1. Riješite diferencijalne jednadžbe:

   (a) (10 bodova) $\frac{y'}{xe^x} = y \ln y$, uz početni uvjet $y(0) = e$,

   (b) (10 bodova) $y' + y = e^{-x} + 2$.

2. (a) (10 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \arcsin\left(\frac{x}{y^2}\right).$$

   (b) (10 bodova) Nađite ekstrem funkcije $f(x,y) = xy$ uz uvjet $x^2 + y^2 - 2 = 0$.

3. (20 bodova) Izračunajte masu tijela $V = \begin{cases} x^2 + y^2 \leq 2x \\ 0 \leq y \\ 0 \leq z \leq 1 \end{cases}$ ako mu je gustoća dana s $\rho(x,y,z) = z\sqrt{x^2+y^2}$. Skicirajte tijelo V.

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4. (a) (8 bodova) Dokažite da je polje $\vec{a} = \vec{i} + (1 + z\cos(zy))\,\vec{j} + (y\cos(zy))\,\vec{k}$ potencijalno.

   (b) (14 bodova) Izračunajte $\int\limits_{\vec{\Gamma}} \vec{a}\, d\vec{r}$ ako je $\vec{\Gamma}$ dio kružnice $x^2 + y^2 = 1$ od točke $(0,1)$ do točke $(-1,0)$ i

$$\vec{a} = -xy\left(x + \frac{y}{2}\right)\vec{i} + xy\left(\frac{x}{2} + y\right)\vec{j}.$$

5. (18 bodova) Izračunajte

$$\iint\limits_{\Sigma} (x + y + z)\; dS$$

gdje je Σ dio sfere sa središtem u ishodištu radijusa 1 u 1. oktantu.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
C	0	$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$
x^α	$\alpha x^{\alpha-1}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{sh} x$	$\operatorname{ch} x$
$\sin x$	$\cos x$	$\operatorname{ch} x$	$\operatorname{sh} x$
$\cos x$	$-\sin x$	$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	$x + C$	$\cos x$	$\sin x + C$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + C$
e^x	$e^x + C$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + C$
a^x	$\frac{a^x}{\ln a} + C$	$\frac{1}{1+x^2}$	$\operatorname{arctg} x + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln\left\lvert x + \sqrt{x^2 \pm 1}\right\rvert + C$

1. a) (10b) $\frac{y'}{xe^x} = y \ln y$, $y(0) = e$

Rj. $$\int \frac{dy}{y \ln y} = \int x e^x dx$$

$$\int \frac{dy}{y\ln y} = \left| \begin{matrix} t = \ln y \\ dt = \frac{dy}{y} \end{matrix} \right| = \int \frac{dt}{t} = \ln t = \underline{\ln(\ln y)}$$

$$\int x e^x dx = \left| \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix} \right| = xe^x - \int e^x dx = \underline{xe^x - e^x + C}$$

$$\boxed{\ln(\ln y) = xe^x - e^x + C}$$

$$y(0) = e \Rightarrow \underbrace{\ln(\underbrace{\ln e}_{1})}_{=0} = 0 - 1 + C \Rightarrow \boxed{C = 1}$$

$$\boxed{\ln(\ln y) = (x-1)e^x + 1}$$

1. b) (10b) $y' + y = e^{-x} + 2 \Rightarrow f(x) = 1$, $g(x) = e^{-x} + 2$

Rj.

$$\int f(x) dx = \int dx = \underline{x}$$

$$\int e^{\int f(x)dx} g(x) dx = \int e^x (e^{-x} + 2) dx = \int (1 + 2e^x) dx$$
$$= \underline{x + 2e^x}$$

$$y(x) = e^{-\int f(x)dx} \left[\int e^{\int f(x)dx} g(x) dx + C \right]$$
$$= e^{-x} \left[x + 2e^x + C \right]$$

$$\boxed{y(x) = \frac{x + C}{e^x} + 2}$$

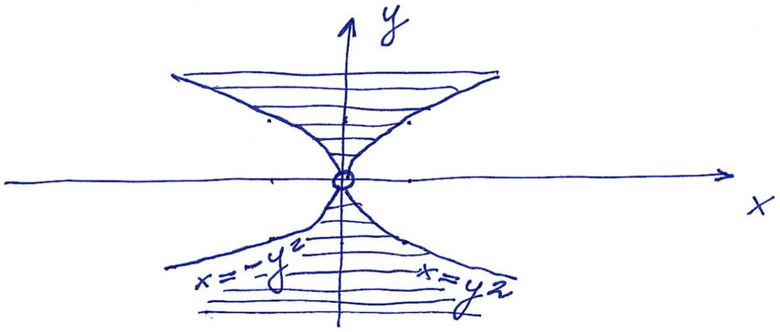
2. a) (10b) $f(x,y) = \arcsin\left(\frac{x}{y^2}\right)$, $D_f = ?$

Rj:

$y \neq 0$ i $-1 \le \frac{x}{y^2} \le 1$

$x \ge -y^2$ i $x \le y^2 \Rightarrow \underline{-y^2 \le x \le y^2}$

$$\boxed{D_f = \{(x,y) \in \mathbb{R}^2 : y \neq 0,\ -y^2 \le x \le y^2\}}$$



b) (10) Ekstremi od $f(x,y) = xy$ uz uvjet $x^2+y^2-2=0$?

Rj: $F(x,y,\lambda) = xy + \lambda x^2 + \lambda y^2 - 2\lambda$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y + 2\lambda x = 0 \\ \frac{\partial F}{\partial y} &= x + 2\lambda y = 0 \end{aligned}\right\} \quad 2\lambda = -\frac{y}{x} = -\frac{x}{y} \Rightarrow \underline{x^2 = y^2}$$

$x^2+y^2=2$

1° slučaj $\underline{x = y}$ $\Rightarrow \underline{\lambda = -\frac{1}{2}}$ i $x^2 = 1 \Rightarrow \underline{x = \pm 1}$

$\underline{T_1(-1,-1),\ T_2(1,1)}$

2° slučaj $x = -y \Rightarrow \underline{\lambda = \frac{1}{2}}$ i $x^2 = 1 \Rightarrow \underline{x = \pm 1}$

$\underline{T_3(-1,1),\ T_4(1,-1)}$

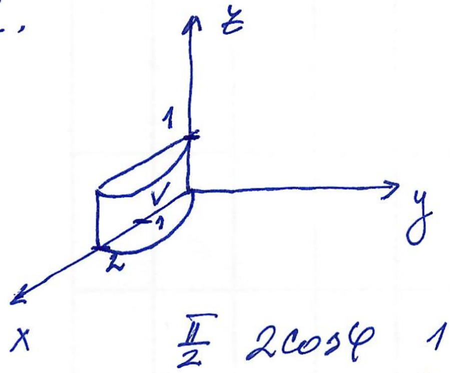
$$\left.\begin{aligned} f(T_1) &= (-1)\cdot(-1) = 1 \\ f(T_2) &= 1 \cdot 1 = 1 \end{aligned}\right\} \quad \boxed{\text{uvjetni maksimumi su } T_1, T_2}$$

$$\left.\begin{aligned} f(T_3) &= -1 \cdot 1 = -1 \\ f(T_4) &= 1 \cdot (-1) = -1 \end{aligned}\right\} \quad \boxed{\text{uvjetni minimumi su } T_3, T_4}$$

3. (2b) $I = \iiint\limits_V z\sqrt{x^2+y^2}\,dx\,dy\,dz = ?$, ako je

$V = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 \le 2x,\ 0\le y,\ 0\le z\le 1\}$.

Rj.



$$V \ldots \quad 0\le\varphi\le\frac{\pi}{2}$$
$$0\le\rho\le 2\cos\varphi$$
$$0\le z\le 1$$

$$I = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} d\rho \int_0^1 z\rho\cdot\rho\,dz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \rho^2 \frac{z^2}{2}\Big|_0^1 d\rho$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\rho^3}{3}\Big|_0^{2\cos\varphi} d\varphi = \frac{1}{6}\int_0^{\frac{\pi}{2}} 8\cos^3\varphi\,d\varphi$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\varphi)\cos\varphi\,d\varphi = \begin{vmatrix} t = \sin\varphi \\ dt = \cos\varphi\,d\varphi \\ \varphi=0 \Rightarrow t=0 \\ \varphi=\frac{\pi}{2} \Rightarrow t=1 \end{vmatrix}$$

$$= \frac{4}{3}\int_0^1 (1-t^2)\,dt = \frac{4}{3}\left(t-\frac{t^3}{3}\right)\Big|_0^1 = \frac{4}{3}\cdot\frac{2}{3} = \boxed{\frac{8}{9}}$$

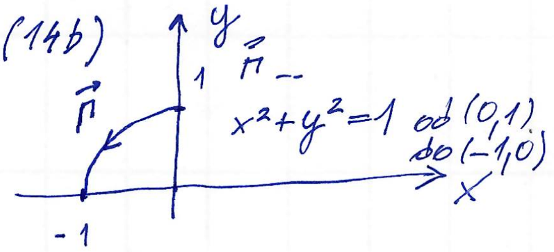
4. a) (8b) $\vec{a} = \vec{i} + (1 + z\cos(zy))\vec{j} + (y\cos(zy))\vec{k}$ potencijalno?

Rj.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1+z\cos(zy) & y\cos(zy) \end{vmatrix} = \left(\cancel{\cos(zy)} - \cancel{(zy)\sin(zy)} - \cancel{\cos(zy)} + \cancel{(zy)\sin(zy)}\right)\vec{i}$$

$$+ (0-0)\vec{j} + (0-0)\vec{k} = \vec{0} \Rightarrow \vec{a} \text{ je POTENCIJALNO POLJE}$$

4. b) (14b)



$$\vec{a} = -xy\left(x+\frac{y}{2}\right)\vec{i} + xy\left(\frac{x}{2}+y\right)\vec{j}$$

$\Gamma \ldots \; x(t) = \cos t \qquad x'(t) = -\sin t$

$y(t) = \sin t \qquad y'(t) = \cos t$

$t \in \left[\frac{\pi}{2}, \pi\right]$

$$\int_\Gamma \vec{a}\,d\vec{r} = \int_{\frac{\pi}{2}}^{\pi} \left[-\cos t \sin t\left(\cos t + \frac{\sin t}{2}\right)\cdot(-\sin t) + \cos t \sin t\left(\frac{\cos t}{2} + \sin t\right)\cos t\right]dt$$

$$= \int_{\frac{\pi}{2}}^{\pi} \sin^2 t\cos^2 t\,dt + \frac{1}{2}\underbrace{\int_{\frac{\pi}{2}}^{\pi}\sin^3 t\cos t\,dt}_{u=\sin t} + \frac{1}{2}\underbrace{\int_{\frac{\pi}{2}}^{\pi}\cos^3 t\sin t\,dt}_{u=\cos t}$$

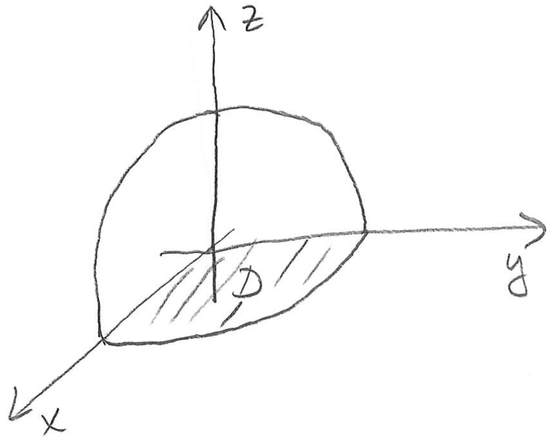
$$+ \int_{\frac{\pi}{2}}^{\pi}\underbrace{\underbrace{\sin^2 t\cos^2 t}_{\frac{1}{4}\sin^2(2t)}}_{\frac{1}{8}(1-\cos(4t))}dt = \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi}(1-\cos(4t))dt$$

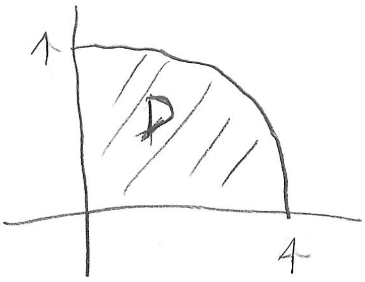
$$+ \frac{1}{8}\sin^4 t\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{8}\cos^4 t\Big|_{\frac{\pi}{2}}^{\pi} = \frac{1}{4}\left(t - \frac{1}{4}\sin(4t)\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$-\frac{1}{8} - \frac{1}{8} = \frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{4} = \boxed{\frac{1}{8}(\pi - 2)}$$

5. (18 bod) $\Sigma \ldots z = f(x,y) = \sqrt{1-x^2-y^2}$

$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$





$D \ldots \; 0 \le \varphi \le \frac{\pi}{2}$

$0 \le r \le 1$

$x = r\cos\varphi$

$y = y\sin\varphi$

$$\iint_{\Sigma} (x+y+z)\,dS = \iint_{D} \left(x+y+\sqrt{1-x^2-y^2}\right)\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^1 \left(r(\cos\varphi+\sin\varphi)+\sqrt{1-r^2}\right)\sqrt{\frac{1}{1-r^2}}\,r\,dr\,d\varphi$$

$$= \int_0^{\pi/2}\int_0^1 (\cos\varphi+\sin\varphi)\frac{r^2}{\sqrt{1-r^2}}\,dr\,d\varphi + \int_0^{\pi/2}\underbrace{\int_0^1 r\,dr}_{1/2}\,d\varphi$$

$$= \left\{\begin{array}{ll} r = \sin t & r=0 \Rightarrow t=0 \\ dr = \cos t\,dt & r=1 \Rightarrow t=\pi/2 \end{array}\right\} = (\sin\varphi - \cos\varphi)\Big|_0^{\pi/2} \cdot \int_0^{\pi/2} \underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}}\,dt +$$

$$+ \frac{1}{2}\varphi\Big|_0^{\pi/2} = 2\cdot\left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\Big|_0^{\pi/2} + \frac{\pi}{4}$$

$$= 2\cdot\left(\frac{\pi}{4}\right) + \frac{\pi}{4} = \frac{3\pi}{4}$$