MATEMATIKA 3 23.6.2021.

1. (20 bodova) Odredite ravnotežni položaj žice duljine $l = 4$ i napetosti $p = 2$ koja se nalazi u homogenom sredstvu s koeficijentom elastičnosti 8 ako je lijevi kraj žice slobodan, a desni pričvršćen i gustoća vanjske sile je dana s

$$f(x) = 52\cos(3x) + 26\sin(3x).$$

2. (20 bodova) Odredite temperaturu izoliranog homogenog štapa duljine 7, toplinskog kapaciteta $\gamma = 4$ i koeficijenta provođenja $\delta = 16$, te s početnom raspodjelom temperature

$$u(x,0) = \sin(3\pi x) + 2\sin(5\pi x).$$

Temperatura na krajevima je $0°C$.

3. (20 bodova) Riješite problem ravnoteže pravokutne membrane
$\Omega = [0,4] \times [0,2]$ opisan Laplaceovom jednadžbom $\Delta u = 0$, uz rubne uvjete

$$u(x,0) = \sin\left(\frac{3\pi}{4}x\right)$$

i

$$u(x,2) = u(0,y) = u(4,y) = 0.$$

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(1.) ravnoteža žice → jednadžba $-pu''(x) + bu(x) = f(x)$, $x \in [0, \ell]$

duljina $\ell = 4$

napetost $p = 2$

homogeno sredstvo s koeficijentom elastičnosti 8 → $b = 8$

lijevi kraj žice slobodan → $u'(0) = 0$

$x=0$

desni kraj žice pričvršćen → $u(4) = 0$

$x=4$

gustoća vanjske sile $f(x) = 52\cos(3x) + 26\sin(3x)$

Rješavamo

$$-2u''(x) + 8u(x) = 52\cos(3x) + 26\sin(3x) \quad /:(-2)$$

$$\Rightarrow u''(x) - 4u(x) = -26\cos(3x) - 13\sin(3x)$$

linearna nehomogena jednadžba 2. reda s konstantnim koeficijentima

Rješenje tražimo u obliku $u = u_H + u_P$

u_H rješava jednadžbu $u''(x) - 4u(x) = 0$.

Pridružujemo njenu karakterističnu jednadžbu: $\lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$

Dakle, $\lambda_1 = -2$, $\lambda_2 = 2$. Vrijedi: $\lambda_1 \neq \lambda_2$ i $\lambda_1, \lambda_2 \in \mathbb{R}$, pa je opće rješenje te jednadžbe dano formulom: $\boxed{u_H(x) = C_1 e^{-2x} + C_2 e^{2x}, \quad C_1, C_2 \in \mathbb{R}.}$

u_P određujemo "algoritmom" prema obliku funkcije smetnje. U ovom slučaju je to

$$f(x) = \underbrace{-26\cos(3x) - 13\sin(3x)}_{P_0(x)\cos(\omega x) + R_0 \sin(\omega x)}$$

Kako je $\lambda_{1,2} \neq \pm \omega i$, partikularno rješenje tražimo u obliku $u_P = A\cos(3x) + B\sin(3x)$

$u_P' = -3A\sin(3x) + 3B\cos(3x)$; $u_P'' = -9A\cos(3x) - 9B\sin(3x)$

Uvrštavamo u jednadžbu:

$$-9A\cos(3x) - 9B\sin(3x) - 4\big(A\cos(3x) + B\sin(3x)\big) = -26\cos(3x) - 13\sin(3x)$$

uz $\cos(3x)$: $-9A - 4A = -26 \Rightarrow -13A = -26 \Rightarrow A = 2$

uz $\sin(3x)$: $-9B - 4B = -13 \Rightarrow -13B = -13 \Rightarrow B = 1$

$\boxed{u_P = 2\cos(3x) + \sin(3x)}$

$$u(x) = C_1 e^{-2x} + C_2 e^{2x} + 2\cos(3x) + \sin(3x) \quad , \; C_1, C_2 \in \mathbb{R}$$

C_1 i C_2 određujemo iz rubnih uvjeta:

Za to nam treba i $u'(x)$. Računamo

$$u'(x) = -2C_1 e^{-2x} + 2C_2 e^{2x} - 6\sin(3x) + 3\cos(3x)$$

$$0 = u'(4) = -2C_1 e^{-2\cdot 4} + 2C_2 e^{2\cdot 4} - 6\sin(3\cdot 4) + 3\cos(3\cdot 4)$$

$$\Rightarrow \quad 0 = -2C_1 e^{-8} + 2C_2 e^{8} - 6\sin 12 + 3\cos 12$$

$$0 = u(0) = C_1 \underbrace{e^{-2\cdot 0}}_{=1} + C_2 \underbrace{e^{2\cdot 0}}_{=1} + 2\underbrace{\cos(3\cdot 0)}_{=1} + \underbrace{\sin(3\cdot 0)}_{=0}$$

$$\Rightarrow \quad 0 = C_1 + C_2 + 2 \quad \Rightarrow \quad C_2 = -2 - C_1$$

$$\Rightarrow \quad 0 = -2C_1 e^{-8} + 2(-2-C_1) e^{8} - 6\sin 12 + 3\cos 12$$

$$\Rightarrow \quad 0 = C_1(-2e^{-8} - 2e^{8}) - 4e^{8} - 6\sin 12 + 3\cos 12$$

$$\Rightarrow \quad C_1 = \frac{-4e^{8} - 6\sin 12 + 3\cos 12}{2e^{-8} + 2e^{8}}$$

$$\Rightarrow \quad u(x) = \frac{-4e^{8} - 6\sin 12 + 3\cos 12}{2e^{-8} + 2e^{8}} e^{-2x} + \left(-2 - \frac{-4e^{8} - 6\sin 12 + 3\cos 12}{2e^{-8} + 2e^{8}}\right) e^{2x} + 2\cos(3x) + \sin(3x)$$

2.

temperatura izoliranog homogenog štapa → Jednadžba $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$c^2 = \frac{\delta}{\gamma} = \frac{16}{4} = 4 \Rightarrow c = 2$

Rješenje jednadžbe: $u(x,t) = \sum_{n=1}^{\infty} E_n e^{-(\lambda_n c)^2 t} \sin(\lambda_n x)$

$\lambda_n = \frac{n\pi}{l}$

$E_n = \frac{2}{l} \int_0^l g(x) \sin(\lambda_n x)\, dx$

duljina 7 → $l = 7$

toplinski kapacitet $\boxed{\gamma = 4}$

koeficijent provođenja $\boxed{\delta = 16}$

početna raspodjela temperature $u(x,0) = \underbrace{\sin(3\pi x) + 2\sin(5\pi x)}_{g(x)}$

temperatura na krajevima 0°C → $u(0,t) = u(7,t) = 0$

Računamo $E_n = \frac{2}{l} \int_0^l g(x) \sin(\lambda_n x)\, dx = \frac{2}{7} \int_0^7 \left[\sin(3\pi x) + 2\sin(5\pi x)\right] \sin\left(\frac{n\pi x}{7}\right) dx =$

$$= \frac{2}{7} \int_0^7 \sin(3\pi x) \sin\left(\frac{n\pi x}{7}\right) dx + \frac{4}{7} \int_0^7 \sin(5\pi x) \sin\left(\frac{n\pi x}{7}\right) dx =$$

$$= \underbrace{\frac{1}{7} \int_{-7}^{7} \sin\left(\frac{21\pi x}{7}\right) \sin\left(\frac{n\pi x}{7}\right) dx}_{= \begin{cases} 1, & n = 21, \\ 0, & n \neq 21 \end{cases}} + \underbrace{\frac{2}{7} \int_{-7}^{7} \sin\left(\frac{35\pi x}{7}\right) \sin\left(\frac{n\pi x}{7}\right) dx}_{= \begin{cases} 2, & n = 35 \\ 0, & n \neq 35 \end{cases}}$$

$$\Rightarrow u(x,t) = e^{-(\lambda_{21} c)^2 t} \sin(\lambda_{21} x) + e^{-(\lambda_{35} c)^2 t} \sin(\lambda_{35} x) =$$

$$= e^{-\left(\frac{21\pi}{7} \cdot 2\right)^2 t} \sin\left(\frac{21\pi}{7} x\right) + e^{-\left(\frac{35\pi}{7} \cdot 2\right)^2 t} \sin\left(\frac{35\pi}{7} x\right) =$$

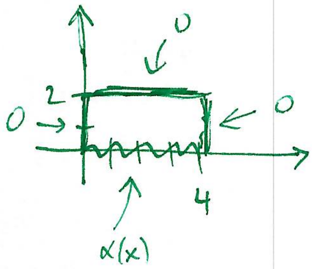
$$= e^{-36\pi^2 t} \sin(3\pi x)\; e^{-100\pi^2 t} \sin(5\pi x)$$

(3.) ravnoteža pravokutne membrane, jednadžba $\Delta u = 0$

$\Omega = [0,4]\times[0,2] \rightarrow a=4,\ b=2$

$u(x,0) = \sin\left(\frac{3\pi}{4}x\right) = \alpha(x)$

$u(x,2) = u(0,y) = u(4,y) = 0$



Rješenje je oblika

$$u(x,y) = \sum_{n=1}^{\infty}\left(A_n \operatorname{ch}\frac{n\pi y}{a} + B_n \operatorname{sh}\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a},$$

a koeficijente računamo po formulama:

$$A_n = \frac{2}{a}\int_0^a \alpha(x)\sin\left(\frac{n\pi x}{a}\right)dx \quad \text{i} \quad B_n = -A_n \operatorname{cth}\left(\frac{n\pi b}{a}\right)$$

Računamo:

$$A_n = \frac{2}{a}\int_0^a \alpha(x)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{4}\int_0^4 \sin\left(\frac{3\pi x}{4}\right)\sin\left(\frac{n\pi x}{4}\right)dx = \begin{cases} 1, & n=3, \\ 0, & n\neq 3. \end{cases}$$

Dakle, $A_3 = 1$, $A_n = 0$, $n \neq 3$.

$$\Rightarrow \quad B_3 = -\operatorname{cth}\left(\frac{3\pi\cdot 2}{4}\right) = -\operatorname{cth}\left(\frac{3\pi}{2}\right) \quad \text{i} \quad B_n = 0 \text{ za } n\neq 3.$$

$$\Rightarrow u(x,y) = \left[A_3 \operatorname{ch}\left(\frac{3\pi y}{4}\right) + B_3 \operatorname{sh}\left(\frac{3\pi y}{4}\right)\right]\sin\frac{3\pi x}{4} = \left[\operatorname{ch}\left(\frac{3\pi y}{4}\right) - \operatorname{cth}\left(\frac{3\pi}{2}\right)\operatorname{sh}\left(\frac{3\pi y}{4}\right)\right]\sin\left(\frac{3\pi x}{4}\right)$$