# Matematika 1 - pismeni ispit

14.2.2022.

Ime i prezime: ______________________________

| 1. | 2. | 3. | 4. | 5. |
|---|---|---|---|---|
| | | | | |

| 1. dio | 2. dio | $\sum$ |
|---|---|---|
| | | |

Ocjena pismenog ispita: ______________________________

1. **(15 bodova)** Odredite točku $T'$ simetričnu točki $T(2,1,1)$ obzirom na ravninu $\pi \ldots x - 2y + 3z + 11 = 0$. Skicirajte.

2. a) **(15 bodova)** Odredite svojstvene vrijednosti i svojstvene vektore matrice $A = \begin{bmatrix} 5 & -1 & 1 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$.

   b) **(10 bodova)** Ispitajte konvergenciju reda

$$\sum_{n=1}^{\infty} \frac{n-1}{n \cdot 3^n}.$$

3. **(20 bodova)** Odredite prirodno područje definicije, nultočke, intervale rasta i pada, ekstreme, asimptote te skicirajte graf funkcije

$$f(x) = \frac{x-2}{\sqrt{x^2+1}}.$$

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4. **(17 bodova)** Odredite

$$\int \frac{(-2e^{3x} + 12e^{2x} - 12e^x + 16)e^x}{(e^x-2)^2(e^{2x}+4)} dx.$$

5. a) **(8 bodova)** Odredite nepravi integral

$$\int_1^{+\infty} \frac{dx}{x^3}.$$

   b) **(15 bodova)** Izračunajte volumen tijela koje nastaje rotacijom lika omeđenog s

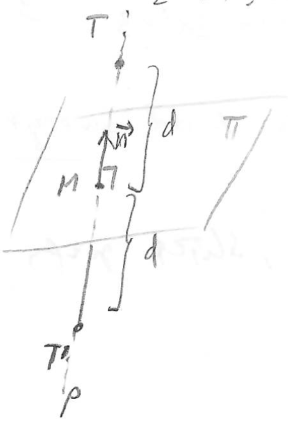
$$y = \frac{x^2}{4},\ y = -x + 3 \text{ i } y = 0$$

   oko osi $x$.

1. (15 bodova)

$T'$ simetrična točki $T(2,1,1)$ obzirom na $\pi \equiv x-2y+3z+11=0$ & skica

Rj: $2-2+3+11 = 14 \neq 0 \Rightarrow T \notin \pi$



1) $p = (T, \vec{n}_\pi) \rightarrow \vec{n}_\pi = (1,-2,3) \rightarrow p \ldots \begin{cases} x = t+2 \\ y = -2t+1 \\ z = 3t+1 \end{cases}$

2) $M = p \cap \pi$

3) $M$ polovište $\overline{TT'}$

$$t+2-2(-2t+1)+3(3t+1)+11=0$$
$$\Rightarrow t+2+4t-2+9t+3+11=0$$
$$\Rightarrow 14t+14=0 \Rightarrow 14t=-14 \Rightarrow t=-1$$
$$\Rightarrow M = (-1+2, -2(-1)+1, 3(-1)+1) = (1,3,-2)$$

$$x_M = \frac{x_T + x_{T'}}{2} \Rightarrow 1 = \frac{2+x_{T'}}{2} \Rightarrow x_{T'} = 0$$

$$y_M = \frac{y_T + y_{T'}}{2} \Rightarrow 3 = \frac{1+y_{T'}}{2} \Rightarrow y_{T'} = 5$$

$$z_M = \frac{z_T + z_{T'}}{2} \Rightarrow -2 = \frac{1+z_{T'}}{2} \Rightarrow z_{T'} = -5$$

$\Rightarrow$ Točka simetrična točki $T$ obzirom na $\pi$ je $T' = (0,5,-5)$.

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2. a) (15 bodova) sv. vrij. i sv. vektori od $A = \begin{pmatrix} 5 & -1 & 1 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$

Rj:

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ -1 & 1 & 3-\lambda \end{vmatrix} = (-1)^{2+2}(2-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (2-\lambda)\left[(5-\lambda)(3-\lambda)+1\right] = (2-\lambda)(15-5\lambda-3\lambda+\lambda^2+1)$$

$$= (2-\lambda)(\lambda^2-8\lambda+16) = (2-\lambda)(\lambda-4)^2$$

$$\lambda_{1,2} = \frac{8 \pm \sqrt{64-4\cdot 16}}{2} = 4$$

$\Rightarrow$ Svojstvene vrijednosti su $\lambda_1 = 2$ ; $\lambda_2 = 4$.

Svojstveni vektori:

za $\lambda = 2$: $(A-2I)v = 0 \Rightarrow \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$3v_1 - v_2 + v_3 = 0 \Rightarrow v_2 = 3v_1 + v_3$

$-v_1 + v_2 + v_3 = 0$

$-v_1 + 3v_1 + v_3 + v_3 = 0$

$\Rightarrow 2v_1 + 2v_3 = 0$

$\Rightarrow v_1 = -v_3$

$v_2 = -3v_3 + v_3 = -2v_3$

$$\Rightarrow v = \begin{pmatrix} -t \\ -2t \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

za $\lambda = 4$: $(A-4I)v = 0 \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 0 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$v_1 - v_2 + v_3 = 0 \Rightarrow v_1 = -v_3$

$-2v_2 = 0 \Rightarrow v_2 = 0$

$-v_1 + v_2 - v_3 = 0$ −||−

$$\Rightarrow v = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

2. b) (10 bodova) konvergencija reda $\sum_{n=1}^{\infty} \frac{n-1}{n\cdot 3^n}$

Rj:

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{\frac{(n+1)-1}{(n+1)\cdot 3^{n+1}}}{\frac{n-1}{n\cdot 3^n}} = \lim_{n\to+\infty} \frac{n\cdot n\cdot \cancel{3^n}}{(n+1)(n-1)\cdot \cancel{3^n}\cdot 3} = \lim_{n\to+\infty} \frac{n^2}{3n^2-3} = \frac{1}{3}$$

Dobili smo $q=\frac{1}{3}<1$ pa po D'Alembertovom kriteriju zaključujemo da red konvergira.

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3.) (20 bodova) $D_f$, nultočke, rast/pad, ekstremi, asimptote, skica grafa

Rj: $f(x) = \frac{x-2}{\sqrt{x^2+1}}$

- domena:

uvjeti: I $\sqrt{x^2+1} \neq 0$, II $x^2+1 \geq 0$ $\Rightarrow x^2+1>0 \Rightarrow x^2>-1$ uvijek vrijedi

$\Rightarrow D_f = \mathbb{R}$

- nultočke: za koje $x$ je $f(x)=0$? $\Rightarrow f(x)=0 \Leftrightarrow \frac{x-2}{\sqrt{x^2+1}}=0 \Rightarrow x-2=0 \Rightarrow x=2$

$\rightarrow$ Nultočka je $x=2$.

- rast/pad i ekstremi:

$$f'(x) = \frac{1\cdot\sqrt{x^2+1} - (x-2)\frac{1}{\cancel{2}\sqrt{x^2+1}}\cdot\cancel{2}x}{\sqrt{x^2+1}^{\,2}} = \frac{\sqrt{x^2+1} - \frac{x^2-2x}{\sqrt{x^2+1}}}{x^2+1} = \frac{\frac{x^2+1-x^2+2x}{\sqrt{x^2+1}}}{x^2+1} =$$

$$= \frac{2x+1}{(x^2+1)^{3/2}}$$

$f'(x)=0 \Leftrightarrow \frac{2x+1}{(x^2+1)^{3/2}}=0 \Rightarrow 2x+1=0 \Rightarrow 2x=-1 \Rightarrow x=\frac{-1}{2}$

$\rightarrow$ stacionarna točka: $x=\frac{-1}{2}$.

| | $-\infty$ | $-\frac{1}{2}$ | $+\infty$ |
|---|---|---|---|
| $f'$ | $-$ | | $+$ |
| $f$ | $\searrow$ | | $\nearrow$ |

$f'(-1) = \frac{-1}{2^{3/2}} < 0$

$f'(1) = \frac{3}{2^{3/2}} > 0$

$\Rightarrow f$ raste na $\langle -\frac{1}{2}, +\infty\rangle$, pada na $\langle -\infty, -\frac{1}{2}\rangle$ i ima lokalni minimum $(-\frac{1}{2}, f(-\frac{1}{2})) = (-\frac{1}{2}, -\frac{5}{\sqrt{5}})$

- asimptote: i) nema V.A. (domena je $\mathbb{R}$)

ii) $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x-2}{\sqrt{x^2+1}} : \frac{x}{x} =$

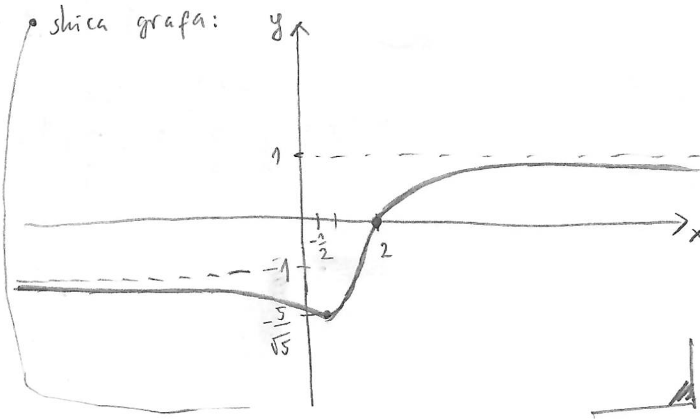
$= \lim_{x\to+\infty} \frac{1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} = 1 \Rightarrow y=1$ desna H.A.

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x-2}{\sqrt{x^2+1}} : \frac{x}{x} = \lim_{x\to-\infty} \frac{1-\frac{2}{x}}{-\sqrt{1+\frac{1}{x^2}}} = -1$

$\Rightarrow y=-1$ lijeva H.A.

iii) nema K.A. (jer ima horizontalne)

- skica grafa:



4.) (17 bodova)

Rj: $\int \frac{(-2e^{3x}+12e^{2x}-12e^{x}+16)e^{x}}{(e^{x}-2)^{2}(e^{2x}+4)}\,dx = \left\{ \begin{array}{l} t=e^{x} \\ dt=e^{x}dx \end{array} \right\} = \int \frac{-2t^{3}+12t^{2}-12t+16}{(t-2)^{2}(t^{2}+4)}\,dt = \circledast$

rastav na parcijalne razlomke:

$$\frac{-2t^{3}+12t^{2}-12t+16}{(t-2)^{2}(t^{2}+4)} = \frac{A}{t-2} + \frac{B}{(t-2)^{2}} + \frac{Ct+D}{t^{2}+4} \quad \Big/ \cdot (t-2)^{2}(t^{2}+4)$$

$$-2t^{3}+12t^{2}-12t+16 = A(t-2)(t^{2}+4) + B(t^{2}+4) + (Ct+D)(t^{2}-4t+4)$$

$$-2t^{3}+12t^{2}-12t+16 = A(t^{3}+4t-2t^{2}-8) + B(t^{2}+4) + Ct^{3}-4Ct^{2}+4Ct+Dt^{2}-4Dt+4D$$

$(t^{3})$: $-2 = A+C \Rightarrow A = -2-C$

$(t^{2})$: $12 = -2A+B-4C+D$ $\Rightarrow 12 = 4+2C+B-4C+D = 4+B-2C+D$

$(t)$: $-12 = 4A+4C-4D$ $\Rightarrow -12 = -8-4C+4C-4D \Rightarrow -4 = -4D \Rightarrow \boxed{D=1}$

$(c)$: $16 = -8A+4B+4D$ $\Rightarrow 16 = 16+8C+4B+4D$

$-4 = 8C+4B$

$7 = B-2C$

$\Rightarrow B = 7+2C$

$-4 = 8C+28+8C$

$-32 = 16C \Rightarrow \boxed{C=-2}$

$\boxed{B=3}$

$\boxed{A=0}$

$$\circledast = \int\left(\frac{0}{t-2} + \frac{3}{(t-2)^{2}} + \frac{-2t+1}{t^{2}+4}\right)dt =$$

$$= 3\int\frac{1}{(t-2)^{2}}\,dt - \int\frac{2t}{t^{2}+4}\,dt + \int\frac{1}{t^{2}+4}\,dt = 3\int\frac{1}{w^{2}}\,dw - \int\frac{1}{z}\,dz + \int\frac{1}{t^{2}+2^{2}}\,dt =$$

$\left\{ \begin{array}{l} w=t-2 \\ dw=dt \end{array} \right\}$ $\left\{ \begin{array}{l} z=t^{2}+4 \\ dz=2t\,dt \end{array} \right\}$

$$= 3\,\frac{w^{-1}}{-1} - \ln|z| + \frac{1}{2}\operatorname{arctg}\frac{t}{2} + C = -\frac{3}{t-2} - \ln(t^{2}+4) + \frac{1}{2}\operatorname{arctg}\frac{t}{2} + C =$$

$$= -\frac{3}{e^{x}-2} - \ln(e^{2x}+4) + \frac{1}{2}\operatorname{arctg}\frac{e^{x}}{2} + C$$

5.) a) (8 bodova)

Rj: $$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b\to+\infty} \int_1^b x^{-3}\,dx = \lim_{b\to+\infty} \frac{x^{-2}}{-2}\Big|_1^b = \lim_{b\to+\infty}\left(\frac{b^{-2}}{-2} - \frac{1^{-2}}{-2}\right) =$$

$$= \lim_{b\to+\infty}\left(\frac{-1}{2b^2} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}$$

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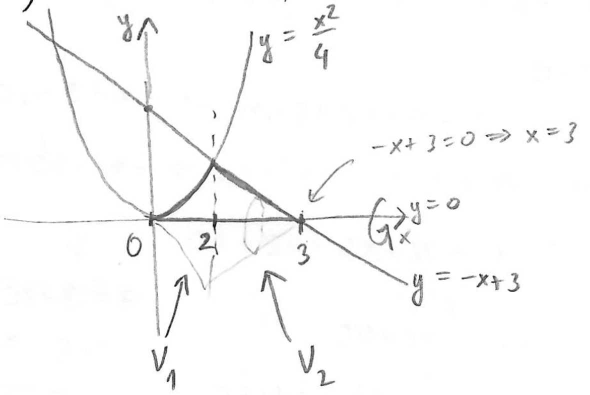
5. b) (15 bodova) $V_x$ za lik omeđen s $y = \frac{x^2}{4}$, $y = -x+3$, $y = 0$




točka presjeka: $\frac{x^2}{4} = -x+3$

$$x^2 + 4x - 12 = 0$$

$$x_{1,2} = \frac{-4 \pm \sqrt{16+4\cdot 12}}{2} = \frac{-4 \pm \sqrt{4(4+12)}}{2} =$$

$$= \frac{-4\pm\sqrt{4\cdot 16}}{2} = \frac{-4\pm 2\cdot 4}{2} = \frac{-4\pm 8}{2} = 2, -6$$

$V_1$ : $y = \frac{x^2}{4}$ (i $y=0$) na $[0,2]$ ; $V_2$ : $y=-x+3$ (i $y=0$) na $[2,3]$

$$\Rightarrow V = V_1 + V_2 = \pi\int_0^2 \left(\frac{x^2}{4}\right)^2 dx + \pi\int_2^3 (-x+3)^2\,dx = \pi\int_0^2 \frac{x^4}{16}\,dx + \pi\int_2^3 (x^2-6x+9)\,dx =$$

$$= \pi\,\frac{x^5}{5\cdot 16}\Big|_0^2 + \pi\left(\frac{x^3}{3} - 6\frac{x^2}{2} + 9x\right)\Big|_2^3 =$$

$$= \pi\left(\frac{2^5}{5\cdot 2^4} - 0\right) + \pi\left(\left(\frac{3^3}{3} - 3\cdot 3^2 + 9\cdot 3\right) - \left(\frac{2^3}{3} - 3\cdot 2^2 + 9\cdot 2\right)\right)$$

$$= \pi\cdot\frac{2}{5} + \pi\left(9 - \cancel{27} + \cancel{27} - \frac{8}{3} + 12 - 18\right) = \frac{2\pi}{5} + \pi\left(\frac{27}{3} - \frac{8}{3} - \frac{18}{3}\right) =$$

$$= \frac{2\pi}{5} + \frac{\pi}{3} = \frac{6\pi + 5\pi}{15} = \frac{11\pi}{15}$$