1. Riješite diferencijalne jednadžbe:
   (a) (12 bodova) $y'' - 4y' + 3y = 3x - 4$,
   (b) (8 bodova) $yy' = \cos(2x)$, pri čemu je $y(0) = 1$.
2. (20 bodova) Dana je funkcija $f(x, y) = y^3 + 2x^2 - 8x - 3y + 7$.
   (a) Odredite lokalne ekstreme funkcije f.
   (b) Nađite jednadžbu tangencijalne ravnine na graf funkcije f u točki $(2, 1)$.
3. (20 bodova) Izračunajte masu tijela omeđenog plohama $z = x^2 + y^2$ i $z = 6 - \sqrt{x^2 + y^2}$ ako mu je gustoća dana s $\rho(x, y, z) = \sqrt{x^2 + y^2}$. Skicirajte tijelo.

---

4. (22 boda) Zadano je polje
$$\vec{a} = 3x^2\vec{i} + e^y \sin z\,\vec{j} + e^y \cos z\vec{k}.$$
   (a) Izračunajte $\operatorname{rot}\vec{a}$. Je li polje potencijalno?
   (b) Odredite potencijal polja $\vec{a}$.
   (c) Izračunajte
$$\int_{\Gamma} \vec{a}\,\mathrm{d}\vec{r}$$
   ako je $\vec{\Gamma}$ spojnica točaka $A(0, 0, 0)$ i $B(1, 1, 0)$.
5. (18 bodova) Izračunajte površinu dijela sfere $x^2 + y^2 + z^2 = 36$ koju isijeca kružni cilindar $x^2 + y^2 = 6y$ ako je $z \geqslant 0$. Skicirajte plohu.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
C	0	ctg	$-\frac{1}{\sin^2 x}$
x^α	$\alpha x^{\alpha-1}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{sh} x$	$\operatorname{ch} x$
$\sin x$	$\cos x$	$\operatorname{ch} x$	$\operatorname{sh} x$
$\cos x$	$-\sin x$	$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	$x + C$	$\cos x$	$\sin x + C$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + C$
e^x	$e^x + C$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + C$
a^x	$\frac{a^x}{\ln a} + C$	$\frac{1}{1+x^2}$	$\operatorname{arctg} x + C$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln \left\lvert x + \sqrt{x^2 \pm 1} \right\rvert + C$

MAT2

ě dif. jdn:

a) $y'' - 4y' + 3y = 3x - 4$

HOMOGENA JDN: $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda_1 = 1, \quad \lambda_2 = 3 \quad \Rightarrow \quad y_H = Ae^{x} + Be^{3x}$$

PARTIKULARNO RJEŠENJE:

Kako je desna strana polinom, tražimo rješenje oblika y = polinom u x.

$3x - 4$ je linearan, pa stavimo $y = ax + b$.

Sada imamo:

$$y'' - 4y' + 3y = 3x - 4$$

$$(ax+b)'' - 4(ax+b)' + 3(ax+b) = 3x - 4$$

$$3ax + 3b - 4a = 3x - 4$$

$\downarrow$

Ovo mora vrijediti za sve x, dakle polinomi (njihovi koeficijenti) moraju biti jednaki.

$$\Rightarrow \begin{cases} 3a = 3 \\ 3b - 4a = -4 \end{cases} \Rightarrow \begin{array}{l} a = 1 \\ b = 0 \end{array}$$

Rješenje dif. jdn. je $y(x) = Ae^{x} + Be^{3x} + 3x$,

gdje su $A, B \in \mathbb{R}$ proizvoljni.

---

b) $yy' = \cos 2x, \quad y(0) = 1$

$$y\frac{dy}{dx} = \cos 2x$$

$$y\,dy = \cos 2x\, dx$$

$$\int y\,dy = \int \cos 2x\, dx$$

$$\frac{y^2}{2} = \frac{1}{2}\sin 2x + C$$

Uvrstimo početni uvjet $y(0) = 1$:

$$\frac{y^2(0)}{2} = \frac{1}{2}\sin 2\cdot 0 + C$$

$$\boxed{\frac{1}{2} = C} \Rightarrow \boxed{Rj.}\ y^2 = 1 + \sin 2x.$$

2.) Dana je $f(x,y) = y^3 + 2x^2 - 8x - 3y + 7$.

a) LOKALNI EKSTREMI:

$\frac{\partial f}{\partial x} = 4x - 8 = 0 \Rightarrow x = 2$

$\frac{\partial f}{\partial y} = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$

> STACIONARNE TOČKE

$A = \frac{\partial^2 f}{\partial x^2} = 4$

$B = \frac{\partial^2 f}{\partial x \partial y} = 0$

$C = \frac{\partial^2 f}{\partial y^2} = 6y$

$AC - B^2 = 24y$

Za točku $(2,1)$

$AC - B^2 = 24 > 0$

$A = 2 > 0$

$\Downarrow$

LOKALNI MINIMUM

Za točku $(2,-1)$

$AC - B^2 = -24 < 0$

$\Downarrow$

SEDLASTA TOČKA

b) Jdn. tang. ravnine:

Općenito glasi $z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$, ako je $z = f(x,y)$.

Dakle:

$z - f(2,1) = \frac{\partial f}{\partial x}(2,1)\,(x-2) + \frac{\partial f}{\partial y}(2,1)\,(y-1)$

$z - (-3) = 0 \cdot (x-2) + 0 \cdot (y-1)$

$\boxed{z = -3}$

(Obzirom da je $(2,1)$ stac. točka, posebno maksimum, ~~...~~ to tang. ravnina mora biti paralelna sa z-osi.)

3.1 Masa tijela omeđenog s $z=x^2+y^2$ - paraboloid
$z=6-\sqrt{x^2+y^2}$ - stožac

gustoće $\rho(x,y,z)=\sqrt{x^2+y^2}$

1. korak: Sijeku se u

$z=x^2+y^2=6-\sqrt{x^2+y^2}$

Stavimo $r=\sqrt{x^2+y^2}$.

$\Downarrow$

$r^2=6-r$

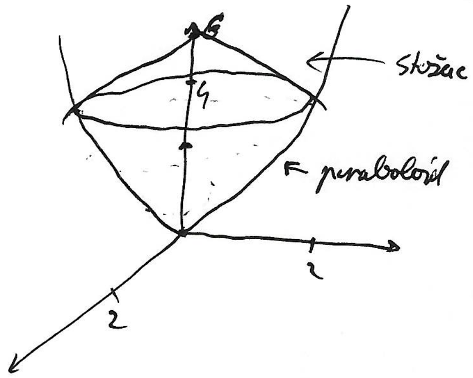
$r^2+r-6=0$

$\Rightarrow r=2$

($r=-3$ nema smisla).

Dakle, plohe se sijeku u $r=\sqrt{x^2+y^2}=2$ = kružnica.

2. korak: Skica:



3. korak: Postavljanje integrala - područje integracije

x, y su zadani s $x^2+y^2\le 4$

z -||- $x^2+y^2\le z\le 6-\sqrt{x^2+y^2}$

Uvodimo cilindrične koordinate. Tada imamo:

$0\le r\le 2$

$0\le \varphi\le 2\pi$

$r^2\le z\le 6-r$

gustoća: $\rho(x,y,z)=\sqrt{x^2+y^2}=r$

4. korak: Integral

$$\int_0^{2\pi}\int_0^2\int_{r^2}^{6-r} r\cdot r\,dr\,d\varphi\,dz=\int_0^{2\pi}\int_0^2 r^2\cdot(-r^2-r+6)\,dr\,d\varphi$$

$$=2\pi\cdot\int_0^2(-r^4-r^3+6r^2)\,dr$$

$$=2\pi\cdot\frac{28}{5}=\frac{56}{5}\pi$$

4.1 Zadano je polje $\vec{a} = 3x^2\vec{i} + e^y \sin z\,\vec{j} + e^y \cos z\,\vec{k}$

a) $\text{rot}\,\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & e^y \sin z & e^y \cos z \end{vmatrix} = (e^y\cos z - e^y \cos z)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k}$

$= \vec{0}$

$\Rightarrow$ Polje jest potencijalno (tj. postoji funkcija f t.d. $\vec{a} = \nabla f$).

b) Sada tražimo tu funkciju f.

1. način: Pomoću integralne formule:

$$\varphi(x,y,z) = \int_{x_0}^{x} a_x(t,y,z) + \int_{y_0}^{y} a_y(x_0,t,z)\,dt + \int_{z_0}^{z} a_z(x_0,y_0,t)\,dt + C.$$

gdje je (x_0,y_0,z_0) proizvoljna (npr. $(0,0,0)$).

Stavimo $\varphi(x,y,z) = \int_0^x 3t^2 dt + \int_0^y e^t \sin z\,dt + \int_0^z e^0 \cos t\,dt + C$

$= x^3 + e^y \sin z - e^0 \sin z + \sin z - \sin 0 + C$

$= x^3 + e^y \sin z + C$, C proizvoljna realna konst.

2. način: Ako integralna formula nije poznata, možemo je izvesti. Tražimo f tako da: $\nabla f = \vec{a}$ tj. $\frac{\partial f}{\partial x} = a_x$, $\frac{\partial f}{\partial y} = a_y$, $\frac{\partial f}{\partial z} = a_z$

a) $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x,y,z) = x^3 +$ konstanta obzirom na x

$= x^3 + C(y,z)$

b) $\frac{\partial f}{\partial y} = \frac{\partial C}{\partial y} = e^y \sin z \Rightarrow C(y,z) = e^y \sin z +$ konst. obz. na y

$= e^y \sin z + D(z)$

c) $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(e^y \sin z + D(z)\right) = e^y \cos z$

$e^y \cos z + \frac{\partial D}{\partial z} = e^y \cos z$

$\frac{\partial D}{\partial z} = 0 \Rightarrow D = \text{const}.$

Dakle, $f(x,y,z) = x^3 + e^y \sin z + \text{const}.$

3. način): Pogodimo funkciju! Izrazi se $\frac{\partial f}{\partial x} = 3x^2$

$$\frac{\partial f}{\partial y} = e^y \sin z$$

$$\frac{\partial f}{\partial z} = e^y \cos z$$

Pogodimo da je moguće rješenje $f(x,y,z) = x^3 + e^y \sin z$. Uistinu $\nabla f = \vec{a}$.

Ako je $g(x,y,z)$ bilo koja druga takva funkcija onda je:

$$\nabla f = \vec{a} = \nabla g$$

$$\Rightarrow \quad \nabla(f-g) = \vec{0} \quad \Rightarrow \quad f - g = const.$$

Dakle, općenito rješenje se razlikuje samo za konstantu,

tj. potencijal je $x^3 + e^y \sin z + const.$

c) Izračunajte $\int_\Gamma \vec{a}\, d\vec{r}$, Γ je dužina AB, $A(0,0,0)$, $B(1,1,0)$.

1. način): Dužina Γ spaja A i B. Parametriziramo:

$$x(t) = t, \quad y(t) = t, \quad z(t) = 0, \quad t \in [0,1]. \qquad d\vec{r}(t) = (1,1,0)\, dt$$

Integral postaje: $\int_0^1 (3t^2, e^t \sin 0, e^t \cos 0) \cdot (1,1,0)\, dt$

$$= \int_0^1 3t^2\, dt = 1.$$

2. način)

$$\int_\Gamma \vec{a} \cdot d\vec{r} = \int_\Gamma \nabla f = f(B) - f(A)$$

$$= f(1,1,0) - f(0,0,0)$$

$$= 1 - 0 = \underline{\underline{1}}$$

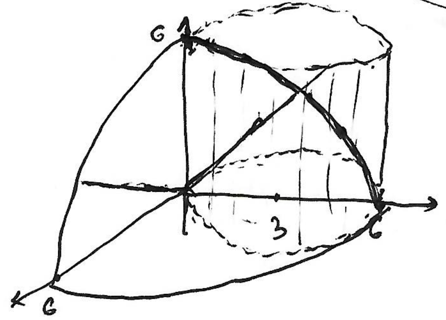
(f je potencijal iz b) dijela)

5.1 Površina dijela sfere $x^2+y^2+z^2=36$,

koju isijeca cilindar $x^2+y^2=6y$, $z\geq 0$.

$x^2+(y-3)^2=9$ - VALJAK SA OSI KROZ (0,3) RADIJUSA BAZE 3

1. Slika:



2. Plošni integral: Površina je jednaka $P=\iint_\Sigma dS$

$$P=\iint_\Sigma dS = \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy \qquad z=f(x,y)=\sqrt{36-x^2-y^2}$$

$$=\iint_{x^2+y^2=6y}\sqrt{1+\left(\frac{-2x}{2\sqrt{36-x^2-y^2}}\right)^2+\left(\frac{-2y}{2\sqrt{36-x^2-y^2}}\right)^2}$$

KRUŽNICA

Nameću se ~~cilindrične~~ POLARNE! koordinate: $x=r\cos\varphi$, $y=r\sin\varphi$

Sada $x^2+y^2=6y$ postaje

$$r^2=6r\sin\varphi$$

$\boxed{r=6\sin\varphi}$ $\quad 0\leq\varphi\leq\pi$ (t.d. $r\geq 0$)

Dakle
$$P=\int_0^\pi\int_0^{6\sin\varphi}\sqrt{1+\frac{r^2}{36-r^2}}\,\underbrace{r\,dr\,d\varphi}_{\text{Jakobijan}} = \int_0^\pi\int_0^{6\sin\varphi}\frac{6r}{\sqrt{36-r^2}}\,dr$$

$$=\left|\begin{matrix}u=36-r^2\\ du=-2r\,dr\end{matrix}\right| = \int_0^\pi\int_{36}^{36\cos^2\varphi}\frac{-3\,du}{\sqrt{u}}=\int_0^\pi\int_{36\cos^2\varphi}^{36}+\frac{3\,du}{\sqrt{u}}$$

$$=\int_0^\pi 6\sqrt{u}\Big|_{36\cos^2\varphi}^{36}d\varphi=\int_0^\pi 6\cdot(6-6|\cos\varphi|)\,d\varphi$$

$$=2\cdot\int_0^{\frac{\pi}{2}}(36-36\cos\varphi)\,d\varphi=72\cdot\left[\frac{\pi}{2}-1\right]$$

$$=36(\pi-2).$$

jednaka površina suprotnog predznaka

$$\int_0^\pi|\cos\varphi|\,d\varphi = 2\int_0^{\frac{\pi}{2}}\cos\varphi\,d\varphi$$