MATEMATIKA 3 1.9.2021.

1. (20 bodova) Pronađite zakon titranja homogene žice duljine 2, gustoće 1 i napetosti 1 koja je pričvršćena na rubovima. Početni položaj i početna brzina su zadani sljedećim funkcijama:

$$u(x,0) = 2\sin(3\pi x) - 4\sin(2\pi x),$$
$$\frac{\partial u}{\partial t}(x,0) = 0.$$

2. (20 bodova) Tanki homogeni štap duljine $l = 4$, toplinskog kapaciteta $\gamma = 1$ i koeficijenta provođenja $\delta = 16$ toplinski je bočno izoliran. Neka je na krajevima temperatura $0°C$ te neka je početna distribucija temperature dana formulom

$$u(x,0) = 2\sin(\frac{3\pi x}{2}) - \sin(2\pi x).$$

Pronađite zakon provođenja topline.

3. (20 bodova) Riješite problem ravnoteže pravokutne membrane $\Omega = [0,3] \times [0,2]$ opisan Laplaceovom jednadžbom $\Delta u = 0$ uz rubne uvjete

$$u(x,2) = 2\sin(\frac{5\pi x}{3}),$$
$$u(x,0) = u(0,y) = u(3,y) = 0.$$

1\. zakon titranja homogene žice

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$l=2,\ \rho=1,\ p=1$

pričvršćena na mnogima: $u(0,t)=u(2,t)=0$

$u(x,0)=2\sin(3\pi x)-4\sin(2\pi x) \quad =\alpha(x)$

$\frac{\partial u}{\partial t}(x,0)=0 \quad =\beta(x)$

$$u(x,t)=\sum_{n=1}^{\infty}\left[E_n\cos\left(\frac{cn\pi t}{l}\right)+F_n\sin\left(\frac{cn\pi t}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

$c^2=\frac{p}{\rho}$

$$E_n=\frac{2}{l}\int_0^l \alpha(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$F_n=\frac{2}{cn\pi}\int_0^l \beta(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$c^2=\frac{1}{1}=1 \Rightarrow c=1$

$\beta(x)=0 \Rightarrow F_n=0,\ n\in\mathbb{N}$

$$E_n=\frac{2}{2}\int_0^2\left[2\sin(3\pi x)-4\sin(2\pi x)\right]\sin\left(\frac{n\pi x}{2}\right)dx$$

$$=\frac{2}{2}\int_0^2\left[2\sin\left(\frac{6\pi x}{2}\right)-4\sin\left(\frac{4\pi x}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)dx=\begin{cases}-4, & n=4\\ 2, & n=6\\ 0, & \text{inače}\end{cases}$$

$$\Rightarrow u(x,t)=-4\cos\left(\frac{4\pi t}{2}\right)\sin\left(\frac{4\pi x}{2}\right)+2\cos\left(\frac{6\pi t}{2}\right)\sin\left(\frac{6\pi x}{2}\right)$$

$$=-4\cos(2\pi t)\sin(2\pi x)+2\cos(3\pi t)\sin(3\pi x)$$

2. temperatura tankog homogenog štapa

$l=4,\ \delta=16,\ \gamma=1$

na krajevima temp. $0°C$: $u(0,t)=u(4,t)=0$

$$u(x,0)=2\sin\left(\frac{3\pi x}{2}\right)-\sin(2\pi x)\quad =g(x)$$

$$u(x,t)=\sum_{n=1}^{\infty}E_n e^{-(\lambda_n c)^2 t}\sin(\lambda_n x)$$

$$\lambda_n=\frac{n\pi}{l}$$

$$c^2=\frac{\delta}{\gamma}$$

$$E_n=\frac{2}{l}\int_0^l g(x)\sin(\lambda_n x)\,dx$$

$$c^2=\frac{16}{1}=16\ \Rightarrow\ c=4$$

$$E_n=\frac{2}{4}\int_0^4\left[2\sin\left(\frac{3\pi x}{2}\right)-\sin(2\pi x)\right]\sin\left(\frac{n\pi x}{4}\right)dx$$

$$=\frac{2}{4}\int_0^4\left[2\sin\left(\frac{6\pi x}{4}\right)-\sin\left(\frac{8\pi x}{4}\right)\right]\sin\left(\frac{n\pi x}{4}\right)dx=\begin{cases}2, & n=6\\ -1, & n=8\\ 0, & \text{inače}\end{cases}$$

$$\Rightarrow u(x,t)=2e^{-\left(\frac{6\pi\cdot 4}{4}\right)^2 t}\sin\left(\frac{6\pi x}{4}\right)-e^{-\left(\frac{8\pi\cdot 4}{4}\right)t}\sin\left(\frac{8\pi x}{4}\right)$$

$$=2e^{-36\pi^2 t}\sin\left(\frac{3\pi x}{2}\right)-e^{-64\pi^2 t}\sin(2\pi x)$$

3.) $\Omega = [0,3]\times[0,2]$

$$u(x,2) = 2\sin\left(\frac{5\pi x}{3}\right)$$

$$u(0,y) = u(3,y) = u(x,0) = 0$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \, sh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$B_n = \frac{1}{sh\left(\frac{n\pi b}{a}\right)} \frac{2}{a}\int_0^a \beta(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$B_n = \frac{1}{sh\left(\frac{n\pi\cdot 2}{3}\right)}\frac{2}{3}\int_0^3 2\sin\left(\frac{5\pi x}{3}\right)\sin\left(\frac{n\pi x}{3}\right)dx = \begin{cases} \frac{2}{sh\left(\frac{10\pi}{3}\right)}, & n=5 \\ 0, & n\neq 5 \end{cases}$$

$$\Rightarrow u(x,y) = \frac{2}{sh\left(\frac{10\pi}{3}\right)} sh\left(\frac{10\pi y}{3}\right)\sin\left(\frac{10\pi x}{3}\right)$$