1. Riješite diferencijalne jednadžbe:

   (a) (10 bodova) $xy' + y - e^x = 0$;

   (b) (10 bodova) $y'' - y = xe^x$.

2. (a) (10 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \frac{\sqrt{y^2 - 4x}}{\ln\left(x^2 + y^2 - 1\right)}.$$

   (b) (10 bodova) Ispitajte lokalne ekstreme funkcije $f(x,y) = e^{x-y}(x^2 - 2y^2)$.

3. (20 bodova) Izračunajte masu tijela $V = \begin{cases} x^2 + y^2 \le 2x \\ 0 \le y \\ 0 \le z \le 1 \end{cases}$, ako je funkcija gustoće dana s $\rho(x,y,z) = z\sqrt{x^2 + y^2}$. Skicirajte tijelo.

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4. (a) (10 boda) Je li polje $\vec{a} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$ potencijalno?

   (b) (15 boda) Zadano je polje $\vec{b} = (1-y)\,\vec{i} + x\vec{j}$. Izračunajte

$$\int_{\vec{\Gamma}} \vec{b}\,\mathrm{d}\vec{r}$$

   ako je $\vec{\Gamma}$ negativno orijentirani luk prvog svoda cikloide $\left(\frac{t-\sin t}{2}, \frac{1-\cos t}{2}\right), t \in [0, 2\pi]$.

5. (15 bodova) Izračunajte

$$\iint_{\vec{\Sigma}} \vec{a}\, d\vec{S}$$

   ako je $\vec{a} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, a $\vec{\Sigma}$ vanjska strana plohe polukugle $x^2 + y^2 + z^2 = 1$, $z \ge 0$. Skicirajte $\vec{\Sigma}$.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
C	0	$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$
x^α	$\alpha x^{\alpha-1}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{sh} x$	$\operatorname{ch} x$
$\sin x$	$\cos x$	$\operatorname{ch} x$	$\operatorname{sh} x$
$\cos x$	$-\sin x$	$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	$x + C$	$\cos x$	$\sin x + C$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \ne -1$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + C$
e^x	$e^x + C$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + C$
a^x	$\frac{a^x}{\ln a} + C$	$\frac{1}{1+x^2}$	$\operatorname{arctg} x + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln\left\lvert x + \sqrt{x^2 \pm 1}\right\rvert + C$

1a) $xy' + y - e^x = 0 /: x$ LIN. DIF. JEDN. PRVOG REDA

$$y' + \frac{y}{x} = \frac{e^x}{x}$$

RJ. $y = e^{-\int f(x)} \left(e^{\int f(x)} g(x)\, dx + C \right)$

$$\int f(x) dx = \int \frac{dx}{x} = \ln |x|$$

$$\Rightarrow \quad y = e^{\ln x^{-1}} \left(e^{\ln x} \cdot \frac{e^x}{x} + c \right)$$

$$y = \frac{1}{x} \left(x \cdot \frac{e^x}{x} + c \right) = \frac{e^x + C}{x}$$

b) $y'' - y = xe^x$

$x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1 \quad \Rightarrow \quad y_H = C_1 e^x + C_2 e^{-x}$

$f(x) = xe^x = P_n(x)\, e^{ax} \quad \Rightarrow \quad a = 1, \quad n = 1$

$$y_p = x \cdot (Ax + B) \cdot e^x = (Ax^2 + Bx) \cdot e^x$$

$$y_p' = (2Ax + B) e^x + (Ax^2 + Bx) e^x = e^x (Ax^2 + x \cdot (2A + B) + B)$$

$$y_p'' = e^x (Ax^2 + x(2A + B) + B) + e^x (2Ax + 2A + B)$$

$$= e^x (Ax^2 + x(4A + B) + 2A + 2B)$$

$y'' - y = xe^x \quad \Rightarrow$

$$e^x (Ax^2 + x(4A + B) + 2A + 2B) - e^x (Ax^2 + Bx) = xe^x$$

$$e^x (4Ax + 2A + 2B) = xe^x$$

$$4A = 1 \Rightarrow A = \frac{1}{4} \Rightarrow B = \frac{-1}{4}$$

$$y_p = \left(\frac{1}{4} x^2 - \frac{1}{4} x \right) e^x$$

$$y = C_1 e^x + C_2 e^{-x} + \frac{x^2 - x}{4} e^x$$

2a) $f(x,y) = \dfrac{\sqrt{y^2-4x}}{\ln(x^2+y^2-1)}$

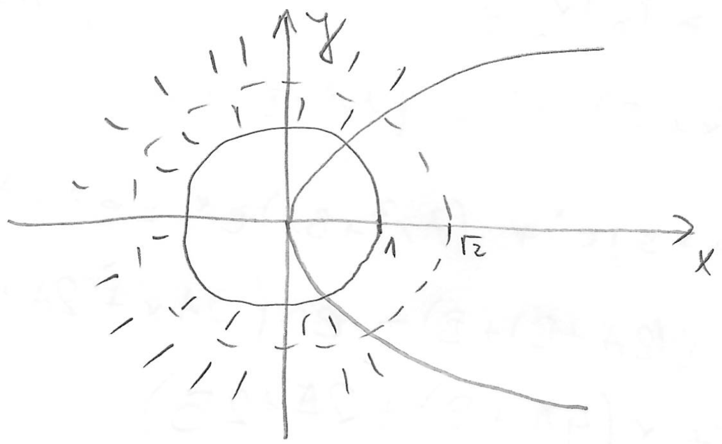
UVJETI: $\begin{cases} y^2-4x \geq 0 & \rightarrow \text{UVJET KORIJENA} \\ x^2+y^2-1>0 & \rightarrow \text{UVJET LOGARITMA} \\ \ln(x^2+y^2-1) \neq 0 & \rightarrow \text{UVJET NAZIVNIKA} \end{cases}$

1. $y^2 = 4x$ PARABOLA

2. $x^2+y^2=1$ KRUŽNICA $r=1$

3. $\ln(x^2+y^2-1) \neq 0$

$x^2+y^2 \neq 2$ KRUŽNICA $r=\sqrt{2}$



b) $f(x,y) = e^{x-y}(x^2-2y^2)$

$\dfrac{\partial f}{\partial x} = e^{x-y}(x^2-2y^2) + e^{x-y}(2x) = e^{x-y}(x^2+2x-2y^2)$

$\dfrac{\partial f}{\partial y} = -e^{x-y}(x^2-2y^2) + e^{x-y}(-4y) = e^{x-y}(2y^2-4y-x^2)$

SUSTAV: $e^{x-y} \neq 0 \Rightarrow$ $\begin{array}{r} x^2+2x-2y^2=0 \\ -x^2-4y+2y^2=0 \\ \hline x=2y \end{array}$ / +

$\Rightarrow -4y^2-4y+2y^2=0$

$2y(2y+4)=0 \Rightarrow y=0, \; y=2 \Rightarrow x=0, \; x=-4$

$$A = \frac{\partial^2 f}{\partial x^2}(x,y) = e^{x-y}(x^2+4x-2y^2+2)$$

$$B = \frac{\partial^2 f}{\partial x \partial y}(x,y) = e^{x-y}(2y^2-4y-2x-x^2)$$

$$C = \frac{\partial^2 f}{\partial y^2}(x,y) = e^{x-y}(x^2-2y^2+8y-4)$$

$T_1(0,0)$ $\quad A = 2 \quad B = 0 \quad C = -4$

$AC - B^2 = -8 < 0$.

SEDLASTA TOČKA

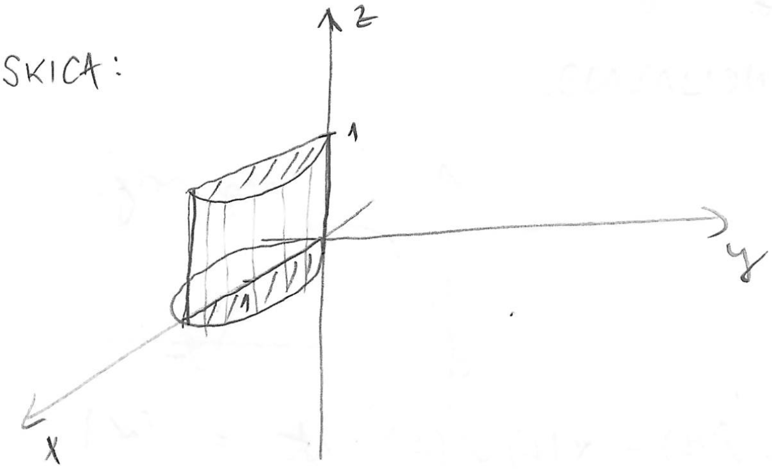
$T_2(-4,-2)$ $\quad A = -6e^{-2} \quad B = 8e^{-2} \quad C = -12e^{-2}$

$AC - B^2 = 72e^{-4} - 64e^{-4} = 8e^{-4} > 0 \Rightarrow$ EKSTREM

$A < 0 \Rightarrow$ MAKSIMUM

3. $V = \begin{cases} x^2+y^2 \le 2x \\ 0 \le y \\ 0 \le z \le 1 \end{cases} \Rightarrow (x-1)^2+y^2 \le 1, \quad S(1,0),\ r=1$

SKICA:



CILINDRIČNE K:

$x = r\cos\varphi$

$y = r\sin\varphi$

$\Rightarrow r^2 \le 2r\cos x$

$\underline{r \le 2\cos\varphi}$

$\underline{0 \le \varphi \le \frac{\pi}{2}}$

$0 \le z \le 1$

$$m = \iiint\limits_V \rho(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^{\pi/2}\int_0^{2\cos\varphi} z\cdot\sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi}\; r\,dr\,d\varphi\,dz =$$

$$= \int_0^1\int_0^{\pi/2}\int_0^{2\cos\varphi} z r^2\,dr\,d\varphi\,dz = \int_0^1 z\,dz \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{2\cos\varphi} d\varphi =$$

$$= \frac{1}{2}z^2\Big|_0^1 \int_0^{\pi/2} \frac{8}{3}\cos^3\varphi\,d\varphi = \frac{4}{3}\int_0^{\pi/2} \cos\varphi(1-\sin^2\varphi)\,d\varphi = \left\{\begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\,d\varphi \\ \varphi = 0 \Rightarrow t = 0 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array}\right.$$

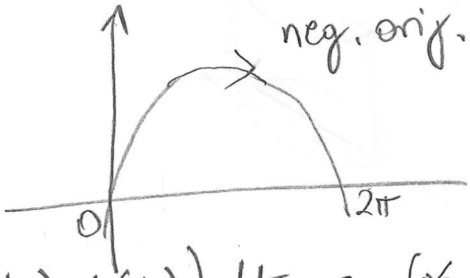
$$= \frac{4}{3}\int_0^1 (1-t^2)\,dt = \frac{4}{3}\left(t - \frac{t^3}{3}\right)\Big|_0^1 = \frac{4}{3}\cdot\frac{2}{3} = \frac{8}{9}$$ //

4a) $\vec{a} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$

$$\mathrm{rot}\,\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix} = (1-1)\vec{i} - (1-1)\vec{j} + (1-1)\vec{k} = 0$$

POLJE JE POTENCIJALNO.

b) $\vec{b} = (1-y)\vec{i} + x\vec{j}$



$$\int_{\Gamma} \vec{b}\,d\vec{r} = \int_0^{2\pi} \Big((1-y(t))\cdot x'(t) + x(t)\,y'(t) \Big)\,dt = (*)$$

$$x(t) = \frac{t-\sin t}{2} \qquad y(t) = \frac{1-\cos t}{2}$$

$$x'(t) = \frac{1-\cos t}{2} \qquad y'(t) = \frac{\sin t}{2}$$

$$(*) = \int_0^{2\pi} \left[\left(1 - \frac{1-\cos t}{2}\right) \cdot \frac{1-\cos t}{2} + \frac{t-\sin t}{2} \cdot \frac{\sin t}{2} \right] dt =$$

$$= \int_0^{2\pi} \left(\frac{1+\cos t}{2} \cdot \frac{1-\cos t}{2} + \frac{t\sin t - \sin^2 t}{4} \right) dt$$

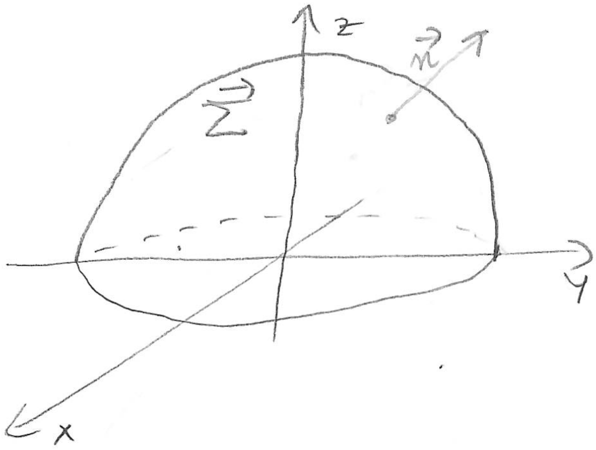
$$= \int_0^{2\pi} \frac{1 \overbrace{- \cos^2 t - \sin^2 t}^{-1} + t\sin t}{4} = \frac{1}{4}\int_0^{2\pi} t \sin t\,dt =$$

$$= \left\{ \begin{matrix} u = t & \sin t\,dt = dv \\ du = dt & -\cos t = v \end{matrix} \right\} = \frac{1}{4}\left(-t\cos t \Big|_0^{2\pi} + \int_0^{2\pi} \cos t \right) =$$

$$= \frac{1}{4}\cdot(-2\pi) + \underbrace{\sin t \Big|_0^{2\pi}}_{0} = \frac{-\pi}{2} \,//$$

5. $\vec{a} = (x^2, y^2, z^2)$, $x^2+y^2+z^2 = 1,\ z \geq 0$

$Z = \sqrt{1-x^2-y^2}$



$f(x,y) = \sqrt{1-x^2-y^2}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\vec{n} = \left(\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}, 1\right)$$

$$\vec{a}\cdot\vec{n} = \frac{-x^3-y^3}{\sqrt{1-x^2-y^2}} + z^2 = \frac{-x^3-y^3}{\sqrt{1-x^2-y^2}} + 1-x^2-y^2$$

$$\iint_{\vec{\Sigma}} \vec{a}\, d\vec{S} = \iint_D \left(\frac{-x^3+y^3}{\sqrt{1-x^2-y^2}} + 1-x^2-y^2\right) dx\,dy =$$

POLARNE K.

$x = r\cos\varphi$

$y = r\sin\varphi$

$0 \leq r \leq 1$

$0 \leq \varphi \leq 2\pi$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{r^3(\cos^3\varphi + \sin^3\varphi)}{\sqrt{1-r^2}} + 1 - r^2\right) r\,dr\,d\varphi$$

$$= \underbrace{\int_0^{2\pi}\int_0^1 \frac{r^4}{\sqrt{1-r^2}}(\cos^3\varphi+\sin^3\varphi)\,dr\,d\varphi}_{(*)} + \underbrace{\int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\varphi}_{(\square)}$$

$$\int_0^{2\pi}\cos^3\varphi\,d\varphi = \int_0^{2\pi}\cos\varphi(1-\sin^2\varphi)\,d\varphi = \left\{\begin{array}{l} t=\sin\varphi \quad dt=\cos\varphi\,d\varphi \\ t=0,\ t=0 \end{array}\right.$$

$$= \int_0^0 (1-t^2)\,dt = \left(t-\frac{t^3}{3}\right)\Big|_0^0 = 0$$

SLICNO i $\int_0^{2\pi}\sin^3\varphi\,d\varphi$

$$\Rightarrow (\square) \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\varphi = \int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 = \int_0^{2\pi}\frac{1}{4}\,d\varphi = \frac{\pi}{2}$$