MATEMATIKA 2 : KOLOKVIJ γ

5.5.2021

1.) Riješite dif. jednadžbu $y' = 4 + y^2$.

[Rj.] Separiramo varijable:

$$\frac{dy}{dx} = 4+y^2$$

$$\frac{dy}{4+y^2} = dx$$

$$\int \frac{dy}{4+y^2} = \int dx$$

$$\frac{1}{2} \arctan \frac{y}{2} = x + C$$

$$y = 2 \tan (2x + 2C)$$

↳ može se staviti $C_1 = 2C$.

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2. a) Domena funkcije $f(x,y) = \arcsin\left(1 - 2\left(\frac{x-y}{x+y}\right)^2\right)$

Domena od $\arcsin x$ je $[-1,1]$ pa zaključujemo da je nužno i dovoljno

$$\left.\begin{aligned} &-1 \le 1 - 2\left(\frac{x-y}{x+y}\right)^2 \le 1 \\ &\text{i nazivnik } x+y \neq 0. \end{aligned}\right\}$$

Nejednakost rastavimo: a) $1 - 2\left(\frac{x-y}{x+y}\right)^2 \le 1$

$$-\left(\frac{x-y}{x+y}\right)^2 \le 0 \quad \rightarrow \text{Vrijedi uvijek}$$

b) $-1 \le 1 - 2\left(\frac{x-y}{x+y}\right)^2$

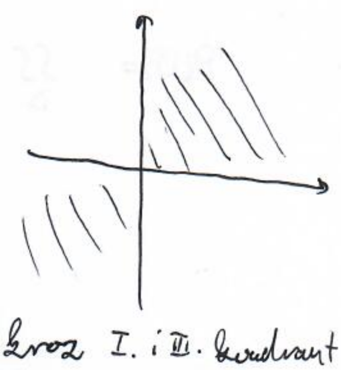
$$-2 \le -2\left(\frac{x-y}{x+y}\right)^2$$

$$1 \ge \left(\frac{x-y}{x+y}\right)^2 \quad \rightarrow \text{množimo s pozitivnim } (x+y)^2$$

$$(x+y)^2 \ge (x-y)^2$$

$$4xy \ge 0$$

$$\boxed{xy \ge 0}$$



Dakle, domena funkcije je $xy \ge 0$ uz uvjet $x+y \neq 0$,

To je I. i III. kvadrant bez pravca $\underbrace{y=-x}$.

ovaj pravac ne prolazi kroz I. i III. kvadrant pa je rješenje ustvari $\boxed{xy \ge 0}$ i $\boxed{(x,y) \neq (0,0)}$

b) $g(x,y) = \ln(x \ln(y-x))$.

$$\frac{\partial g}{\partial y} = \frac{1}{x\ln(y-x)} \cdot x \cdot \frac{1}{y-x} = \frac{y-x}{\ln(y-x)}$$

$$\frac{\partial g}{\partial y}(e,2e) = \frac{e}{\ln(e)} = e.$$

3.) Lokalni ekstremi funkcije $f(x,y) = 5x^2+10x-4y^3+48y+7$

$$\frac{\partial f}{\partial x} = 10x+10 = 0 \Rightarrow x=-1$$

$$\frac{\partial f}{\partial y} = -12y^2+48 = 0 \Rightarrow y = \pm 2$$

Stacionarne točke su $(-1,2)$ i $(-1,-2)$.

$$\frac{\partial^2 f}{\partial x^2} = 10$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

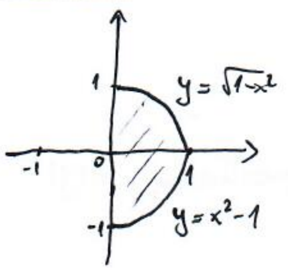
$$\frac{\partial^2 f}{\partial y^2} = -24y$$

Uz oznake kao na vježbama vrijedi:

a) $(-1,2) \rightarrow$ $A=10>0$
$AC-B^2 = -480 < 0 \Rightarrow$ sedlasta točka

b) $(-1,-2) \rightarrow$ $\left.\begin{array}{l} A=10>0 \\ AC-B^2 = 480>0 \end{array}\right\} \Rightarrow (-1,-2)$ je točka minimuma.

4.) Skica



$\rightarrow$ površina četvrtine kruga

$$P(D) = \iint_D dx\,dy = \iint_{D_1} dx\,dy + \iint_{D_2} dx\,dy$$ gdje je D_1 gornji dio $(y \geq 0)$, D_2 donji dio

$$= \frac{\pi}{4} + \int_0^1 \int_{x^2-1}^0 dx\,dy = \frac{\pi}{4} + \int_0^1 (1-x^2)\,dx$$

$$= \frac{\pi}{4} + \left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{\pi}{4} + \frac{2}{3}.$$

4. Alternativno rješenje:

$$\iint_D dx\,dy = \int_0^1 \int_{x^2-1}^{\sqrt{1-x^2}} dy\,dx = \int_0^1 y \Big|_{x^2-1}^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(\sqrt{1-x^2} + 1 - x^2\right) dx$$

$$= \underbrace{\int_0^1 \sqrt{1-x^2}\,dx}_{} + \underbrace{\int_0^1 (1-x^2)\,dx}_{=\frac{2}{3}}$$

Možemo izračunati direktno sa supstitucijom $x = \sin t$, $dx = \cos t$

$$= \frac{2}{3} + \int_0^{\frac{\pi}{2}} \cos^2 t\,dt$$

$$= \frac{2}{3} + \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt = \frac{2}{3} + \left[\frac{t}{2} + \frac{1}{4}\sin 2t\right]\Big|_0^{\frac{\pi}{2}}$$

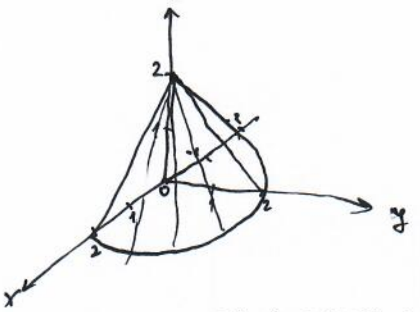
$$= \frac{2}{3} + \frac{\pi}{4}.$$

5.) Skica: funkcija $f(x,y) = \sqrt{x^2+y^2}$ predstavlja stožac prema gore s vrhom u $(0,0)$

$\Rightarrow z = 2 - \sqrt{x^2+y^2}$ je stožac prema dolje s visine 2

Siječe xy ravninu kad je $z=0$, dakle

$$z = 2 - \sqrt{x^2+y^2} = 0$$
$$x^2+y^2 = 4$$



Tijelo V je opisano s $\quad 0 \le z \le 2 - \sqrt{x^2+y^2}$, $x^2+y^2 \le 4$, $y \ge 0$ — pogodno za cilindrične koordinate

$\Rightarrow \quad x = \rho\cos t,\ y = \rho\sin t,\ z = z \quad \Rightarrow$ Tijelo V u cil. koord. je $\quad 0 \le z \le 2-\rho$, $0 \le \rho < 2$, $0 \le \varphi \le \pi$

$$\iiint_V y^3\,dx\,dy\,dz = \int_0^{\pi}\int_0^2\int_0^{2-\rho} \rho^3 \sin^3\varphi \cdot \rho\,dz\,d\rho\,d\varphi$$ ← prvo je po z integral jer ovisi o ρ

$$= \int_0^{\pi}\int_0^2 \left(\rho^4 \sin^3\varphi \cdot z\Big|_0^{2-\rho}\right) d\rho\,d\varphi$$

$$= \int_0^{\pi}\int_0^2 (2\rho^4 - \rho^5)\sin^3\varphi\,d\rho\,d\varphi = \int_0^{\pi} \sin^3\varphi \left[\frac{2\rho^5}{5} - \frac{\rho^6}{6}\right]\Big|_0^2 d\varphi$$

$$= \int_0^{\pi} \sin^3\varphi\,\frac{32}{15}\,d\varphi = \frac{32}{15}\int_0^{\pi} (1-\cos^2\varphi)\sin\varphi\,d\varphi$$ Supstituiramo $u = \cos\varphi$, $du = -\sin\varphi\,d\varphi$

$$= \frac{32}{15} \cdot \int_{+1}^{-1} (u^2-1)\,du = \frac{32}{15}\left[\frac{u^3}{3} - u\right]\Big|_{+1}^{-1} = \frac{128}{45}.$$