MATEMATIKA I
23.6.2021. ~~A~~

1. (a) (8 bodova) Koliki kut zatvaraju vektori $\vec{a} = 3\vec{i} + 2\vec{j} - 2\vec{k}$ i $\vec{b} = 2\vec{i} + \vec{j} + 4\vec{k}$?

   (b) (12 bodova) Odredite presjek ravnine $\pi \ldots 2x + y - z + 1 = 0$ i pravca $p \ldots \frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-2}{1}$.

2. (a) (10 bodova) Riješite matričnu jednadžbu $X(A - I) = B$, gdje je $A = \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$ i $B = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$.

   (b) (10 bodova) Odredite jednadžbe tangenti na krivulju $f(x) = 4 - x^2$ koje prolaze točkom $A(0, 8)$.

3. (20 bodova) Odredite ekstreme, intervale rasta i pada, asimptote, te skicirajte graf funkcije:

$$f(x) = \frac{2}{\sqrt{x^2 + 4}}$$

4. (20 bodova) Odredite

$$\int \frac{e^x \mathrm{d}x}{e^{2x} - e^x - 6}.$$

5. (a) (10 bodova) Izračunajte:

$$\int_0^\pi (x + 2) \sin(3x) \, \mathrm{d}x.$$

   (b) (10 bodova) Odredite površinu lika omeđenog krivuljama $y = \arcsin x$, $y = \frac{\pi}{2}$ i $x = 0$. Skicirajte lik.

a) (8b) $\vec{a} = 3\vec{i} + 2\vec{j} - 2\vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} + 4\vec{k}$ $\qquad \alpha = \sphericalangle(\vec{a}, \vec{b})$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3\cdot 2 + 2\cdot 1 - 2\cdot 4}{\sqrt{9+4+4}\sqrt{4+1+16}} = 0 \Rightarrow \boxed{\alpha = \frac{\pi}{2}}$$

b) (12b) $\pi \dots 2x + y - z + 1 = 0$

$$p \dots \frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-2}{1}$$

$$p \dots \begin{cases} x = 1 + t \\ y = -2 - 2t \\ z = 2 + t \end{cases}$$

$2(1+t) - 2 - 2t - 2 - t + 1 = 0$

$\Rightarrow \cancel{2} + \cancel{2t} - \cancel{2} - \cancel{2t} - 2 - t + 1 = 0$

$\Rightarrow t = -1 \Rightarrow \boxed{T(0, 0, 1)} \in p \cap \pi$

2. a) (10b) $X(A - I) = B \Rightarrow X = B(A - I)^{-1}$

$$A - I = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \qquad \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right] \begin{matrix} + \\ \cdot(-1) \end{matrix} \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\Rightarrow (A - I)^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\boxed{X} = \cancel{\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}$$

(10b) $f(x) = 4 - x^2$, $A(0,8)$, tangente na f kroz A

$f'(x) = -2x$

$t \ldots y - 4 + x_0^2 = -2x_0(x - x_0)$

$A \in t \Rightarrow 8 - 4 + x_0^2 = +2x_0^2 \Rightarrow x_0^2 = 4$

$\Rightarrow (x_0)_{1,2} = \pm 2 \Rightarrow (y_0)_{1,2} = 0$

$T_1(-2,0)$, $T_2(2,0)$ dirališta tangenti

$t_1 \ldots y = +4(x+2)$ tj. $\boxed{y = +4x + 8}$

$t_2 \ldots y = -4(x-2)$ tj. $\boxed{y = -4x + 8}$

3. (20b) $f(x) = \dfrac{2}{\sqrt{x^2+4}}$, $D_f = \mathbb{R}$

EKSTREMI:

$$f'(x) = -\frac{2 \cdot \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4}}}{x^2+4} = -\frac{2x}{\sqrt{(x^2+4)^3}} = 0 \Rightarrow x = 0 \text{ S.T.}$$

	$-\infty$	0	$+\infty$
$f'(x)$	$+$	$-$	
$f(x)$	$\nearrow$	$\searrow$	

U $x=0$ funkcija ima lokalni maksimum, $f(0) = 1$.

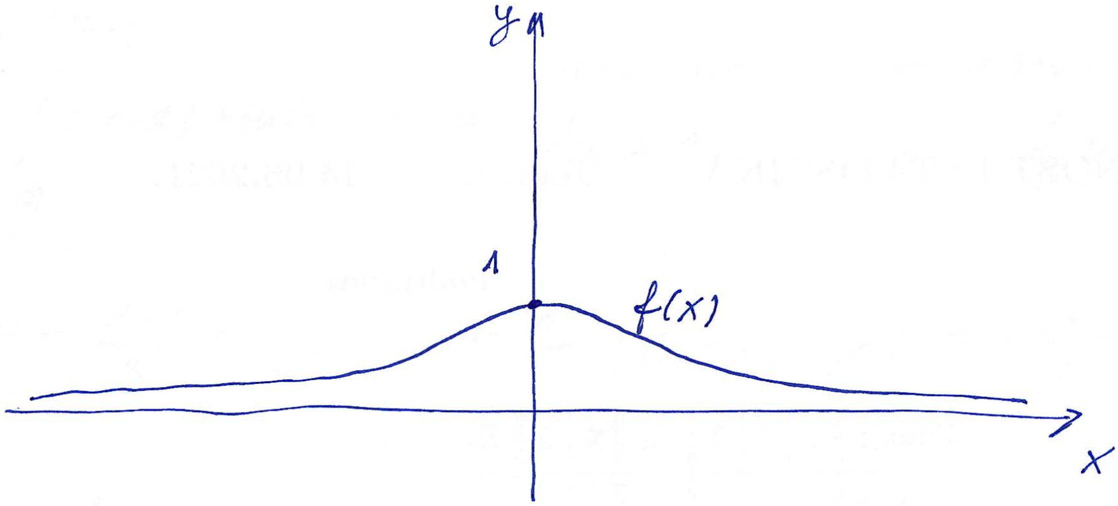
INTERVAL RASTA : $\langle -\infty, 0 \rangle$

—"— PADA : $\langle 0, +\infty \rangle$

ASIMPTOTE:

$D_f = \mathbb{R} \Rightarrow f$ NEMA V.A.

$$\lim_{x \to \pm\infty} \frac{2}{\sqrt{x^2+4}} = 0 \Rightarrow y = 0 \text{ je obostrana H.A.}$$



4. (20b)

$$\int \frac{e^x dx}{e^{2x} - e^x - 6} = \left| \begin{array}{l} t = e^x \\ dt = e^x dx \end{array} \right| = \int \frac{dt}{t^2 - t - 6} = (*)$$

$$t^2 - t - 6 = 0 \Rightarrow t_{1,2} = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} \begin{array}{l} \nearrow t_1 = -2 \\ \searrow t_2 = 3 \end{array}$$

$$\frac{1}{(t+2)(t-3)} = \frac{A}{t+2} + \frac{B}{t-3} \quad \Big/ \cdot (t+2)(t-3)$$

$$1 = A(t-3) + B(t+2) = (A+B)t + (-3A+2B)$$

$$\begin{array}{l} A + B = 0 \quad / \cdot 3 \\ -3A + 2B = 1 \quad +) \end{array} \Rightarrow 5B = 1 \Rightarrow B = \frac{1}{5} \Rightarrow A = -\frac{1}{5}$$

$$(*) = -\frac{1}{5}\int \frac{dx}{t+2} + \frac{1}{5}\int \frac{dx}{t-3} = \boxed{-\frac{1}{5}\ln|t+2| + \frac{1}{5}\ln|t-3| + C}$$

$$= \boxed{\frac{1}{5}\ln\left|\frac{t-3}{t+2}\right| + C}$$

$$= \boxed{\frac{1}{5}\ln\left|\frac{e^x - 3}{e^x + 2}\right| + C}$$

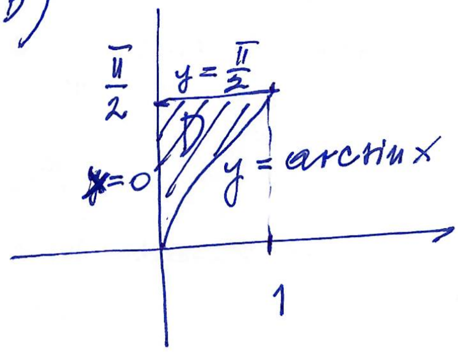
a) (10b)

$$\int_0^{\pi}(x+2)\sin(3x)\,dx = \left| \begin{matrix} u = x+2 & dv = \sin(3x)dx \\ du = dx & v = -\frac{1}{3}\cos(3x) \end{matrix} \right|$$

$$= -\frac{x+2}{3}\cos(3x)\Big|_0^{\pi} + \frac{1}{3}\int_0^{\pi}\cos(3x)dx = -\frac{\pi+2}{3}(-1)+\frac{2}{3}$$

$$+\frac{1}{9}\sin(3x)\Big|_0^{\pi} = \boxed{\frac{\pi+4}{3}}$$

b) (10b)



$$P(D) = \frac{\pi}{2} - \int_0^1 \arcsin x\,dx$$

$$= \left| \begin{matrix} u = \arcsin x & dv = dx \\ du = \frac{dx}{\sqrt{1-x^2}} & v = x \end{matrix} \right|$$

$$= \frac{\pi}{2} - x\arcsin x\Big|_0^1 + \int_0^1 \frac{x\,dx}{\underbrace{\sqrt{1-x^2}}_{t=1-x^2}} = \frac{\pi}{2} - \frac{\pi}{2} + \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}}$$

$$= \cancel{\frac{1}{2}}\,\frac{\sqrt{t}}{\cancel{\frac{1}{2}}}\Big|_0^1 = \boxed{1}$$

ili

$$P(D) = \int_0^{\frac{\pi}{2}} \sin y\,dy = -\cos y\Big|_0^{\frac{\pi}{2}} = -0-(-1) = \boxed{1}$$