1. (15 bodova) Leži li točka $T(-4,6,1)$ u ravnini $\pi ... x+5y-z+2=0$? Odredite koordinate točke T' koja je simetrična točki T s obzirom na ravninu π. Skicirajte.

2. (a) (15 bodova) Odredite svojstvene vrijednosti matrice $\begin{bmatrix} 3 & -3 & 2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{bmatrix}$.

   (b) (10 bodova) Ispitajte konvergenciju reda $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n \cdot \operatorname{arctg}^n n}$.

3. (20 bodova) Odredite prirodnu domenu, intervale rasta i pada, ekstreme, područja konveksnosti i konkavnosti, asimptote, te skicirajte graf funkcije

$$f(x) = \frac{x^2}{x-1}.$$

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4. (a) (5 bodova) Odredite

$$\int x \ln x \, dx.$$

   (b) (10 bodova) Izračunajte

$$\int_{\ln 5}^{\ln 10} \sqrt{e^x - 1} \, dx.$$

5. (a) (15 bodova) Izračunajte površinu lika omeđenog krivuljom $y = 3\sqrt{x}$, njenom tangentom u točki s apscisom 4 i osi x. Skicirajte lik.

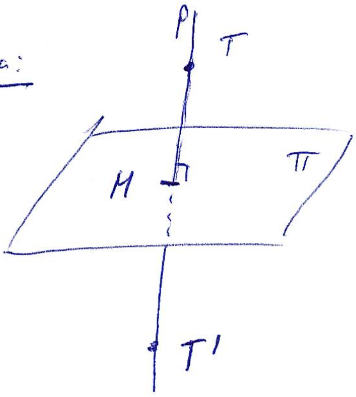
   (b) (10 bodova) Primjenom integralnog računa izračunajte opseg kružnice $x^2 + y^2 = R^2$. Skicirajte.

1. (15 b) $T(-4,6,1)$

$\pi \ldots \ x + 5y - z + 2 = 0$

$-4 + 5\cdot 6 - 1 + 2 = -4 + 30 + 1 = 27 \neq 0 \Rightarrow T \notin \pi$

skica:



① $p = (T, \vec{n})$

② $M = \frac{1}{2}|TT'|$ & $M = p \cap \pi$

$\vec{n} = (1, 5, -1)$

$$\rightarrow p \ldots \frac{x+4}{1} = \frac{y-6}{5} = \frac{z-1}{-1} \ldots \begin{cases} x = t-4 \\ y = 5t+6 \\ z = -t+1 \end{cases}$$

$p \cap \pi: \quad t - 4 + 5(5t+6) - (-t+1) + 2 = 0$

$\Rightarrow t - 4 + 25t + 30 + t - 1 + 2 = 0$

$27t + 27 = 0 \Rightarrow \underline{t = -1} \Rightarrow M = (-5, 1, 2)$

$$x_M = \frac{x_T + x_{T'}}{2} \Rightarrow -5 = \frac{-4 + x_{T'}}{2} \Rightarrow x_{T'} = -6$$

$$y_M = \frac{y_T + y_{T'}}{2} \Rightarrow 1 = \frac{6 + y_{T'}}{2} \Rightarrow y_{T'} = -4 \qquad \Rightarrow \underline{T' = (-6, -4, 3)}$$

$$z_M = \frac{z_T + z_{T'}}{2} \Rightarrow 2 = \frac{1 + z_{T'}}{2} \Rightarrow z_{T'} = 3$$

2. a) (15 b)

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -3 & 2 \\ 1 & -1-\lambda & 2 \\ 1 & -3 & 4-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 1 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & -1-\lambda \\ 1 & -3 \end{vmatrix} =$$

$$= (3-\lambda)\left[(-1-\lambda)(4-\lambda) + 6\right] + 3(4 - \lambda - 2) + 2(-3 + 1 + \lambda)$$

$$= (3-\lambda)(-4 + \lambda - 4\lambda + \lambda^2 + 6) + 3(2-\lambda) + 2(\lambda - 2) = (3-\lambda)(\lambda-1)(\lambda-2) - (\lambda-2) =$$

$$= (\lambda-2)(3\lambda - 3 - \lambda^2 + \lambda - 1) = (\lambda-2)(-\lambda^2 + 4\lambda - 4) = (\lambda-2)(\lambda-2)^2 = -(\lambda-2)^3$$

sv.vr.: $\underline{2}$

② b) (10 b) $\quad a_n = \dfrac{3^n}{2^n \operatorname{arctg}^n n}$

$$\sqrt[n]{a_n} = \frac{3}{2\operatorname{arctg} n} \quad \Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{3}{2\operatorname{arctg} n} = \frac{3}{2} \lim_{n\to\infty} \frac{1}{\operatorname{arctg} n} =$$

$$= \frac{3}{2}\,\frac{1}{\frac{\pi}{2}} = \frac{3}{2}\,\frac{2}{\pi} = \frac{3}{\pi} < 1$$

$\Rightarrow$ Red konv. prema Cauchyjevom kriteriju.

③ (20 bodova) D_f, rast/pad, ekstremi, konveksnost/konkavnost, asimptote, graf

$$f(x) = \frac{x^2}{x-1}$$

• D_f $\quad x-1 \neq 0 \Rightarrow x \neq 1 \quad \Rightarrow \quad D_f = \mathbb{R} \setminus \{1\}$

• rast/pad
ekstr.
konv./kav.

$$f'(x) = \frac{2x(x-1) - x^2 \cdot 1}{(x-1)^2} = \frac{2x^2-2x-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}$$

$$f'(x) = 0 \iff x^2 - 2x = 0 \iff x(x-2) = 0 \Rightarrow$$

stac. točke:
$x_1 = 0$
$x_2 = 2$

$$f''(x) = \frac{(2x-2)(x-1)^2 - (x^2-2x)\cdot 2(x-1)\cdot 1}{(x-1)^4} = \frac{\cancel{(x-1)}(2x^2-4x+2-2x^2+4x)}{(x-1)^{\cancel{4}\,3}} =$$

$$= \frac{2}{(x-1)^3}$$

$f''(x) \neq 0, \ \forall x \in D_f$

		0		1		2	
f'	+	0	−		−	0	+
f	↗		↘		↘		↗
f''	−		−		+		+
f	∩		∩		∪		∪

[intervali rasta: $\langle -\infty, 0\rangle$, $\langle 2, +\infty\rangle$
intervali pada: $\langle 0, 1\rangle$, $\langle 1, 2\rangle$

[int. konveksnosti: $\langle 1, +\infty\rangle$
int. konkavnosti: $\langle -\infty, 1\rangle$

[lok. max u $x = 0$, lok. min u $x = 2$ $\qquad f(0) = 0$, $f(2) = 4$

• asimptote:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2}{x-1} = +\infty$$

nema H.A.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2}{x-1} = \frac{1}{0^-} = -\infty \quad \Rightarrow \quad x = 1 \text{ je } \underline{\text{V.A.}}$$

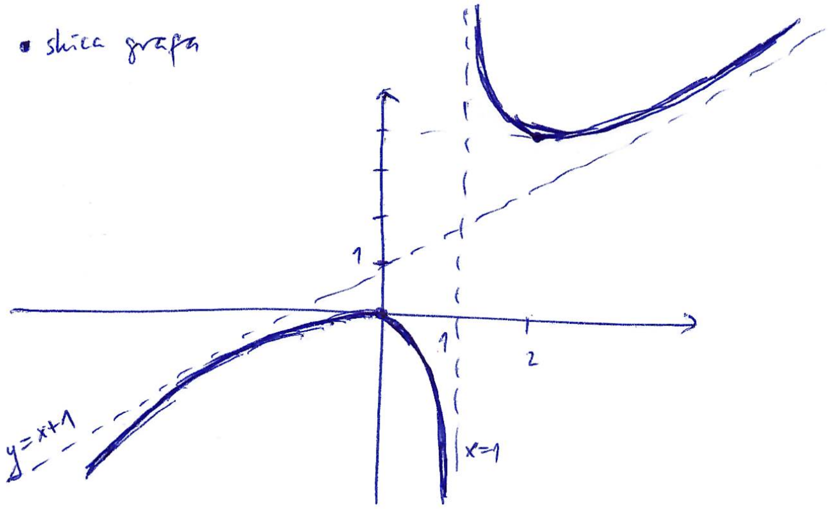
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2}{x-1} = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2}{x^2-x} = 1 = k$$

$$\lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \left(\frac{x^2}{x-1} - x\right) = \lim_{x\to\pm\infty} \frac{x^2 - x(x-1)}{x-1} = \lim_{x\to\pm\infty} \frac{x^2-x^2+x}{x-1} = \lim_{x\to\pm\infty} \frac{x}{x-1} = 1 = l$$

$\Rightarrow y = x + 1$ je $\underline{\text{K.A.}}$

• skica grafa



4. a) (5 b)

$$\int x \ln x \, dx = \left| \begin{matrix} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{matrix} \right| = \frac{x^2 \ln x}{2} - \int \frac{1}{x}\frac{x^2}{2}dx =$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

4. b) $$\int_{\ln 5}^{\ln 10} \sqrt{e^x - 1}\, dx = \left\{ \begin{matrix} t^2 = e^x - 1 \Rightarrow t = \sqrt{e^x - 1} \\ \Rightarrow e^x = t^2 + 1 \\ 2t\,dt = e^x dx \Rightarrow dx = \frac{2t\,dt}{t^2+1} \end{matrix} \right.$$

$x = \ln 5 \mapsto t = 2$

$x = \ln 10 \mapsto t = 3$

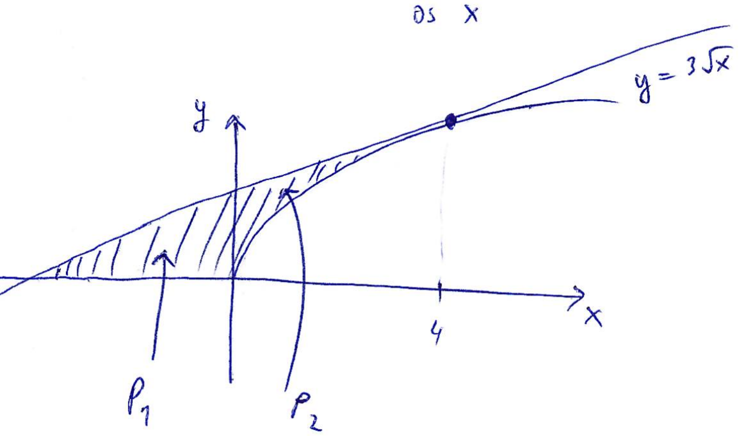
$$= \int_2^3 \frac{2t^2 dt}{t^2+1} = 2\int_2^3 \frac{t^2+1}{t^2+1}dt - 2\int_2^3 \frac{dt}{t^2+1} =$$

$$= 2t\Big|_2^3 - 2\,\mathrm{arctg}\Big|_2^3 = 2 - 2\,\mathrm{arctg}\,3 + 2\,\mathrm{arctg}\,2$$

(5.) a) (10 b) $y = 3\sqrt{x}$

njena tangenta u točki s $x=4$

os x



P_1 površina ispod pravca

od točke a = presjek pravca i osi x do $x=0$

P_2 površina između pravca i krivulje od

točke $x=0$ do točke $x=4$

$$x_0 = 4, \quad y_0 = 3\sqrt{4} = 6$$

$$y' = 3\,\frac{1}{2\sqrt{x}} \Rightarrow k_T = \frac{3}{2\sqrt{4}} = \frac{3}{4} \qquad \Rightarrow t \ldots \; y - y_0 = k_T(x - x_0)$$

$$y - 6 = \frac{3}{4}(x-4)$$

$$\boxed{y = \frac{3}{4}x + 3}$$

$$\frac{3}{4}x + 3 = 0 \iff \frac{3}{4}x = -3 \iff x = -4$$

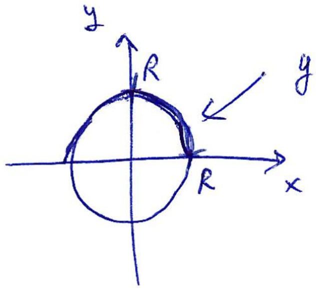
$$\Rightarrow P_1 = \int_{-4}^{0}\left(\frac{3}{4}x + 3\right)dx = \left(\frac{3}{4}\frac{x^2}{2} + 3x\right)\Bigg|_{-4}^{0} = 0 - \left(\frac{3}{\cancel{4}}\frac{\cancel{16}^{\,2}}{\cancel{2}} + 3\cdot(-4)\right) = -(6-12) = 6$$

$$P_2 = \int_{0}^{4}\left(\frac{3}{4}x + 3 - 3\sqrt{x}\right)dx = \left(\frac{3}{4}\frac{x^2}{2} + 3x - 3\,\frac{x^{3/2}}{3/2}\right)\Bigg|_{0}^{4} =$$

$$= \frac{3}{\cancel{4}_1}\frac{\cancel{16}^{\,2}}{\cancel{2}} + 3\cdot 4 - \cancel{3}\cdot\frac{2}{\cancel{3}}\,4^{3/2} - 0 = 6 + 12 - 2\cdot 4\cdot 2 = 18 - 16 = 2$$

$$\Rightarrow P = P_1 + P_2 = \boxed{8}$$

5. b) (10b) opseg kružnice $x^2+y^2=R^2$



$$y=\sqrt{R^2-x^2} \rightarrow y'(x)=\frac{1}{2\sqrt{R^2-x^2}}(-2x)=\frac{-x}{\sqrt{R^2-x^2}}$$

4 · duljina luka krivulje od 0 do R

$$s=\int_a^b\sqrt{1+(y'(x))^2}\,dx \Rightarrow O=4s$$

$$\Rightarrow O=4\int_0^R\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx=4\int_0^R\sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}\,dx=4\int_0^R\sqrt{\frac{R^2}{R^2-x^2}}\,dx=$$

$$=4\int_0^R\frac{R}{\sqrt{R^2-x^2}}\,dx=4R\int_0^R\frac{1}{R\sqrt{1-\left(\frac{x}{R}\right)^2}}\,dx=4R\int_0^1\frac{1}{\sqrt{1-t^2}}\,dt=$$

$$\left\{\begin{array}{l} t=\frac{x}{R} \\ dt=\frac{1}{R}dx \\ 0\to 0,\ R\to 1 \end{array}\right\}$$

$$=4R\arcsin t\Big|_0^1=4R\Big(\underbrace{\arcsin 1}_{=\frac{\pi}{2}}-\underbrace{\arcsin 0}_{=0}\Big)=4R\frac{\pi}{2}=2R\pi$$