MATEMATIKA 3 27.1.2021.

1. (25 bodova) Riješite problem slobodnih oscilacija grede duljine 2, pri čemu je $a^2 = 4$ te su krajevi nehomogeni zglobovi, to jest, vrijedi:

$$\begin{aligned} u(0,t) &= 2, \\ \frac{\partial^2 u}{\partial x^2}(0,t) &= 2, \\ u(2,t) &= 4, \\ \frac{\partial^2 u}{\partial x^2}(2,t) &= 8. \end{aligned}$$

početna brzina je $\psi(x) = 0$, a početni položaj je

$$\varphi(x) = \sin(\pi x) + \frac{1}{2}x^3 + x^2 - 2x + 2.$$

2. (15 bodova) Odredite temperaturu izoliranog homogenog štapa duljine 14, toplinskog kapaciteta $\gamma = 2$ i koeficijenta provođenja $\delta = 18$, te s početnom raspodjelom temperature

$$u(x,0) = \sin(2\pi x) + 2\sin(3\pi x).$$

Temperatura na krajevima je $0°C$.

3. (20 bodova) Riješite problem oscilacija homogene izotropne pravokutne membrane $\Omega = [0,2] \times [0,3]$, gustoće $\rho = 2.5$ i napetosti $p = 40$. Početna brzina membrane je jednaka nuli, rubovi su homogeno pričvršćeni, a početni položaj membrane je zadan funkcijom

$$\alpha(x,y) = 0.4(2x - x^2)y.$$

MAT 3 27.1. 2021.

1. slobodne oscilacije grede → jednadžba: $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^4 u}{\partial x^4}$

duljine 2 → $l = 2$

$\boxed{a^2 = 4}$

krajevi su $\boxed{\text{nehomogeni zglobovi}}$:

$u(0,t) = 2$

$\frac{\partial^2 u}{\partial x^2}(0,t) = 2$

$u(2,t) = 4$

$\frac{\partial^2 u}{\partial x^2}(2,t) = 8$

homogenizacija, tj. rješenje je oblika

$u(x,t) = v(x,t) + w(x)$, gdje

$v(x,t)$ rješava jednadžbu $\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^4 v}{\partial x^4}$

uz $v(0,t) = v(2,t) = \frac{\partial^2 v}{\partial x^2}(0,t) = \frac{\partial^2 v}{\partial x^2}(2,t) = 0$,

a $w(x)$ rješava jednadžbu $w^{(4)}(x) = 0$

uz $w(0) = 2$, $w''(0) = 2$, $w(2) = 4$, $w''(2) = 8$.

početna brzina $\psi = 0$

početni položaj $\varphi(x) = \sin(\pi x) + \frac{1}{2}x^3 + x^2 - 2x + 2$

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{a n^2 \pi^2 t}{l^2}\right) + B_n \sin\left(\frac{a n^2 \pi^2 t}{l^2}\right) \right] \sin\left(\frac{n\pi x}{l}\right)$$

koeficijente računamo po formulama:

$$A_n = \frac{2}{l} \int_0^l \varphi_1(x) \sin\left(\frac{n\pi x}{l}\right) dx,$$

$$B_n = \frac{2l}{a n^2 \pi^2} \int_0^l \psi_1(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\varphi_1(x) = \varphi(x) - w(x)$$

$w^{(4)}(x) = 0 \quad | \int dx$

$\Rightarrow w'''(x) = C_1 \quad | \int dx$

$\Rightarrow w''(x) = C_1 x + C_2 \quad | \int dx$

$\Rightarrow w'(x) = C_1 \frac{x^2}{2} + C_2 x + C_3 \quad | \int dx$

$\Rightarrow w(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$

$2 = w(0) = C_4 \Rightarrow \boxed{C_4 = 2}$

$2 = w''(0) = C_2 \Rightarrow \boxed{C_2 = 2}$

$4 = w(2) = C_1 \frac{8}{6} + C_2 \frac{4}{2} + C_3 \cdot 2 + C_4$

$\Rightarrow 4 = \frac{4}{3} C_1 + 2 \cdot 2 + 2C_3 + 2$

$\Rightarrow -2 = \frac{4}{3} C_1 + 2C_3 \quad \Rightarrow -2 = 4 + 2C_3 \Rightarrow \boxed{C_3 = -3}$

$8 = w''(2) = C_1 \cdot 2 + C_2 \Rightarrow 8 = 2C_1 + 2 \Rightarrow \boxed{C_1 = 3}$

$w(x) = 3\frac{x^3}{6} + 2\frac{x^2}{2} - 3x + 2 \Rightarrow \boxed{w(x) = \frac{x^3}{2} + x^2 - 3x + 2}$

$$\Rightarrow \varphi_1(x) = \varphi(x) - w(x) = \sin(\pi x) + \frac{1}{2}x^3 + x^2 - 2x + 2 - \left(\frac{x^3}{2} + x^2 - 3x + 2\right) =$$

$$= \sin(\pi x) + \cancel{\frac{1}{2}x^3} + \cancel{x^2} - 2x + \cancel{2} - \cancel{\frac{x^3}{2}} - \cancel{x^2} + 3x - \cancel{2} = \sin(\pi x) + x$$

$$\psi_1(x) = \psi(x) = 0 \Rightarrow \boxed{B_n = 0,\ n \in \mathbb{N}}$$

$$A_n = \frac{2}{\ell}\int_0^\ell \varphi_1(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{2}\int_0^2 \left[\sin(\pi x) + x\right]\sin\left(\frac{n\pi x}{2}\right)dx = \underbrace{\int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx}_{I_1} + \underbrace{\int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx}_{I_2}$$

$$I_1 = \int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^2 \sin\left(\frac{2\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx = \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}$$

$$I_2 = \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx = \left\{\begin{matrix} u = x & dv = \sin\left(\frac{n\pi x}{2}\right)dx \\ du = dx & v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) \end{matrix}\right\} = \frac{-2}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \int_0^2 \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)dx =$$

$$= \frac{-2}{n\pi}2\cos\left(\frac{n\pi\cdot 2}{2}\right) - 0 + \frac{2}{n\pi}\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2 = \frac{-4}{n\pi}\overset{(-1)^n}{\overset{\parallel}{\cos(n\pi)}} + \frac{4}{n^2\pi^2}\left[\overset{0}{\overset{\parallel}{\sin(n\pi)}} - \overset{0}{\overset{\parallel}{\sin 0}}\right] =$$

$$= \frac{-4(-1)^n}{n\pi}$$

$$\Rightarrow A_n = \begin{cases} 1 - \frac{4(-1)^n}{n\pi}, & n = 2 \\ \frac{-4(-1)^n}{n\pi}, & n \neq 2 \end{cases} = \begin{cases} 1 - \frac{2}{\pi}, & n = 2 \\ \frac{-4(-1)^n}{n\pi}, & n \neq 2 \end{cases}$$

$$v(x,t) = \sum_{n\neq 2} \frac{-4(-1)^n}{n\pi}\cos\left(\frac{4n^2\pi^2 t}{2}\right)\sin\left(\frac{n\pi x}{2}\right) + \left(1 - \frac{2}{\pi}\right)\cos\left(\frac{4\cdot 4\pi^2 t}{2}\right)\sin\left(\frac{2\pi x}{2}\right)$$

$$= \sum_{n\neq 2} \frac{-4(-1)^n}{n\pi}\cos(2n^2\pi^2 t)\sin\left(\frac{n\pi x}{2}\right) + \left(1 - \frac{2}{\pi}\right)\cos(8\pi^2 t)\sin(\pi x)$$

$$\underline{u(x,t) = \sum_{n\neq 2} \frac{-4(-1)^n}{n\pi}\cos(2n^2\pi^2 t)\sin\left(\frac{n\pi x}{2}\right) + \left(1 - \frac{2}{\pi}\right)\cos(8\pi^2 t)\sin(\pi x) + \frac{x^3}{2} + x^2 - 3x + 2}$$

(2.) temperatura izoliranog homogenog štapa → jednadžba $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

duljina 14 → $l = 14$

$c^2 = \frac{\delta}{\gamma} = \frac{18}{2} = 9 \Rightarrow c = 3$

toplinski kapacitet $\gamma = 2$

$\lambda_n = \frac{n\pi}{l}$

koeficijent provođenja $\delta = 18$

početna raspodjela temperature $u(x,0) = \sin(2\pi x) + 2\sin(3\pi x) = g(x)$

temperatura na krajevima $0°C$ → $u(0,t) = u(l,t) = 0$

rješenje: $u(x,t) = \sum_{n=1}^{\infty} E_n e^{-(\lambda_n c)^2 t} \sin(\lambda_n x)$

$E_n = \frac{2}{l} \int_0^l g(x) \sin(\lambda_n x)\, dx$

Računamo:

$$E_n = \frac{2}{l}\int_0^l g(x)\sin(\lambda_n x)\,dx = \frac{2}{14}\int_0^{14}\left[\sin(2\pi x) + 2\sin(3\pi x)\right]\sin\left(\frac{n\pi x}{14}\right)dx =$$

$$= \underbrace{\frac{2}{14}\int_0^{14}\sin\left(\frac{28\pi x}{14}\right)\sin\left(\frac{n\pi x}{14}\right)dx}_{=\begin{cases}1, & n=28,\\ 0, & n\neq 28\end{cases}} + \underbrace{\frac{2}{14}\cdot 2\int_0^{14}\sin\left(\frac{42\pi x}{14}\right)\sin\left(\frac{n\pi x}{14}\right)dx}_{=\begin{cases}2, & n=42,\\ 0, & n\neq 42\end{cases}} = \begin{cases}1, & n=28,\\ 2, & n=42,\\ 0, & \text{inače.}\end{cases}$$

$$\Rightarrow u(x,t) = E_{28}\, e^{-(\lambda_{28} c)^2 t}\sin(\lambda_{28} x) + E_{42}\, e^{-(\lambda_{42} c)^2 t}\sin(\lambda_{42} x) =$$

$$= e^{-(2\pi\cdot 3)^2 t}\sin(2\pi x) + 2e^{-(3\pi\cdot 3)^2 t}\sin(3\pi x) =$$

$$= \boxed{e^{-36\pi^2 t}\sin(2\pi x) + 2e^{-81\pi^2 t}\sin(3\pi x)}$$

3.

oscilacije homogene izotropne pravokutne membrane → jednadžba $\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$

$\Omega = [0,2] \times [0,3] \rightarrow a=2, \; b=3$

$c^2 = \frac{p}{\rho} = \frac{40}{2.5} = 16 \Rightarrow c = 4$

gustoća $\boxed{\rho = 2.5}$

napetost $\boxed{p = 40}$

rješenje:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\cos(\lambda_{mn}t) + B_{mn}\sin(\lambda_{mn}t)\right]\sin\left(\frac{m\pi x}{2}\right)\sin\left(\frac{n\pi y}{b}\right)$$

početna brzina jednaka nuli → $\beta(x,y) = 0$

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a \alpha(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

rubovi homogeno pričvršćeni → $u|_{\partial\Omega} = 0$

$$B_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a \beta(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

početni položaj zadan funkcijom $\alpha(x,y) = 0.4\,(2x - x^2)\,y$

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

$\beta(x,y) = 0 \Rightarrow \boxed{B_{mn} = 0}$, za sve m,n

$$A_{mn} = \frac{4}{2\cdot 3}\int_0^3\int_0^2 0.4\,(2x-x^2)\,y\,\sin\left(\frac{m\pi x}{2}\right)\sin\left(\frac{n\pi y}{3}\right)dx\,dy =$$

$$= \frac{0.8}{3}\underbrace{\int_0^3 y\sin\left(\frac{n\pi y}{3}\right)dy}_{I_1}\;\underbrace{\int_0^2 (2x-x^2)\sin\left(\frac{m\pi x}{2}\right)dx}_{I_2}$$

$$I_1 = \int_0^3 y\sin\left(\frac{n\pi y}{3}\right)dy = \left\{\begin{matrix} u = y & dv = \sin\left(\frac{n\pi y}{3}\right)dy \\ du = dy & v = -\frac{3}{n\pi}\cos\left(\frac{n\pi y}{3}\right)\end{matrix}\right\} = \frac{-3}{n\pi}\,y\cos\left(\frac{n\pi y}{3}\right)\Big|_0^3 - \int_0^3 \frac{-3}{n\pi}\cos\left(\frac{n\pi y}{3}\right)dy =$$

$$= \frac{-3}{n\pi}\cdot 3\cos(n\pi) - 0 + \frac{3}{n\pi}\,\frac{3}{n\pi}\sin\left(\frac{n\pi y}{3}\right)\Big|_0^3 = \frac{-9(-1)^n}{n\pi} + \frac{9}{n^2\pi^2}\left(\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin 0}_{0}\right) =$$

$$= \frac{-9(-1)^n}{n\pi}$$

$$I_2 = \int_0^2 (2x-x^2)\sin\left(\frac{m\pi x}{2}\right)dx = \left\{\begin{matrix} u = 2x-x^2 & dv = \sin\left(\frac{m\pi x}{2}\right)dx \\ du = (2-2x)dx & v = \frac{-2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)\end{matrix}\right\} = \frac{-2}{m\pi}(2x-x^2)\cos\left(\frac{m\pi x}{2}\right)\Big|_0^2 - \int_0^2 \frac{2}{m\pi}(2-2x)\cos\left(\frac{m\pi x}{2}\right)dx$$

$$= \underbrace{\cancel{\frac{-2}{m\pi}(4-4)\cos(m\pi)}}_{=0} - \cancel{\frac{-2}{m\pi}(0-0)\cos 0} - \frac{2}{m\pi}\int_0^2 (2-2x)\cos\left(\frac{m\pi x}{2}\right)dx = \left\{\begin{matrix} u = 2-2x & dv = \cos\left(\frac{m\pi x}{2}\right)dx \\ du = -2dx & v = \frac{2}{m\pi}\sin\left(\frac{m\pi x}{2}\right)\end{matrix}\right\} =$$

$$= -\frac{2}{m\pi}\left(\underbrace{\frac{2}{m\pi}(2-2x)\sin\left(\frac{m\pi x}{2}\right)\Big|_0^2}_{=\sin(m\pi)-\sin 0 = 0} - \int_0^2 -2\,\frac{2}{m\pi}\sin\left(\frac{m\pi x}{2}\right)dx\right) = -\frac{8}{m^2\pi^2}\,\frac{2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)\Big|_0^2 =$$

$$= -\frac{16}{m^3\pi^3}\left(\cos(m\pi)-\cos 0\right) = \frac{-16}{m^3\pi^3}\left(\underbrace{(-1)^m - 1}\right) = \begin{cases} 0, & m \text{ paran,} \\ \frac{32}{m^3\pi^3}, & m \text{ neparan} \end{cases}$$

m paran: $1-1=0$

m neparan: $-1-1=-2$

$$\Rightarrow A_{mn} = \frac{0.8}{3} I_1 \cdot I_2 = \frac{0.8}{3} \frac{-9(-1)^n}{n\pi} \frac{-16}{m^3\pi^3}\left((-1)^m - 1\right) = \begin{cases} 0, & m \text{ paran} \\ \frac{96 \cdot 0.8 \cdot (-1)^n}{n m^3 \pi^4}, & m \text{ neparan} \end{cases}$$

$(m = 2k+1)$

$$\Rightarrow u(x,y,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{(2k+1)n} \cos(\lambda_{2k+1,n}\, t) \sin\left(\frac{(2k+1)\pi x}{2}\right) \sin\left(\frac{n\pi y}{3}\right) =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{96 \cdot 0.8 \cdot (-1)^n}{n(2k+1)^3\pi^4} \cos\left(4\pi\sqrt{\frac{(2k+1)^2}{4} + \frac{n^2}{9}}\, t\right) \sin\left(\frac{(2k+1)\pi x}{2}\right) \sin\left(\frac{n\pi y}{3}\right)$$