1. Riješite diferencijalne jednadžbe:

   (a) (12 bodova) $y'' + 3y' + 2y = 6x + 1$,

   (b) (8 bodova) $y' \tan x = 3y$.

2. (a) (10 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x, y) = \frac{\ln(1 - x^2 - y^2)}{\sqrt{1 - |y - x|}}.$$

   (b) (10 bodova) Ispitajte lokalne ekstreme funkcije $f(x, y) = x^3 + 3xy^2 - 15x - 12y$.

3. (20 bodova) Izračunajte masu gornjeg dijela kugle $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$ ako je funkcija gustoće dana s $\rho(x, y, z) = z^2\sqrt{x^2 + y^2}$.

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4. (22 boda) Zadano je polje

$$\vec{a} = (y \cos(xy) + e^z)\, \vec{i} + x \cos(xy)\, \vec{j} + xe^z\, \vec{k}.$$

   (a) Izračunajte $\operatorname{rot} \vec{a}$. Je li polje potencijalno?

   (b) Odredite potencijal polja $\vec{a}$ ako postoji.

   (c) Izračunajte

$$\int_{\vec{\Gamma}} \vec{a}\, \mathrm{d}\vec{r}$$

   ako je $\vec{\Gamma}$ spojnica točaka $A(0, 0, 0)$ i $B(\pi, 1, 0)$.

5. (18 bodova) Izračunajte

$$\iint_{\Sigma} xz\, dS$$

gdje je Σ dio stošca $z = \sqrt{x^2 + y^2}$ isječen plohom $x^2 + y^2 = 2x$. Skicirajte plohu Σ.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
C	0	$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$
x^α	$\alpha x^{\alpha-1}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{sh} x$	$\operatorname{ch} x$
$\sin x$	$\cos x$	$\operatorname{ch} x$	$\operatorname{sh} x$
$\cos x$	$-\sin x$	$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	$x + C$	$\cos x$	$\sin x + C$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + C$
e^x	$e^x + C$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + C$
a^x	$\frac{a^x}{\ln a} + C$	$\frac{1}{1+x^2}$	$\operatorname{arctg} x + C$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln \left\lvert x + \sqrt{x^2 \pm 1} \right\rvert + C$

1. Riješite dif. jednadžbe:

a) $y'' + 3y' + 2y = 6x + 1$

HOMOGENO: $\rightsquigarrow x^2 + 3x + 2 = 0$

$$x_{1/2} = \frac{-3 \pm \sqrt{9 - 4 \cdot 2}}{2} \Rightarrow \begin{matrix} x_1 = -1 \\ x_2 = -2 \end{matrix}$$

$$y_H = Ae^{-x} + Be^{-2x}$$

PARTIKULARNO: DESNA STRANA JE POLINOM PA TRAŽIMO RJEŠENJE U OBLIKU $y_P = Ax + B$

(polinom je linearan jer se u partikularnom rješenju ne pojavljuje polinom)

UVRSTIMO:

$$(Ax+B)'' + 3(Ax+B)' + 2(Ax+B) = 6x+1$$

$$3A + 2Ax + 2B = 6x + 1 \Rightarrow \begin{cases} 2A = 6 \\ 3A + 2B = 1 \end{cases} \Rightarrow A = 3,\ B = -1.$$

Dakle, $y = y_P + y_H = 3x - 1 + Ae^{-x} + Be^{-2x}$, $A, B \in \mathbb{R}$ proizvoljni.

b) $y' \tan x = 3y$

$$\frac{y'}{y} = 3 \cdot \frac{1}{\tan x} = 3 \cdot \frac{\cos x}{\sin x}$$

$$\frac{dy}{y} = 3 \cdot \frac{\cos x}{\sin x}\, dx \quad / \int$$

$$\ln|y| = 3 \cdot \int \frac{\cos x}{\sin x} dx = \left| \begin{matrix} u = \sin x \\ du = \cos x\, dx \end{matrix} \right| = 3 \cdot \int \frac{du}{u} = 3 \ln|u| + c$$

$$= 3 \ln|\sin x| + c$$

Dakle, $|y| = e^c \cdot e^{3 \ln|\sin x|} = e^c \cdot |\sin x|^3$

$y = A \cdot \sin^3 x$, $A \in \mathbb{R}$ proizvoljan.

2.a) Odredi i skiciraj domenu

$$f(x,y) = \frac{\ln(1-x^2-y^2)}{\sqrt{1-|x-y|}}$$

Rj: MORA BITI:

$$\begin{cases} 1-x^2-y^2>0 \\ 1-|x-y|>0 \end{cases}$$ (ne može biti $1-|x-y|=0$ jer je u nazivniku)

PRVO, $1-x^2-y^2>0$

$\Leftrightarrow \quad x^2+y^2<1 \quad \rightarrow$ KRUG RADIJUSA 1, BEZ RUBA

DRUGO:

$$1-|x-y|>0$$

$$\Leftrightarrow \quad |x-y|<1$$

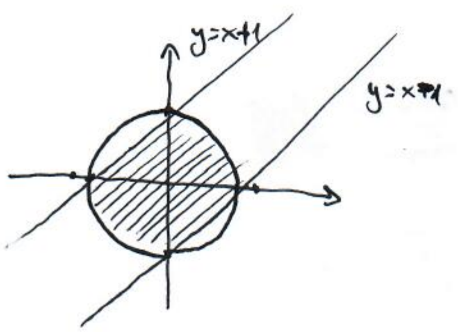
$$\Leftrightarrow \quad -1<x-y<1 \quad /(-1)$$

$$\Leftrightarrow \quad -1<y-x<1$$

$$\Leftrightarrow \quad \underbrace{x-1<y<x+1}$$

SKUP RJEŠENJA JE IZMEĐU TA DVA PRAVCA $\begin{cases} y=x-1 \\ y=x+1 \end{cases}$

SKICA:



2. b) ISPITAJ EKSTREME FUNKCIJE

$$f(x,y) = x^3 + 3xy^2 - 15x - 12y$$

Rj:

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15 = 0$$

$$\frac{\partial f}{\partial y} = 6xy - 12 = 0 \Rightarrow y = \frac{2}{x}$$

Dobivamo jednadžbu:

$$3x^2 + 3\cdot\left(\frac{2}{x}\right)^2 - 15 = 0 \quad /:3 \quad /\cdot x^2$$

$$x^4 - 5x^2 + 4 = 0$$

$$t = x^2 \Rightarrow t^2 - 5t + 4 = 0 \Leftrightarrow \begin{cases} t_1 = 1 \\ t_2 = 4 \end{cases}$$

$$\Rightarrow \quad x_1 = 1 \quad y_1 = 2$$
$$x_2 = -1 \quad y_2 = -2$$
$$x_3 = 2 \quad y_3 = 1$$
$$x_4 = -2 \quad y_4 = -1$$

Dakle, 4 su stacionarne točke $P_1(1,2)$, $P_2(-1,-2)$, $P_3(2,1)$, $P_4(-2,-1)$.

$$A = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$C = \frac{\partial^2 f}{\partial y^2} = 6x$$

$$AC - B^2 = 36x^2 - 36y^2$$

P_1	P_2	P_3	P_4
$A > 0$	$A < 0$	$A > 0$	$A < 0$
$AC - B^2 < 0$	$AC - B^2 < 0$	$AC - B^2 > 0$	$AC - B^2 < 0$
SEDLASTA TOČKA	SEDLASTA TOČKA	LOKALNI MINIMUM	LOKALNI MAKSIMUM

3.1 $\begin{cases} x^2+y^2+z^2 \le 4 \\ z \ge 0 \end{cases}$

Funkcija gustoće $\rho(x,y,z) = z^2\sqrt{x^2+y^2}$

Rj: Masa je dana formulom $m = \iiint\limits_{\Omega} \rho(x,y,z)\,dx\,dy\,dz$, gdje je Ω područje integracije

Dakle:

$$m = \iiint\limits_{\substack{x^2+y^2+z^2\le 4 \\ z\ge 0}} z^2\sqrt{x^2+y^2}\,dx\,dy\,dz$$

KAKO SE RADI O SFERI, PRIGODNE SU SFERNE KOORDINATE:

$x = r\cos\varphi\sin\theta$

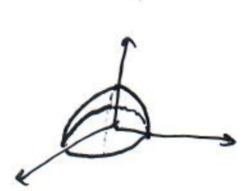
$y = r\sin\varphi\sin\theta$

$z = r\cos\theta$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \frac{\pi}{2}$

$0 \le r \le 2$



$$m = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} r^2\cos^2\theta\sqrt{r^2\cos^2\varphi\sin^2\theta + r^2\sin^2\varphi\sin^2\theta}\cdot \underbrace{r^2\sin\theta}_{\text{JAKOBIJAN}}\,dr\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} r^5\cdot\cos^2\theta\sqrt{\sin^2\theta}\cdot\sin\theta\,dr\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} r^5\cos^2\theta\,|\sin\theta|\,\sin\theta\,dr\,d\varphi\,d\theta$$

veće od 0 za $0\le\theta\le\frac{\pi}{2}$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} r^5\cos^2\theta\sin^2\theta\,dr\,d\varphi\,d\theta$$

$$= \int_0^{2\pi} d\varphi\cdot\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta\int_0^{2} r^5\,dr$$

$$= 2\pi\cdot\frac{2^6}{6}\cdot\int_0^{\frac{\pi}{2}}\frac{\sin^2 2\theta}{4}\,d\theta = \frac{64\pi}{3}\int_0^{\frac{\pi}{2}}\frac{1-\cos4\theta}{8}\,d\theta$$

$$= \frac{64\pi}{3}\cdot\frac{\pi}{16} = \frac{4\pi^2}{3}.$$

4. Zadano je polje $\vec{a} = \left(y\cos(xy) + e^{z}\right)\vec{i} + x\cos(xy)\,\vec{j} + xe^{z}\,\vec{k}$

a) $$\operatorname{rot}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = (0-0)\,\vec{i} + (e^{z} - e^{z})\,\vec{j} + \left(\cos(xy) - xy\sin(xy) - \cos(xy) + xy\sin(xy)\right)\vec{k}$$

$$= \underline{\underline{0}}.$$

$\operatorname{rot}\vec{a} = 0$ POVLAČI DA POLJE JEST POTENCIJALNO.

b) Odredite potencijal:

MOŽEMO UZETI $(x_0, y_0, z_0) = (0,0,0)$ I UVRSTITI U FORMULU:

$$\varphi(x,y,z) = \int_{x_0}^{x} a_x(t,y,z)\,dt + \int_{y_0}^{y} a_y(x_0,t,z)\,dt + \int_{z_0}^{z} (x_0,y_0,t)\,dt + C$$

$$= \int_0^x \left(y\cos ty + e^{z}\right)dt + \int_0^y 0\cdot dt + \int_0^z 0\cdot dt + C$$

$$= \left(y\cdot\frac{1}{y}\sin ty + te^{z}\right)\Big|_0^x + C$$

$$= \sin xy + xe^{z}$$

c) $\vec{\Gamma}$ je spojnica točaka $A(0,0,0)$ i $B(\pi,1,0)$.

$$\int_{\vec{\Gamma}} \vec{a}\,d\vec{r} = \varphi(B) - \varphi(A) \quad \text{jer je } \vec{a} \text{ potencijalno.}$$

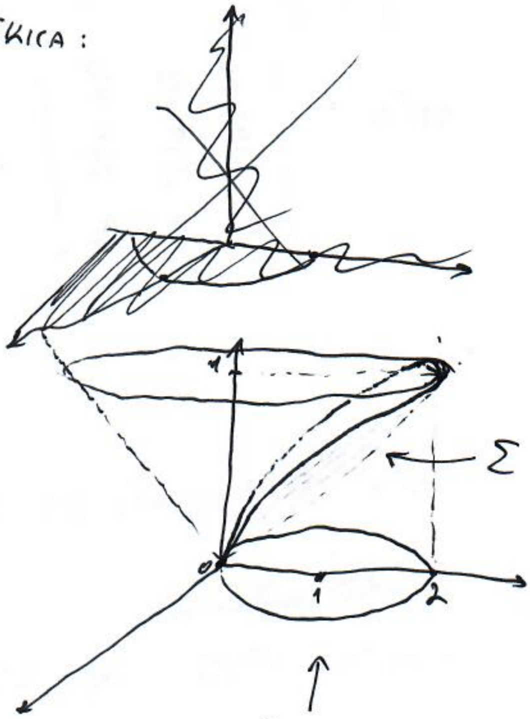
$$= \varphi(\pi,1,0) - \varphi(0,0,0)$$

$$= \pi.$$

5.) $\iint\limits_{\Sigma} xz\, dS$

Σ je ISJEČAK STOŠCA $z=\sqrt{x^2+y^2}$
i VALJKA $x^2+y^2=2x$.

SKICA:



PROJEKCIJA

$D \dots x^2+y^2=2x$ KRUG!

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dx\,dy$$

$$= \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\; dx\,dy$$

$$= \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1}\; dx\,dy$$

$$= \sqrt{2}\cdot dx\,dy$$

DAKLE:

$$\iint\limits_{\Sigma} xz\,dS = \iint\limits_{D} x\cdot\sqrt{x^2+y^2}\cdot\sqrt{2}\,dx\,dy$$

→ POGODNO ZA POLARNE KOORDINATE:

$r^2 = 2r\cos\varphi$

$r = 2\cos\varphi$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\cos\varphi} r\cos\varphi\cdot r\cdot\sqrt{2}\,r\,dr\,d\varphi$$

$$= \sqrt{2}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\cdot \frac{r^4}{4}\Big|_0^{2\cos\varphi}\,d\varphi$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\cdot\frac{16\cos^4\varphi}{4}\,d\varphi$$

$$= 4\sqrt{2}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\varphi\,d\varphi$$

$$= 4\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\varphi)^2\cos\varphi\,d\varphi = \left|\begin{matrix} u=\sin\varphi \\ du=\cos\varphi\,d\varphi\end{matrix}\right|$$

$$= 4\sqrt{2}\int_{-1}^{1}(1-u^2)^2\,du = 4\sqrt{2}\cdot\int_{-1}^{1}(1-2u^2+u^4)\,du$$

$$= 4\sqrt{2}\cdot\left[u-\frac{2}{3}u^3+\frac{u^5}{5}\right]\Big|_{-1}^{1} = 4\sqrt{2}\left[2-\frac{4}{3}+\frac{2}{5}\right] = \frac{64\sqrt{2}}{15}$$