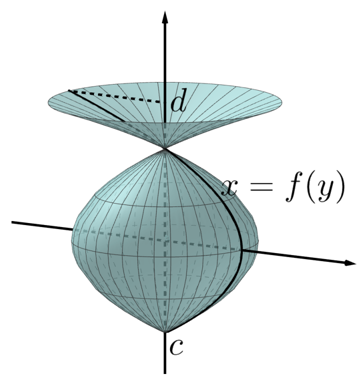
$$V_y = 2\pi \int_a^b x f(x) \, dx$$



$$V_y = 2\pi \int_a^b x \left(f(x) - g(x)\right) \, dx$$

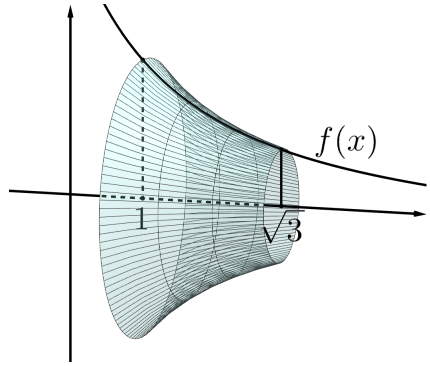
Ako je $x = f(y)$ i tijelo je dobiveno rotacijom oko osi y na segmentu $[c, d]$, onda je

$$V = \pi \int_c^d f(y)^2 \, dy$$



Zadatak 9.7. Odredite volumen tijela koje nastaje rotacijom funkcije $f(x) = \dfrac{1}{x\sqrt{1+x^2}}$ oko osi x na segmentu $\left[1, \sqrt{3}\right]$

$$\begin{aligned} V_x &= \pi \int_1^{\sqrt{3}} f(x)^2 \, dx \\ &= \pi \int_1^{\sqrt{3}} \frac{1}{x^2(1+x^2)} \, dx = (*) \end{aligned}$$



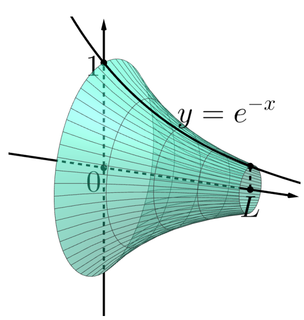
$$
\begin{aligned}
\frac{1}{x^2(1+x^2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2} \quad \Big/ \cdot x^2(1+x^2) \\
1 &= A(x^3+x) + B(1+x^2) + Cx^3 + Dx^2 \\
1 &= x^3(A+C) + x^2(B+D) + Ax + B
\end{aligned}
$$

$$
\begin{aligned}
B &= 1 \\
A &= 0 \\
A+C &= 0 \quad \Longrightarrow \quad C=0 \\
B+D &= 0 \quad \Longrightarrow \quad D=-1
\end{aligned}
$$

$$
\begin{aligned}
(*) &= \pi \int_1^{\sqrt{3}} \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx \\
&= \pi \left(-\frac{1}{x} - \operatorname{arctg} x \right) \Big|_1^{\sqrt{3}} \\
&= \pi \left(-\frac{1}{\sqrt{3}} - \underbrace{\operatorname{arctg} \sqrt{3}}_{\frac{\pi}{3}} + \frac{1}{1} + \underbrace{\operatorname{arctg} 1}_{\frac{\pi}{4}} \right) \\
&= \pi \left(1 - \frac{\sqrt{3}}{3} - \frac{\pi}{12} \right)
\end{aligned}
$$

Zadatak 9.8. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y = e^{-x}$, pravcem $x = L$ i koordinatnim osima x i y, oko osi x.

$$
\begin{aligned}
V_x &= \pi \int_0^L e^{-2x}\, dx \\
&= -\frac{\pi}{2} \cdot e^{-2x} \Big|_0^L \\
&= -\frac{\pi}{2}(e^{-2L} - 1) \\
&= \frac{\pi}{2} - \frac{\pi}{2} e^{-2L}
\end{aligned}
$$

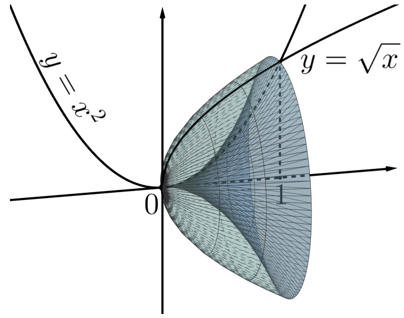


Zadatak 9.9. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2$ i $y = \sqrt{x}$, oko osi x.

Odredimo granice integriranja:

$$\begin{aligned} x^2 &= \sqrt{x} \qquad \Big/^2 \\ x^4 &= x \\ x^4 - x &= 0 \\ x(x^3-1) &= 0 \\ x(x-1)(x^2+x+1) &= 0 \end{aligned}$$

$x_1 = 0$ i $x_2 = 1$



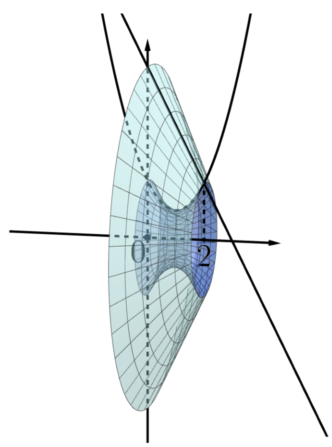
$$\begin{aligned} V_x &= \pi \int_0^1 \left(x - x^4\right)\, dx \\ &= \pi \left(\frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Bigg|_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{5}\right) \\ &= \frac{3\pi}{10} \end{aligned}$$

Zadatak 9.10. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2 - 2x + 2$, $y = -2x + 6$ i osi y, oko osi x.

Odredimo sjecišta krivulja:

$$\begin{aligned} x^2 - 2x + 2 &= -2x + 6 \\ x^2 - 4 &= 0 \end{aligned}$$

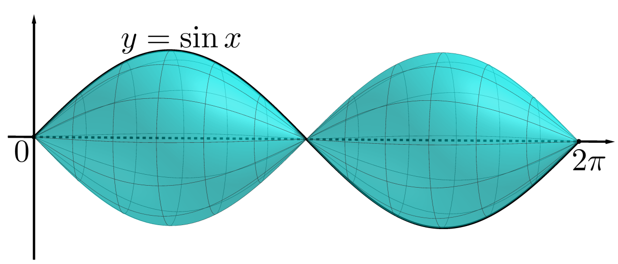
$x_1 = -2$ i $x_2 = 2$



$$V_x = \pi \int_0^2 \left[(-2x+6)^2 - \left(x^2 - 2x + 2\right)^2\right]\, dx$$

$$
\begin{aligned}
&= \pi \int_0^2 \left[\cancel{4x^2} - 24x + 36 - x^4 - \cancel{4x^2} - 4 + 4x^3 - 4x^2 + 8x\right] \, dx \\
&= \pi \int_0^2 \left[-x^4 + 4x^3 - 4x^2 - 16x + 32\right] \, dx \\
&= \pi \left[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 - 8x^2 + 32x\right] \Bigg|_0^2 \\
&= \pi \left[-\frac{1}{5} \cdot 32 + 16 - \frac{4}{3} \cdot 8 - 32 + 64\right] - 0 \\
&= \pi \left(48 + \frac{-256}{15}\right) - 0 \\
&= \frac{464\pi}{15}
\end{aligned}
$$

Zadatak 9.11. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y = \sin x$ i osi x, na segmentu $[0, 2\pi]$, oko osi x.



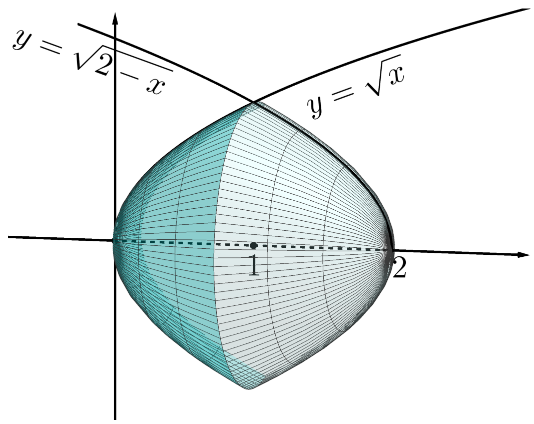
Kako imamo dva jednaka volumena, dovoljno je računati jedan.

$$
\begin{aligned}
V_x &= 2\left(\pi \int_0^\pi \sin^2 x \, dx\right) \\
&= 2\pi \int_0^\pi \frac{1 - \cos(2x)}{2} \, dx \\
&= \pi \left(x - \frac{1}{2}\sin(2x)\right) \Bigg|_0^\pi
\end{aligned}
$$

$$= \pi\left(\pi - \frac{1}{2}\sin(2\pi) - 0 + \frac{1}{2}\sin 0\right)$$

$$= \pi^2$$

Zadatak 9.12. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = \sqrt{x}$, $y = \sqrt{2-x}$ i osi x, oko osi x.

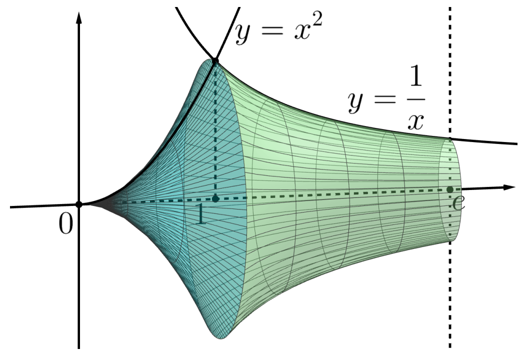


$$\begin{aligned}
V_x &= \pi\int\limits_0^1 \sqrt{x}^2\,dx + \pi\int\limits_1^2 \sqrt{2-x}^2\,dx \\
&= \pi\int\limits_0^1 x\,dx + \pi\int\limits_1^2 (2-x)\,dx \\
&= \pi\cdot\frac{1}{2}x^2\Big|_0^1 + \pi\left(2x - \frac{x^2}{2}\right)\Big|_1^2 \\
&= \frac{\pi}{2} + \pi\left(4 - 2 - 2 + \frac{1}{2}\right) \\
&= \pi
\end{aligned}$$

Zadatak 9.13. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2$, $y = \dfrac{1}{x}$, $x = e$ i osi x, oko osi x.

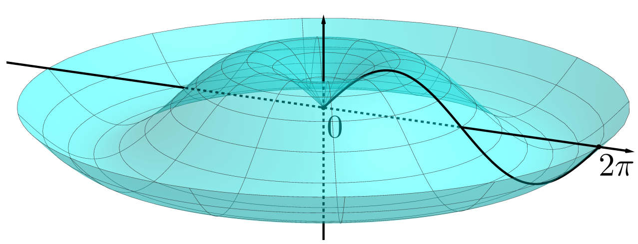
Odredimo sjecišta krivulja:

$$\begin{aligned} x^2 &= \frac{1}{x} \\ x^3 &= 1 \\ x &= 1 \end{aligned}$$



$$\begin{aligned} V_x &= \pi \int_0^1 x^4\,dx + \pi \int_1^e \frac{1}{x^2}\,dx \\ &= \pi \frac{x^5}{5}\Big|_0^1 - \pi \frac{1}{x}\Big|_1^e \\ &= \frac{\pi}{5} - \frac{\pi}{e} + \pi \\ &= \pi\left(\frac{6}{5} - \frac{1}{e}\right) \end{aligned}$$

Zadatak 9.14. Odredite volumen tijela koje nastaje rotacijom lika omeđenog kruvuljom $y = \sin x$ i osi x, na segmentu $[0, 2\pi]$, oko osi y.

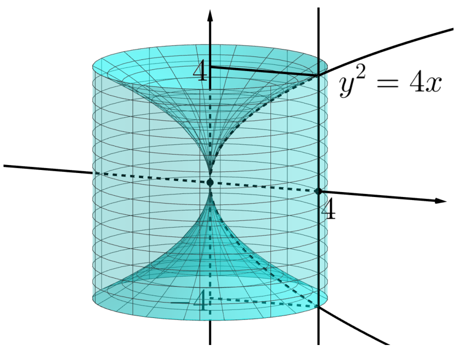


$$\begin{aligned} V_y &= 2\pi \int_0^\pi x \sin x\,dx - 2\pi \int_\pi^{2\pi} x \sin x\,dx \\ &= \left\{ \begin{array}{lcl} u = x & \Rightarrow & du = dx \\ dv = \sin x\,dx & \Rightarrow & v = -\cos x \end{array} \right\} \\ &= 2\pi \left(-x \cos x \Big|_0^\pi + \int_0^\pi \cos x\,dx + x\cos x \Big|_\pi^{2\pi} - \int_\pi^{2\pi} \cos x\,dx \right) \end{aligned}$$

$$= \quad 2\pi \left(-\pi\cos(\pi) + 0 + \sin x \Big|_0^{\pi} + 2\pi\cos(2\pi) - \pi\cos(\pi) - \sin x \Big|_{\pi}^{2\pi} \right)$$

$$= \quad 8\pi^2$$

Zadatak 9.15. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y^2 = 4x$ i $x = 4$, oko osi y.



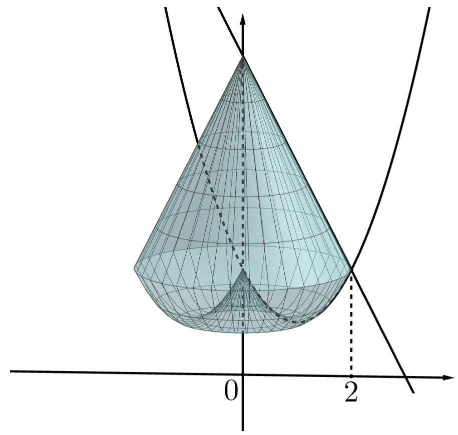
$$\begin{aligned}
V &= \pi \int_{-4}^{4} \left[4^2 - \left(\frac{y^2}{4} \right)^2 \right] dy \\
&= \pi \int_{-4}^{4} \left[16 - \frac{y^4}{16} \right] dy \\
&= \pi \left(16y - \frac{y^5}{80} \right) \Big|_{-4}^{4} \\
&= \pi \left(64 - \frac{1024}{80} + 64 - \frac{1024}{80} \right) \\
&= \pi \left(128 - \frac{1024}{40} \right) \\
&= \frac{512}{5}\pi
\end{aligned}$$

Zadatak 9.16. Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2 - 2x + 2$, $y = -2x + 6$ i osi y, oko osi y.

Odredimo sjecišta krivulja:

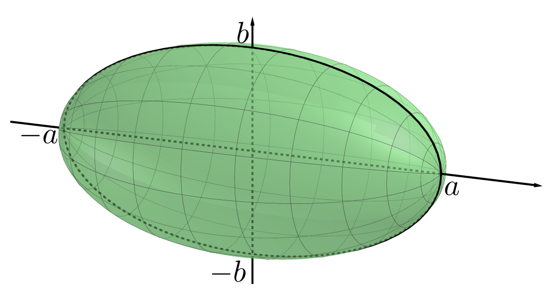
$$\begin{aligned} x^2 - 2x + 2 &= -2x + 6 \\ x^4 - 4 &= 0 \end{aligned}$$

$x_1 = -2$ i $x_2 = 2$



$$\begin{aligned} V_y &= 2\pi \int_0^2 x\left[-2x + 6 - \left(x^2 - 2x + 2\right)\right]\,dx \\ &= 2\pi \int_0^2 x\left(-x^2 + 4\right)\,dx \\ &= 2\pi \int_0^2 \left(-x^3 + 4x\right)\,dx \\ &= 2\pi \left(-\frac{1}{4}x^4 + 2x^2\right)\Big|_0^2 \\ &= 2\pi\left(-4 + 8\right) \\ &= 8\pi \end{aligned}$$

Zadatak 9.17. Odredite volumen tijela koje nastaje rotacijom elispe $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ oko osi x.



$$
\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1 \quad \Big/\cdot b^2\\
\frac{b^2x^2}{a^2}+y^2 &= b^2\\
y^2 &= b^2-\frac{b^2x^2}{a^2}\\
y^2 &= b^2\left(1-\frac{x^2}{a^2}\right)
\end{aligned}
$$

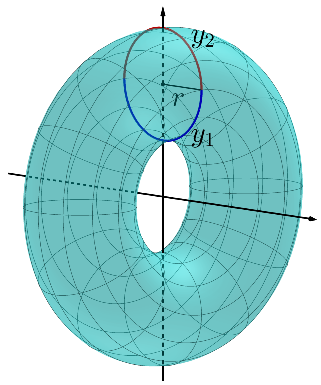
$$
\begin{aligned}
V_x &= 2\pi\int\limits_0^a \left(b^2\left(1-\frac{x^2}{a^2}\right)\right)\,dx\\
&= 2b^2\pi\int\limits_0^a \left(1-\frac{x^2}{a^2}\right)\,dx\\
&= 2b^2\pi\left(x-\frac{x^3}{3a^2}\right)\Big|_0^a\\
&= 2b^2\pi\left(a-\frac{a^{\cancel{3}}}{3\cancel{a^{2}}}\right)-0\\
&= \frac{4}{3}ab^2\pi
\end{aligned}
$$

Uočimo da, ako je $a=b=r$, promatrano tijelo postaje kugla, a rezultat zadatka nam daje da je $V=\frac{4}{3}r^3\pi$.

Zadatak 9.18. Odredite volumen tijela koje nastaje rotacijom kruga $x^2+(y-b)^2=r^2$, gdje je $r\le b$, oko osi x.

Rotacijom kruga dobije se torus.

$$
\begin{aligned}
x^2+(y-b)^2 &= r^2\\
(y-b)^2 &= r^2-x^2 \quad \Big/\sqrt{}\\
y_1 &= b-\sqrt{r^2-x^2}\\
y_2 &= b+\sqrt{r^2-x^2}
\end{aligned}
$$



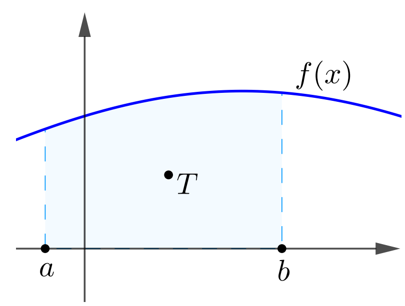
$$
\begin{aligned}
V_x &= \pi \int\limits_{-r}^{r} \left(y_2^2 - y_1^2\right)\, dx \\
&= \pi \int\limits_{-r}^{r} \left(\left(b + \sqrt{r^2 - x^2}\right)^2 - \left(b - \sqrt{r^2 - x^2}\right)^2\right)\, dx \\
&= \pi \int\limits_{-r}^{r} \left(\cancel{b^2} + 2b\sqrt{r^2 - x^2} + \cancel{r^2} - \cancel{x^2} - \cancel{b^2} + 2b\sqrt{r^2 - x^2} - \cancel{r^2} + \cancel{x^2}\right)\, dx \\
&= 4b\pi \int\limits_{-r}^{r} \sqrt{r^2 - x^2}\, dx \\
&= \left\{\begin{matrix} x = r\sin t & x = -r & \mapsto & t = -\dfrac{\pi}{2} \\ dt = r\cos t\, dt & x = r & \mapsto & t = \dfrac{\pi}{2} \end{matrix}\right\} \\
&= 4b\pi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t} \cdot r\cos t\, dt \\
&= 4r^2 b\pi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t} \cdot \cos t\, dt \\
&= 4r^2 b\pi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\, dt \\
&= 4r^2 b\pi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2t)}{2}\, dt
\end{aligned}
$$

$$
\begin{aligned}
&= 2r^2 b\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2t))\ dt \\
&= 2r^2 b\pi \left(t + \frac{1}{2}\sin(2t)\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 2r^2 b\pi \left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi) + \frac{\pi}{2} - \frac{1}{2}\sin(-\pi)\right) \\
&= 2r^2 b\pi^2
\end{aligned}
$$

9.3 Težište

Neka je $f : [a,b] \to \mathbb{R}$ neprekidna. Koordinate težišta T **lika** omeđenog grafom ove funkcije i osi x dane su formulama:

$$x_T = \frac{\int_a^b x f(x)\ dx}{\int_a^b f(x)\ dx} \qquad \text{i} \qquad y_T = \frac{\int_a^b f(x)^2\ dx}{2\int_a^b f(x)\ dx}$$



Zadatak 9.19. Odredite težište homogenog lika gustoće 1 omeđenog grafom funkcije $f(x) = \sin x$ na $[0,\pi]$ i osi x.

$$x_T = \frac{\int_0^\pi x\sin x\ dx}{\int_0^\pi \sin x\ dx}$$

$$
\begin{aligned}
\int_0^\pi x\sin x\ dx &= \left\{ \begin{array}{lll} u = x & \Rightarrow & du = dx \\ dv = \sin x\ dx & \Rightarrow & v = -\cos x \end{array} \right\} \\
&= -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\ dx
\end{aligned}
\qquad
\begin{aligned}
\int_0^\pi \sin x\ dx &= -\cos x\Big|_0^\pi \\
&= 1-(-1)
\end{aligned}
$$

$$= \quad \pi + \sin x \Big|_0^{\pi} \qquad\qquad = \quad 2$$

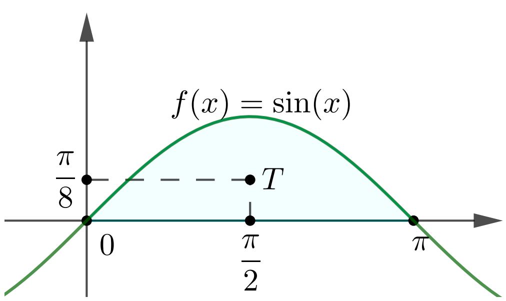
$$= \quad \pi$$

$$x_T = \frac{\pi}{2}$$

$$y_T = \frac{\int\limits_0^{\pi} \sin^2 x \, dx}{2\int\limits_0^{\pi} \sin x \, dx}$$

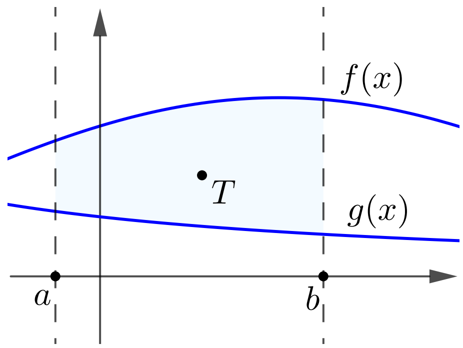
$$\begin{aligned}
\int\limits_0^{\pi} \sin^2 x \, dx &= \int\limits_0^{\pi} \frac{1-\cos 2x}{2} \, dx \\
&= \left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\Big|_0^{\pi} \\
&= \frac{1}{2}\pi \\
y_T &= \frac{\pi}{8}
\end{aligned}$$

Imamo da su koordinate težišta $T = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$.



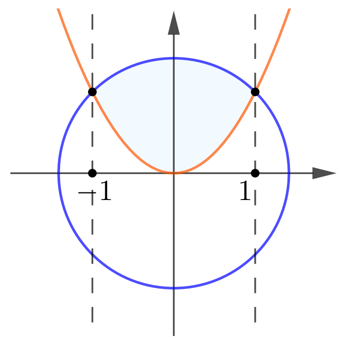
Koordinate težišta lika omeđenog grafovima neprekidnih funkcija f i g na segmentu $[a, b]$ i pravcima $x = a$ i $x = b$ dano je formulama:

$$x_T = \frac{\int\limits_a^b x\left(f(x) - g(x)\right) \, dx}{\int\limits_a^b \left(f(x) - g(x)\right) \, dx} \qquad \text{i} \qquad y_T = \frac{\int\limits_a^b \left(f(x)^2 - g(x)^2\right) \, dx}{2\int\limits_a^b \left(f(x) - g(x)\right) \, dx}$$



Zadatak 9.20. Odredite težište homogenog lika gustoće 1 omeđenog kružnicom $x^2+y^2=2$ i parabolom $y=x^2$.

Odredimo koordinate sjecišta krivulja:



$$\begin{aligned} x^2+y^2 &= 2 \\ y &= \sqrt{2-x^2} \\ x^2 &= \sqrt{2-x^2} \qquad \Big/^2 \\ x^4+x^2-2 &= 0 \end{aligned}$$

$x_1^2 = 1$, $\cancel{x_2^2 = -2}$

$x_1 = -1$, $x_2 = 1$

$$x_T = \frac{\int_{-1}^{1} x\left(\sqrt{2-x^2}-x^2\right)\,dx}{\int_{-1}^{1}\left(\sqrt{2-x^2}-x^2\right)\,dx}$$

$$\int_{-1}^{1} x\left(\sqrt{2-x^2}-x^2\right)\,dx = 0 \qquad \text{jer je podintegralna funkcija neparna.}$$

$$x_T = 0$$

$$y_T = \frac{\int_{-1}^{1}\left(\left(\sqrt{2-x^2}\right)^2-\left(x^2\right)^2\right)\,dx}{2\int_{-1}^{1}\left(\sqrt{2-x^2}-x^2\right)\,dx}$$

$$\int_{-1}^{1}\left(\left(\sqrt{2-x^2}\right)^2-\left(x^2\right)^2\right)\,dx = \int_{-1}^{1}\left(2-x^2-x^4\right)\,dx$$

$$
\begin{aligned}
&= \left(2x - \frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Bigg|_{-1}^{1} \\
&= \left(2 - \frac{1}{3} - \frac{1}{5} + 2 - \frac{1}{3} - \frac{1}{5}\right) \\
&= \frac{44}{15}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} \left(\sqrt{2-x^2} - x^2\right)\, dx &= \left\{ \begin{matrix} x = \sqrt{2}\sin t & \Rightarrow & x = -1 \mapsto & t = -\frac{\pi}{4} \\ dx = \sqrt{2}\cos t dx & \Rightarrow & x = 1 \mapsto & t = \frac{\pi}{4} \end{matrix} \right\} \\
&= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 t\, dt - \frac{1}{3}x^3\Bigg|_{-1}^{1} \\
&= 2\left(\frac{1}{2}t + \frac{1}{4}\sin(2t)\right)\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \frac{2}{3} \\
&= 2\left(\frac{\pi}{8} + \frac{\pi}{8} + \frac{1}{4}(1+1)\right) - \frac{2}{3} \\
&= \frac{3\pi + 2}{6}
\end{aligned}
$$

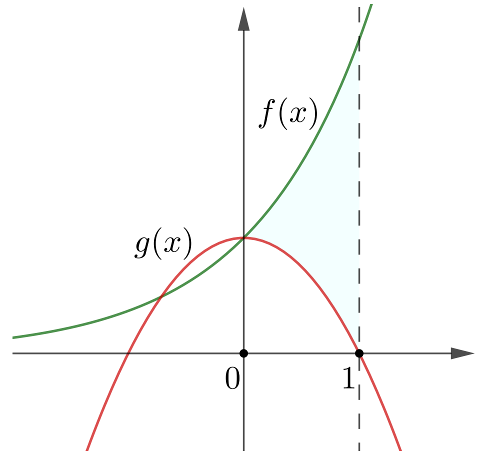
$$
y_T = \frac{\frac{44}{15}}{2 \cdot \frac{3\pi+2}{6}}
$$

Imamo da su koordinate težišta $T = \left(0, \dfrac{44}{15\pi + 10}\right)$.

Zadatak 9.21. Odredite težište homogenog lika gustoće 1 omeđenog grafovima funkcija $f(x) = e^x$ i $g(x) = 1 - x^2$ na segmentu $[0, 1]$.

$$x_T = \frac{\int_0^1 x\left(e^x - 1 + x^2\right)\ dx}{\int_0^1 \left(e^x - 1 + x^2\right)\ dx}$$

$$y_T = \frac{\int_0^1 \left(e^{2x} - 1 + 2x^2 - x^4\right)\ dx}{2\int_0^1 \left(e^x - 1 + x^2\right)\ dx}$$



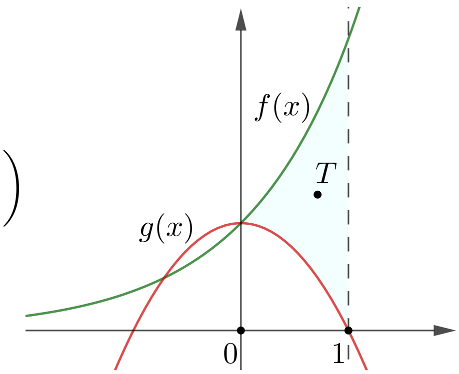
$$\begin{aligned}
\int_0^1 x\left(e^x - 1 + x^2\right)\ dx &= \int_0^1 xe^x\ dx + \int_0^1 \left(-x + x^3\right)\ dx \\
&= \left\{\begin{array}{lcl} u = x & \Rightarrow & du = dx \\ dv = e^x\ dx & \Rightarrow & v = e^x \end{array}\right\} \\
&= xe^x\Big|_0^1 - \int_0^1 e^x\ dx + \left(-\frac{1}{2}x^2 + \frac{1}{4}x^4\right)\Big|_0^1 \\
&= e - e^x\Big|_0^1 - \frac{1}{4} \\
&= e - e + 1 - \frac{1}{4} \\
&= \frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \left(e^{2x} - 1 + 2x^2 - x^4\right)\ dx &= \left(\frac{1}{2}e^{2x} - x + \frac{2}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1 \\
&= \left(\frac{1}{2}e^2 - 1 + \frac{2}{3} - \frac{1}{5}\right) - \frac{1}{2} \\
&= \frac{15e^2 - 31}{30}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \left(e^x - 1 + x^2\right)\ dx &= \left(e^x - x + \frac{1}{3}x^3\right)\Big|_0^1 \\
&= \left(e - 1 + \frac{1}{3}\right) - 1
\end{aligned}$$

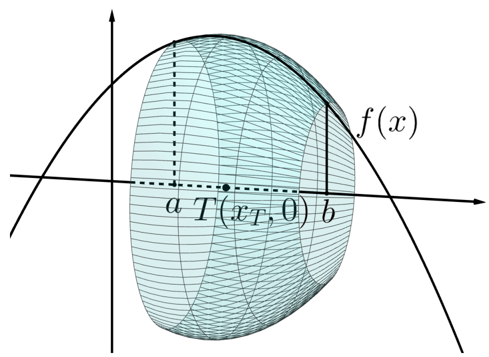
$$= \quad \frac{3e-5}{3}$$

Koordinate težišta: $T = \left(\frac{9}{12e-20}, \frac{15e^2-31}{60e-100} \right)$



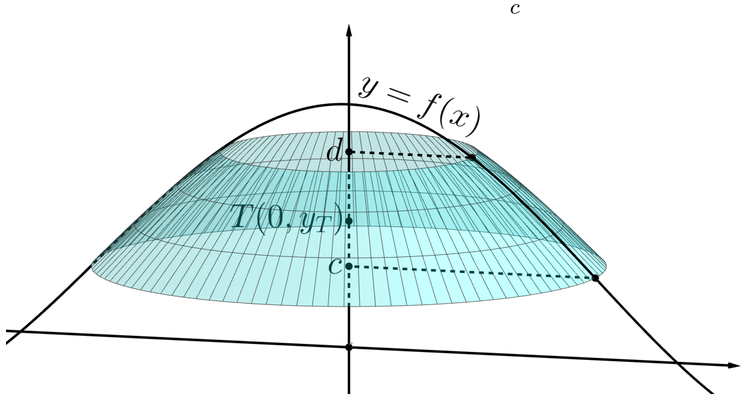
Neka je $f : [a,b] \to \mathbb{R}$ neprekidna. Koordinate težišta T **tijela** dobivenog rotacijom grafa funkcije f oko osi x dane su formulama:

$$x_T = \frac{\int_a^b x f(x)^2 \, dx}{\int_a^b f(x)^2 \, dx} \quad \text{i} \quad y_T = 0 \text{ zbog simetrije.}$$



Tijelo dobiveno rotacijom oko osi y ima koordinate težišta:

$$x_T = 0 \text{ zbog simetrije} \quad \text{i} \quad y_T = \frac{\int_c^d y f^{-1}(y)^2 \, dy}{\int_c^d f^{-1}(y)^2 \, dy}$$



Zadatak 9.22. Odredite težište homogenog tijela gustoće 1 koje se dobije rotacijom funkcije $f(x) = e^x$ na segmentu $[0,1]$:

a) oko osi x

$$x_T = \frac{\int\limits_0^1 xe^{2x}\,dx}{\int\limits_0^1 e^{2x}\,dx}$$

$$\begin{aligned}
\int\limits_0^1 xe^{2x}\,dx &= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = e^{2x}\,dx & \Rightarrow & v = \frac{1}{2}e^{2x} \end{matrix}\right\} \\
&= \frac{1}{2}xe^{2x}\Big|_0^1 - \frac{1}{2}\int\limits_0^1 e^{2x}\,dx \\
&= \frac{1}{2}\cdot 1\cdot e^2 - 0 - \frac{1}{4}e^{2x}\Big|_0^1 \\
&= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \\
&= \frac{e^2+1}{4} \\
\int\limits_0^1 e^{2x}\,dx &= \frac{1}{2}e^{2x}\Big|_0^1 \\
&= \frac{e^2}{2} - \frac{1}{2} \\
&= \frac{e^2-1}{2}
\end{aligned}$$

Imamo da su koordinate težišta $T = \left(\dfrac{e^2+1}{2e^2-2}, 0\right)$.

b) oko osi y

$$\begin{aligned}
y &= f(x) & & \\
y &= e^x \quad \Big/ \ln & x = 0 &\mapsto y = 1 \\
\ln y &= x & x = 1 &\mapsto y = e \\
f^{-1}(y) &= \ln y & &
\end{aligned}$$

$$y_T = \frac{\int\limits_1^e y \ln^2 y \, dy}{\int\limits_1^e \ln^2 y \, dy}$$

$$\begin{aligned}
\int\limits_1^e y \ln^2 y \, dy &= \left\{ \begin{matrix} u = \ln^2 y & \Rightarrow & du = 2 \ln y \cdot \dfrac{1}{y} \, dy \\ dv = y \, dy & \Rightarrow & v = \dfrac{1}{2} y^2 \end{matrix} \right\} \\
&= \frac{1}{2} y^2 \ln^2 y \Big|_1^e - \int\limits_1^e y \ln y \, dy \\
&= \frac{e^2}{2} - \int\limits_1^e y \ln y \, dy \\
&= \left\{ \begin{matrix} u = \ln y & \Rightarrow & du = \dfrac{1}{y} \, dy \\ dv = y \, dy & \Rightarrow & v = \dfrac{1}{2} y^2 \end{matrix} \right\} \\
&= \frac{e^2}{2} - \left(\frac{1}{2} y^2 \ln y \Big|_1^e - \frac{1}{2} \int\limits_1^e y \, dy \right) \\
&= \frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{4} y^2 \Big|_1^e \right) \\
&= \frac{e^2 - 1}{4}
\end{aligned}$$

$$\begin{aligned}
\int\limits_1^e \ln^2 y \, dy &= \left\{ \begin{matrix} u = \ln^2 y & \Rightarrow & du = 2 \ln y \cdot \dfrac{1}{y} \, dy \\ dv = dy & \Rightarrow & v = y \end{matrix} \right\} \\
&= y \ln^2 y \Big|_1^e - 2 \int\limits_1^e \ln y \, dy \\
&= e - 2 \int\limits_1^e \ln y \, dy \\
&= \left\{ \begin{matrix} u = \ln y & \Rightarrow & du = \dfrac{1}{y} \, dy \\ dv = dy & \Rightarrow & v = y \end{matrix} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= e - 2\left(y\ln y\Big|_1^e - \int_1^e dy \right) \\
&= e - 2\left(e - y\Big|_1^e \right) \\
&= e - 2\,(e - e + 1) \\
&= e - 2
\end{aligned}
$$

Imamo da su koordinate težišta $T = \left(0, \dfrac{e^2-1}{4e-8}\right)$.

Zadatak 9.23. Odredite težište homogenog tijela gustoće 1 dobivenog rotacijom područja omeđenog krivuljama $f(x) = x^2$ i $g(x) = \frac{x^2}{2}$ na segmentu $[0,1]$.

a) oko osi x

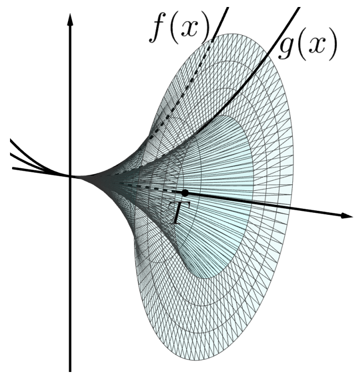
$$
x_T = \frac{\int_0^1 x\left(x^4 - \frac{x^4}{4}\right)\,dx}{\int_0^1 \left(x^4 - \frac{x^4}{4}\right)\,dx}
$$

$$
y_T = 0
$$

$$
\begin{aligned}
\int_0^1 x\left(x^4 - \frac{x^4}{4}\right)\,dx &= \frac{3}{4}\int_0^1 x^5\,dx \\
&= \frac{3}{4}\left(\frac{1}{6}x^6\right)\Bigg|_0^1 \\
&= \frac{1}{8}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 \left(x^4 - \frac{x^4}{4}\right)\,dx &= \frac{3}{4}\int_0^1 x^4\,dx \\
&= \frac{3}{4}\left(\frac{1}{5}x^5\right)\Bigg|_0^1 \\
&= \frac{3}{20}
\end{aligned}
$$

Koordinate težišta: $T = \left(\dfrac{5}{6}, 0\right)$



b) oko osi y

$$\begin{aligned} y &= f(x) & y &= g(x) \\ y &= x^2 \quad \Big/ \sqrt{} & y &= \frac{x^2}{2} \quad \Big/ \cdot 2 \\ \sqrt{y} &= x & 2y &= x^2 \quad \Big/ \sqrt{} \\ f^{-1}(y) &= \sqrt{y} & g^{-1}(y) &= \sqrt{2y} \end{aligned}$$

$$\begin{aligned} x = 0 &\mapsto y = 0 \\ x = 1 &\mapsto y = 1 \end{aligned}$$

$$x_T = 0$$

$$y_T = \frac{\int\limits_0^{\frac{1}{2}} y\left(2y - y\right)\, dy + \int\limits_{\frac{1}{2}}^{1} y\left(1 - y\right)\, dy}{\int\limits_0^{\frac{1}{2}} \left(2y - y\right)\, dy + \int\limits_{\frac{1}{2}}^{1} \left(1 - y\right)\, dy}$$

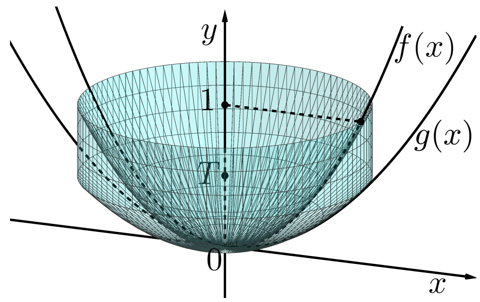
$$\begin{aligned} \int_0^{\frac{1}{2}} y\left(2y - y\right)\, dy + \int_{\frac{1}{2}}^{1} y\left(1 - y\right)\, dy &= \int_0^{\frac{1}{2}} y^2\, dy + \int_{\frac{1}{2}}^{1} \left(y - y^2\right)\, dy \\ &= \frac{1}{3}y^3\Big|_0^{\frac{1}{2}} + \left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_{\frac{1}{2}}^{1} \\ &= \tfrac{1}{8} \end{aligned}$$

$$\int_0^{\frac{1}{2}} (2y - y)\; dy + \int_{\frac{1}{2}}^{1} (1 - y)\; dy \;=\; \int_0^{\frac{1}{2}} y\, dy + \int_{\frac{1}{2}}^{1} (1 - y)\; dy$$

$$= \frac{1}{2}y^2\Big|_0^{\frac{1}{2}} + \left(y - \frac{1}{2}y^2\right)\Big|_{\frac{1}{2}}^{1}$$

$$= \tfrac{2}{8}$$

Koordinate težišta: $T = \left(0, \frac{1}{2}\right)$



9.4 Duljina luka ravninske krivulje

Duljinu luka krivulje $y = y(x)$, $x \in [a, b]$ određujemo po formuli:

$$s = \int_a^b \sqrt{1 + y'(x)^2}\, dx$$

Zadatak 9.24. Odredite duljinu luka krivulje $y = \operatorname{ch} x$ na segmentu $[0, 1]$.

$$y' = \operatorname{sh} x$$

$$\begin{aligned}
s &= \int_0^1 \sqrt{1 + \operatorname{sh}^2 x}\, dx \\
&= \int_0^1 \sqrt{\operatorname{ch}^2 x}\, dx \\
&= \operatorname{sh} x\Big|_0^1 \\
&= \operatorname{sh} 1 - \operatorname{sh} 0 \\
&= \frac{e - e^{-1}}{2}
\end{aligned}$$

Zadatak 9.25. Odredite duljinu luka krivulje $y^3 = x^2$ na segmentu $[0, 1]$.

$$\begin{aligned} y &= \sqrt[3]{x^2} = x^{\frac{2}{3}} \\ y' &= \frac{2}{3}x^{-\frac{1}{3}} \\ s &= \underbrace{\int\limits_0^1 \sqrt{1 + \frac{4}{9}x^{-\frac{2}{3}}}\, dx}_{\text{Nije jednostavno integrirati!}} \end{aligned}$$

Krivulju ćemo promatrati kao $x = x(y)$

$$\begin{aligned} y^3 &= x^2 \\ x &= \sqrt{y^3} = y^{\frac{3}{2}} \\ x' &= \frac{3}{2}y^{\frac{1}{2}} \end{aligned}$$

$x = 0 \quad \mapsto \quad y = 0, \qquad x = 1 \quad \mapsto \quad y = 1$

$$\begin{aligned} s &= \int\limits_0^1 \sqrt{1 + \frac{9}{4}y}\, dy \\ &= \left\{ \begin{matrix} t = 1 + \frac{9}{4}y & dt = \frac{9}{4}\, dy \Rightarrow & \frac{4}{9}\, dt = dy \\ y = 0 \mapsto t = 1 & & y = 1 \mapsto t = \frac{13}{4} \end{matrix} \right\} \\ &= \frac{4}{9} \int\limits_1^{\frac{13}{4}} \sqrt{t}\, dt \\ &= \frac{4}{9} \left(\frac{2}{3} t^{\frac{3}{2}} \right) \Bigg|_1^{\frac{13}{4}} \\ &= \frac{8}{27} \left(\frac{13\sqrt{13}}{8} - 1 \right) \end{aligned}$$

Zadatak 9.26. Odredite duljinu luka krivulje $y = \ln x$ na segmentu $\left[\sqrt{3}, \sqrt{8}\right]$.

$$y = \ln x$$

$$
\begin{aligned}
y' &= \frac{1}{x} \\
s &= \int\limits_{\sqrt{3}}^{\sqrt{8}} \sqrt{1+\frac{1}{x^2}}\,dx \\
&= \int\limits_{\sqrt{3}}^{\sqrt{8}} \sqrt{\frac{x^2+1}{x^2}}\,dx \\
&= \int\limits_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{x^2+1}}{x}\,dx \\
&= \left\{\begin{matrix} t^2 = 1+x^2 & \Rightarrow & x = \sqrt{t^2-1} & x=\sqrt{3} & \mapsto & t=2 \\ \cancel{2}t\,dt = \cancel{2}x\,dx & \Rightarrow & \dfrac{t\,dt}{\sqrt{t^2-1}} = dx & x=\sqrt{8} & \mapsto & t=3 \end{matrix}\right\} \\
&= \int\limits_{2}^{3} \frac{t}{\sqrt{t^2-1}} \cdot \frac{t\,dt}{\sqrt{t^2-1}} \\
&= \int\limits_{2}^{3} \frac{t^2}{t^2-1}\,dt \\
&= \int\limits_{2}^{3} \left(\frac{t^2-1+1}{t^2-1}\right)\,dt \\
&= \int\limits_{2}^{3} dt + \int\limits_{2}^{3} \frac{1}{(t-1)(t+1)}\,dt = (*) \\
\frac{1}{t^2-1} &= \frac{A}{t-1} + \frac{B}{t+1} \quad \Big/ \cdot (t^2-1) \\
1 &= A(t+1) + B(t-1) \\
1 &= t(A+B) + A - B \\
\\
0 &= A+B \\
1 &= A-B \\
A &= \frac{1}{2}
\end{aligned}
$$

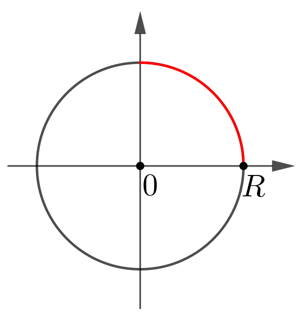
$$
\begin{aligned}
B &= -\frac{1}{2} \\
(*) &= \int_2^3 dt + \frac{1}{2}\int_2^3 \frac{dt}{t-1}\,dt - \frac{1}{2}\int_2^3 \frac{dt}{t+1}\,dt \\
&= t\Big|_2^3 + \frac{1}{2}\ln|t-1|\Big|_2^3 - \frac{1}{2}\ln|t+1|\Big|_2^3 \\
&= 3-2+\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 3 \\
&= 1+\frac{1}{2}\ln\frac{2\cdot 3}{4} \\
&= 1+\frac{1}{2}\ln\frac{3}{2}
\end{aligned}
$$

Zadatak 9.27. Odredite opseg kružnice radijusa R.

$x^2+y^2=R^2$

$y=\sqrt{R^2-x^2}$

$y'=\dfrac{-\cancel{2}x}{\cancel{2}\sqrt{R^2-x^2}}$



Zbog simetričnosti je dovoljno računati na segmentu $[0,R]$.

$$
\begin{aligned}
s &= \int_0^R \sqrt{1+\frac{x^2}{R^2-x^2}}\,dx \\
&= \int_0^R \sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}\,dx \\
&= R\int_0^R \frac{1}{\sqrt{R^2-x^2}}\,dx \\
&= R\cdot\arcsin\frac{x}{R}\Big|_0^R
\end{aligned}
$$

$$
\begin{aligned}
&= R\cdot\left(\arcsin\frac{R}{R}-\arcsin\frac{0}{R}\right)\\
&= R\cdot(\underbrace{\arcsin 1}_{\frac{\pi}{2}}-\arcsin 0)\\
&= \frac{R\pi}{2}\\
o_\circ &= 4\cdot\frac{R\pi}{2}\\
o_\circ &= 2R\pi
\end{aligned}
$$

Zadatak 9.28. Odredite duljinu krivulje $y = e^x$ na segmentu $\left[\ln\sqrt{3}, \ln\sqrt{15}\,\right]$.

$$y = e^x$$
$$y' = e^x$$

$$
\begin{aligned}
s &= \int\limits_{\ln\sqrt{3}}^{\ln\sqrt{8}} \sqrt{1+e^{2x}}\,dx\\
&= \left\{\begin{matrix} 1+e^{2x}=t^2 & \Rightarrow & e^{2x}=t^2-1 & x=\ln\sqrt{3} \;\mapsto\; t=\sqrt{1+e^{\ln\sqrt{3}^2}}=2\\ \cancel{2}e^{2x}\,dx=\cancel{2}t\,dt & \Rightarrow & dx=\dfrac{t\,dt}{t^2-1} & x=\ln\sqrt{15} \;\mapsto\; t=\sqrt{1+e^{\ln\sqrt{15}^2}}=4 \end{matrix}\right\}\\
&= \int\limits_2^4 \frac{t^2}{t^2-1}\,dt\\
&= \int\limits_2^4 \frac{t^2-1+1}{t^2-1}\,dt\\
&= \int\limits_2^4 dt + \int\limits_2^4 \frac{1}{(t-1)(t+1)}\,dt = (*)\\
\frac{1}{t^2-1} &= \frac{A}{t-1}+\frac{B}{t+1} \quad \Big/\cdot(t^2-1)\\
1 &= A(t+1)+B(t-1)\\
1 &= t(A+B)+A-B
\end{aligned}
$$

$$
\begin{aligned}
0 &= A+B \\
1 &= A-B \\
A &= \frac{1}{2} \\
B &= -\frac{1}{2} \\
(*) &= t\Big|_2^4 + \frac{1}{2}\int\limits_2^4 \frac{1}{t-1}\,dt - \frac{1}{2}\int\limits_2^4 \frac{1}{t+1}\,dt \\
&= (4-2) + \frac{1}{2}(\ln 3 - \ln 1) - \frac{1}{2}(\ln 5 - \ln 3) \\
&= 2 + \ln 3 - \frac{\ln 5}{2}
\end{aligned}
$$

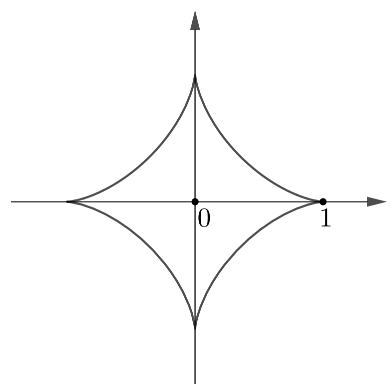
Zadatak 9.29. Odredite duljinu krivulje $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \quad \Big/'$$

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot y' = 0 \quad \Big/ \cdot \frac{3}{2}$$

$$y^{-\frac{1}{3}} \cdot y' = -x^{-\frac{1}{3}} \quad \Big/ : y^{-\frac{1}{3}}$$

$$y' = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$



Zbog simetričnosti je dovoljno računati na segmentu $[0,1]$.

$$
\begin{aligned}
s &= \int\limits_0^1 \sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,dx \\
&= \int\limits_0^1 \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,dx \\
&= \int\limits_0^1 \sqrt{\frac{1}{x^{\frac{2}{3}}}}\,dx \\
&= \int\limits_0^1 x^{-\frac{1}{3}}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1 \, dx \\
&= \frac{3}{2} \\
o &= 4 \cdot \frac{3}{2} \\
o &= 6
\end{aligned}
$$

2. način:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

$$y^{\frac{2}{3}} = 1 - x^{\frac{2}{3}}$$

$$y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$y' = \frac{3}{2}\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot \frac{-2}{3}x^{-\frac{1}{3}}$$

$$y' = -\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}$$

$$
\begin{aligned}
s &= \int\limits_0^1 \sqrt{1 + \left(1 - x^{\frac{2}{3}}\right) \cdot x^{-\frac{2}{3}}} \, dx \\
&= \int\limits_0^1 \sqrt{1 + x^{-\frac{2}{3}} - 1} \, dx \\
&= \int\limits_0^1 \sqrt{x^{-\frac{2}{3}}} \, dx \\
&= \int\limits_0^1 x^{-\frac{1}{3}} \, dx \\
&= \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1 \, dx \\
&= \frac{3}{2}
\end{aligned}
$$

$$
\begin{aligned}
o &= 4 \cdot \frac{3}{2} \\
o &= 6
\end{aligned}
$$

9.5 Oplošje rotacijske plohe

9.6 Težište luka krivulje