MATEMATIKA 2, 29. 6. 2022.

1. Riješite diferencijalne jednadžbe:
   (a) **(12 bodova)** $y'' - 4y' + 4y = \sin(2x)$,
   (b) **(8 bodova)** $y' + 2xy = x$.

2. (a) **(10 bodova)** Odredite i skicirajte prirodnu domenu funkcije
$$f(x,y) = \arcsin(x^2 + y^2 - 2x - 2y) + \ln(xy).$$
   (b) **(12 bodova)** Nađite tangencijalnu ravninu na plohu $z = 2x^2 + 2y^2$ koja je okomita na pravac
$$p \ldots \frac{x-1}{-2} = \frac{y+2}{4} = \frac{z}{1}.$$

3. **(18 bodova)** Izračunajte površinu lika omeđenog krivuljama $x^2 + y^2 = 2x$, $x^2 + y^2 = 4x$, $y = \sqrt{3}x$ i $y = -\sqrt{3}x$. Skicirajte lik.

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4. (a) **(11 bodova)** Odredite gradijent polja $\varphi(x,y,z) = \sin^2 x + yz$, a zatim i usmjerenu derivaciju polja φ u smjeru vektora $\vec{s} = 3\vec{i} + 4\vec{j}$ u točki $T\left(\frac{\pi}{4}, 0, 3\right)$.
   (b) **(14 bodova)** Izračunajte
$$\int_\Gamma x^3\, ds$$
   ako je Γ kružnica $x^2 + y^2 = 4y$. Skicirajte krivulju.

5. **(15 bodova)** Izračunajte tok vektorskog polja $\vec{a} = -x^3\vec{i} + 3x^2y\vec{j} + 4z\vec{k}$ kroz zatvorenu plohu koja se sastoji od dijelova ploha $z = x^2 + y^2$ i $z = 6 - \sqrt{x^2 + y^2}$. Skicirajte plohu.

Prvi dio čine prva tri zadatka. **Drugi dio** čine 4. i 5. zadatak.

Za polaganje ispita treba skupiti 50 bodova (od tog barem 30 bodova iz prvog dijela i barem 16 bodova iz drugog dijela).

$y'' - 4y' + 4y = \sin(2x)$ [illegible]

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \lambda_{1,2} = 2$

$y_H = C_1 e^{2x} + C_2 x e^{2x}$ 4

$f(x) = \sin(2x)$, $\pm 2i \neq \lambda_{1,2}$

$y_p = A\cos(2x) + B\sin(2x)$

$y_p' = -2A\sin(2x) + 2B\cos(2x)$

$y_p'' = -4A\cos(2x) - 4B\sin(2x)$ 3

$\cancel{-4A\cos(2x)} - \cancel{4B\sin(2x)} + 8A\sin(2x) - 8B\cos(2x)$
$\cancel{+4A\cos(2x)} + \cancel{4B\sin(2x)} = \sin(2x)$

$\Rightarrow 8A = 1 \Rightarrow A = \frac{1}{8}$ 3
$-8B = 0 \Rightarrow B = 0$ $\Big\}$ $y_p = \frac{1}{8}\cos(2x)$

$$\boxed{y = y_H + y_p = C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{8}\cos(2x)}$$ 2

b) (8b) $y' + 2xy = x$

$f(x) = 2x \quad g(x) = x$

$\int f(x)dx = 2\int x\,dx = x^2$ 1

$\int e^{\int f(x)dx} g(x)dx = \int e^{x^2} x\,dx = \left| \begin{matrix} t = x^2 \\ dt = 2x\,dx \end{matrix} \right|$

$= \frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \frac{1}{2}e^{x^2}$ 4

$y = e^{-x^2}\left(\frac{1}{2}e^{x^2} + C\right) = \boxed{\frac{1}{2} + \frac{C}{e^{x^2}}}$ 3

Ili

$\frac{dy}{dx} = x(1-2y)$

$\int \frac{dy}{1-2y} = \int x\,dx$ 2

$-\frac{1}{2}\ln(1-2y) = \frac{x^2}{2} + C_1 \,/\cdot 2\,/ e$ 4

$\frac{1}{1-2y} = Ce^{x^2}$

$1 - 2y = \frac{C}{e^{x^2}}$

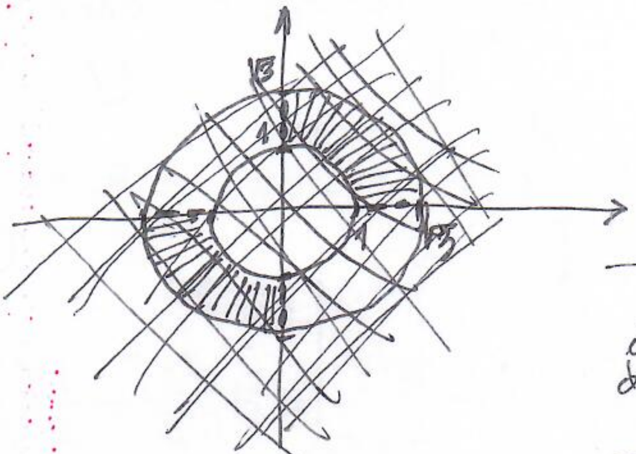
$$\boxed{y = \frac{1}{2} + \frac{C}{e^{x^2}}}$$ 2

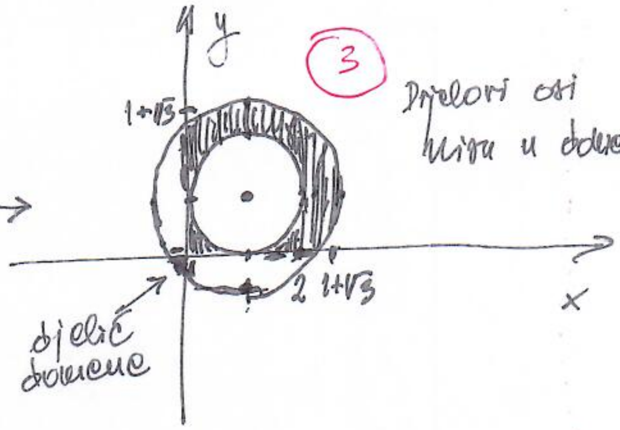
2. a) (10b) $f(x,y) = \arcsin(x^2+y^2-2x-2y) \cdot \ln(xy)$ ...

$$\underbrace{-1 \le \underbrace{x^2+y^2-2x-2y \le 1}}$$ (i) $\quad xy > 0 \rightarrow$ 1. kvadrant $\rightarrow$ 3. (2)

$(x-1)^2+(y-1)^2 \ge 1$ (1) $\quad (x-1)^2+(y-1)^2 \le 3$ (3)

$$\boxed{D_f = \{(x,y) \in \mathbb{R}^2 : 1 \le (x-1)^2+(y-1)^2 \le 3,\ xy > 0\}}$$ (2)





(3) Dijelovi osi nisu u domeni

b) (12b) $z = 2x^2+2y^2 = f(x,y)$, $\Pi_t \perp p \ldots \frac{x-1}{-2} = \frac{y+2}{4} = \frac{z}{1}$

$\Pi_t \ldots z - 2x_0^2 - 2y_0^2 = 4x_0(x-x_0) + 4y_0(y-y_0)$

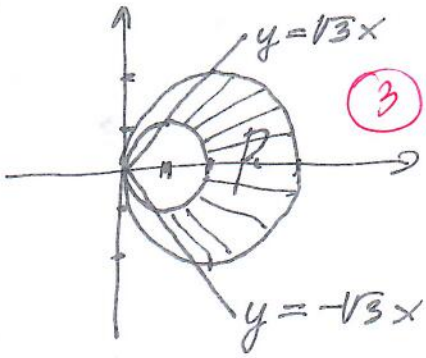
$\Pi_t \ldots -4x_0x - 4y_0y + z + 2x_0^2 + 2y_0^2 = 0$ (4)

$$\frac{-4x_0}{-2} = \frac{-4y_0}{4} = \frac{1}{1}$$ (4) $\Rightarrow \underline{x_0 = \frac{1}{2},\ y_0 = -1}$ (2)

$\Pi_t \ldots -2x + 4y + z + \frac{1}{2} + 2 = 0 \quad | \cdot 2$

$$\boxed{\Pi_t \ldots -4x + 8y + 2z + 5 = 0}$$ (2)

3. (18b)



3

$$P=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} r\,dr = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{r^2}{2}\Big|_{2\cos\varphi}^{4\cos\varphi} d\varphi$$ 6 1

$$=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(16\cos^2\varphi-4\cos^2\varphi)\,d\varphi$$ 1

$$=6\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^2\varphi\,d\varphi = 3\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(1+\cos(2\varphi))\,d\varphi = 3\left(\varphi+\frac{1}{2}\sin(2\varphi)\right)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$ 2 2

$$=3\left(\frac{\pi}{3}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}-\left(-\frac{\pi}{3}\right)-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right)=3\left(\frac{2\pi}{3}+\frac{\sqrt{3}}{2}\right)=\boxed{2\pi+\frac{3\sqrt{3}}{2}}$$ 2 1

4. a) (11b)

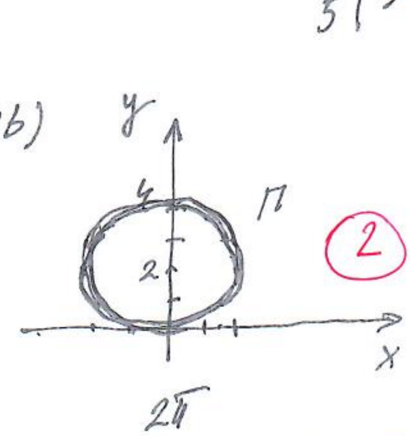
$$\operatorname{grad}\varphi=\boxed{2\sin\cos x\,\vec{i}+z\vec{j}+y\vec{k}=\sin(2x)\vec{i}+z\vec{j}+y\vec{k}}$$ 3

$$\vec{s}_0=\frac{1}{\sqrt{9+16}}(3\vec{i}+4\vec{j})=\frac{1}{5}(3\vec{i}+4\vec{j})$$ 2

$$\left(\frac{\partial\varphi}{\partial\vec{s}}\right)\Big|_{T(\frac{\pi}{4},0,3)}=\left(\operatorname{grad}\varphi\cdot\vec{s}_0\right)\Big|_{T(\frac{\pi}{4},0,3)}=\frac{1}{5}\left(3\sin(2x)+4z\right)\Big|_{T(\frac{\pi}{4},0,3)}$$ 4

$$=\frac{1}{5}(3\cdot1+4\cdot3)=\boxed{3}$$ 2

b) (14b)



2

$$\Gamma\ldots\ x(t)=2\cos t,\quad y(t)=2+2\sin t,\quad t\in[0,2\pi]$$ 3

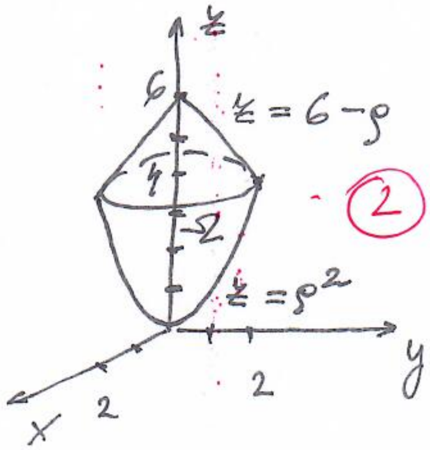
$$x'(t)=-2\sin t,\quad y'(t)=2\cos t$$ 1

$$\int_\Gamma x^3\,ds=\int_0^{2\pi}8\cos^3t\sqrt{4\sin^2t+4\cos^2t}\,dt=16\int_0^{2\pi}(1-\sin^2t)\cos t\,dt$$ 3 2

$$=\left|\begin{matrix}u=\sin t & t=0\Rightarrow u=0\\ du=\cos t\,dt & t=2\pi\Rightarrow u=0\end{matrix}\right|=\int_0^0=\boxed{0}$$ 2 1

3.(15b) Presjek ploha: $x^2+y^2 = 6-\sqrt{x^2+y^2}$ $\quad (t=\sqrt{x^2+y^2})$

$\Rightarrow t^2 = 6-t \Rightarrow t^2+t-6=0 \Rightarrow t = \frac{-1+\sqrt{1+24}}{2} = \underline{2}$ 2



2

$\operatorname{div}\vec{a} = -3x^2+3x^2+4 = 4$ 2

$$\iint_{\partial\Omega} \vec{a}\,d\vec{S} = \iiint \operatorname{div}\vec{a}\,dx\,dy\,dz$$

$$= 4\int_0^{2\pi} d\varphi \int_0^2 d\rho \int_{\rho^2}^{6-\rho} \rho\,dz \quad 4$$

$$= 4\int_0^{2\pi} d\varphi \int_0^2 (6\rho-\rho^2-\rho^3)\,d\rho \;\; 2 \;\; = 4\cdot 2\pi\cdot\left(3\rho^2-\frac{\rho^3}{3}-\frac{\rho^4}{4}\right)\Big|_0^2 \quad 2$$

$$= 8\pi\left(12-\frac{8}{3}-4\right) = 8\pi\cdot\frac{16}{3} = \boxed{\frac{128\pi}{3}} \quad 1$$