MATEMATIKA 3 8.9.2021.

1. (20 bodova) Pronađite zakon titranja homogene žice duljine 3, gustoće 1 i napetosti 4 koja je pričvršćena na rubovima. Početni položaj i početna brzina su zadani sljedećim funkcijama:

$$\begin{aligned} u(x,0) &= \sin(2\pi x) - 3\sin(\frac{2\pi x}{3}), \\ \frac{\partial u}{\partial t}(x,0) &= 0. \end{aligned}$$

2. (20 bodova) Riješite problem slobodnih oscilacija štapa duljine $l = 5$ čiji su krajevi nehomogeni zglobovi, to jest

$$\begin{aligned} u(0,t) &= 0, \\ \frac{\partial^2 u}{\partial x^2}(0,t) &= 5, \\ u(l,t) &= 0, \\ \frac{\partial^2 u}{\partial x^2}(l,t) &= 0. \end{aligned}$$

Početna brzina je $\psi(x) = 0$, a početni položaj je $\varphi(x) = \sin(\frac{4\pi x}{5}) - \frac{x^3}{6} + \frac{5x^2}{2} - \frac{25x}{3}$.

3. (20 bodova) Riješite problem ravnoteže kružne membrane polumjera $R = 2$ i napetosti $p = 4$ ako je zadana gustoća vanjske sile $f(r) = 3r - 5$ i rubni uvjet $u|_{r=2} = 0$.

1. zakon titranja homogene žice

8.9.2021. | MAT3

$l=3,\ \rho=1,\ p=4$

pričvršćena na rubovima: $u(0,t)=u(3,t)=0$

$u(x,0) = \sin(2\pi x) - 3\sin\left(\frac{2\pi x}{3}\right)$

$\frac{\partial u}{\partial t}(x,0)=0$

$c^2=\frac{p}{\rho}$

$$u(x,t)=\sum_{n=1}^{\infty}\left[E_n\cos\left(\frac{cn\pi t}{l}\right)+F_n\sin\left(\frac{cn\pi t}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

$$E_n=\frac{2}{l}\int_0^l \alpha(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$F_n=\frac{2}{cn\pi}\int_0^l \beta(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$\beta(x)=0 \Rightarrow F_n=0,\ n\in\mathbb{N}$

$$E_n=\frac{2}{3}\int_0^3\left[\sin(2\pi x)-3\sin\left(\frac{2\pi x}{3}\right)\right]\sin\left(\frac{n\pi x}{3}\right)dx$$

$$=\frac{2}{3}\int_0^3\left[\sin\left(\frac{6\pi x}{3}\right)-3\sin\left(\frac{2\pi x}{3}\right)\right]\sin\left(\frac{n\pi x}{3}\right)dx=\begin{cases}-3, & n=2,\\ 1, & n=6,\\ 0, & \text{inače.}\end{cases}$$

$$\Rightarrow u(x,t)=-3\cos\left(\frac{2\cdot 2\cdot\pi t}{3}\right)\sin\left(\frac{2\pi x}{3}\right)+\cos\left(\frac{2\cdot 6\pi t}{3}\right)\sin\left(\frac{6\pi x}{3}\right)$$

$$=-3\cos\left(\frac{4\pi t}{3}\right)\sin\left(\frac{2\pi x}{3}\right)+\cos(4\pi t)\sin(2\pi x)$$

2. slobodne oscilacije štapa

$\ell = 5$

nehomogeni zglobovi: $u(0,t) = 0$

$\frac{\partial^2 u}{\partial x^2}(0,t) = 5$

$u(5,t) = 0$

$\frac{\partial^2 u}{\partial x^2}(5,t) = 0$

$\psi(x) = 0$

$\varphi(x) = \sin\left(\frac{4\pi x}{5}\right) - \frac{x^3}{6} + \frac{5x^2}{2} - \frac{25x}{3}$

$$\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0$$

$$u(x,t) = v(x,t) + w(x)$$

- $w^{(4)}(x) = 0$

$w(0) = 0, \quad w''(0) = 5, \quad w(5) = 0, \quad w''(5) = 0$

- $v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{a n^2 \pi^2 t}{\ell^2}\right) + B_n \sin\left(\frac{a n^2 \pi^2 t}{\ell^2}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right)$

$\boxed{\psi(x) = 0 \Rightarrow B_n = 0, n \in \mathbb{N}}$

$A_n = \frac{2}{\ell}\int_0^\ell v(x,0) \sin\left(\frac{n\pi x}{\ell}\right) dx$

$w^{(4)}(x) = 0 \;/\int dx$

$w'''(x) = C_1 \;/\int dx \Rightarrow w''(x) = C_1 x + C_2 \;/\int dx$

$\Rightarrow w'(x) = C_1 \frac{x^2}{2} + C_2 x + C_3 \;/\int dx$

$\Rightarrow w(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$

$0 = w(0) = C_4$

$5 = w''(0) = C_2$

$0 = w(5) = C_1 \frac{125}{6} + 5\frac{25}{2} + C_3 \cdot 5 + 0 \Rightarrow$

$0 = w''(5) = 5C_1 + 5 \Rightarrow C_1 = -1$

$0 = -\frac{125}{6} + \frac{125}{2} + 5C_3 \;/:5$

$\Rightarrow 0 = -\frac{25}{6} + \frac{25}{2} + C_3$

$\Rightarrow C_3 = \frac{25}{6} - \frac{25}{2} = -\frac{50}{6} = -\frac{25}{3}$

$\boxed{w(x) = -\frac{x^3}{6} + \frac{5x^2}{2} - \frac{25}{3}x}$

$v(x,0) = \varphi(x) - w(x) = \sin\left(\frac{4\pi x}{5}\right)$

$A_n = \frac{2}{5}\int_0^5 \sin\left(\frac{4\pi x}{5}\right)\sin\left(\frac{n\pi x}{5}\right)dx = \begin{cases} 1, & n=4 \\ 0, & n \neq 4 \end{cases}$

$\Rightarrow v(x,t) = \cos\left(\frac{a \cdot 16\pi^2 t}{25}\right) \sin\left(\frac{4n\pi}{5}\right)$

$\Rightarrow$

$u(x,t) = \cos\left(\frac{16 a \pi^2 t}{25}\right)\sin\left(\frac{4n\pi}{5}\right) - \frac{x^3}{6} + \frac{5x^2}{2} - \frac{25x}{3}$

3. ravnoteža kružne membrane

$R=2$

$p=4$

$f(r)=3r-5$

$u\big|_{r=2}=0$

$$-4\Delta u = 3r-5 \quad \Big/ :(-4) \quad \Rightarrow \quad \Delta u = -\frac{3r-5}{4}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{3r-5}{4} \quad \Big| \cdot r$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{3r^2}{4}+\frac{5r}{4} \quad \Big| \int dr$$

$$r\frac{\partial u}{\partial r} = -\frac{3}{4}\frac{r^3}{3}+\frac{5}{4}\frac{r^2}{2}+C_1 \quad \Big| :r$$

$$\frac{\partial u}{\partial r} = -\frac{1}{4}r^2+\frac{5}{8}r+\frac{C_1}{r} \quad \Big| \int dr$$

$$u(r) = -\frac{1}{4}\frac{r^3}{3}+\frac{5}{8}\frac{r^2}{2}+\underbrace{C_1 \ln r}_{\Rightarrow\, C_1=0}+C_2$$

$$0=u(2) = -\frac{1}{4}\frac{8}{3}+\frac{5}{8}\frac{4}{2}+C_2 = -\frac{2}{3}+\frac{5}{4}+C_2 \Rightarrow C_2=\frac{2}{3}-\frac{5}{4}=\frac{8-15}{12}=\frac{-7}{12}$$

$$\Rightarrow u(r) = -\frac{r^3}{12}+\frac{5r^2}{16}-\frac{7}{12}$$