MATEMATIKA II 6.7.2022.

1. (20 bodova) Teška žica mase $m = 10$, duljine $l = 5$, te napetosti $p = 25$, nalazi se u homogenom sredstvu koeficijenta elastičnosti $q = 100$, te se deformira pod utjecajem vlastite težine. Odredite ravnotežni položaj žice ako su joj oba kraja pričvršćena.

2. a) (8 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \sqrt{4 - x^2 - y^2} - \ln(y - x).$$

   b) (12 bodova) Izračunajte $\frac{\partial f}{\partial x}(0,1) + \frac{\partial f}{\partial y}(0,1)$, ako je $f(x,y) = (x^2 + y^3)\cos\frac{x}{y}$.

3. (20 bodova) Izračunajte masu tijela određenog plohom $z = 2 - \sqrt{x^2 + y^2}$ i ravninama $z = 0$ i $y = 0$ ako je gustoća tijela $\rho(x,y,z) = y^3$. Skicirajte tijelo.

4. a) (6 bodova) Izračunajte div$\vec{a}$ u točki $T(0,1,1)$, ako je

$$\vec{a}(x,y,z) = e^{2x}\vec{i} + z\cos(xy)\vec{j} + \ln\frac{y}{z}\vec{k}.$$

   Je li polje $\vec{a}$ solenoidalno?

   b) (14 bodova) Izračunajte

$$\int_{\vec{\Gamma}} \vec{a}\, \mathrm{d}\vec{r},$$

   ako je $\vec{a} = yz\vec{j}$, a krivulja $\vec{\Gamma}$ je zadana parametrizacijom $x(t) = t$, $y(t) = e^t$ i $z(t) = \cos t$ za $t \in [0, \frac{\pi}{2}]$.

5. (20 bodova) Izračunajte $\int\int_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$ ako je $\vec{a} = 2y\vec{i} + z\vec{j} - xy\vec{k}$, a $\vec{\Sigma}$ je dio ravnine $x + 2y + 2z = 2$ u 1. oktantu orijentirane normalom koja zatvara šiljasti kut s vektorom $\vec{k}$. Skicirajte plohu.

MAT2, 6.7.2022.

1. (20b) $f(x) = -\frac{mg}{l} = -\frac{10 \cdot 10}{5} = -20$

$-25u'' + 100u = -20 \quad | : (-25)$

$u'' - 4u = \frac{4}{5} \quad \Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2 \Rightarrow u_H = C_1 e^{2x} + C_2 e^{-2x}$

$f(x) = \frac{4}{5} = P_0(x) e^{0x} \quad \Rightarrow \quad u_p = Q_0(x) e^{0x} = A, \quad u_p' = u_p'' = 0$

$0 - 4A = \frac{4}{5} \quad \Rightarrow A = -\frac{1}{5} \quad \Rightarrow \quad u_p = -\frac{1}{5}$

$u(x) = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{5}$

$u(0) = u(5) = 0$

$$\left.\begin{aligned} C_1 + C_2 &= \frac{1}{5} \\ C_1 e^{10} + C_2 e^{-10} &= \frac{1}{5} \end{aligned}\right\} \Rightarrow \quad C_1 = \frac{\begin{vmatrix} \frac{1}{5} & 1 \\ \frac{1}{5} & e^{-10} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ e^{10} & e^{-10} \end{vmatrix}} = \frac{\frac{1}{5}(e^{-10} - 1)}{e^{-10} - e^{10}}$$

$$C_2 = \frac{\begin{vmatrix} 1 & \frac{1}{5} \\ e^{10} & \frac{1}{5} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ e^{10} & e^{-10} \end{vmatrix}} = \frac{\frac{1}{5}(1 - e^{10})}{e^{-10} - e^{10}}$$

$$C_1 = \frac{1}{5} \cdot \frac{\frac{1 - e^{10}}{\cancel{e^{10}}}}{\frac{1 - e^{20}}{\cancel{e^{10}}}} = \frac{1}{5} \frac{\cancel{1 - e^{10}}}{\cancel{(1 - e^{10})}(1 + e^{10})} = \frac{1}{5} \frac{1}{1 + e^{10}}$$

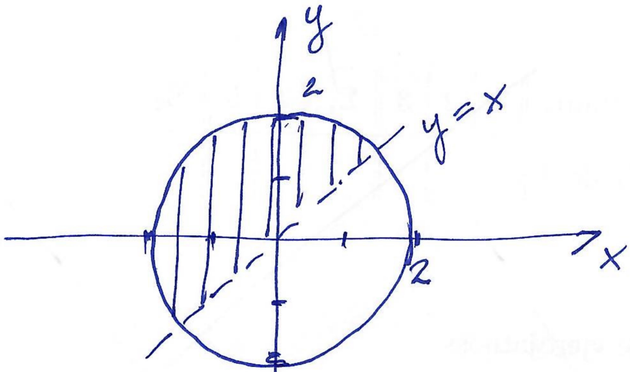
$$C_2 = \frac{1}{5} \frac{1 - e^{10}}{\frac{1 - e^{20}}{e^{10}}} = \frac{1}{5} \frac{e^{10}}{1 + e^{10}}$$

$$\boxed{u(x) = \frac{1}{5}\left(\frac{e^{2x}}{1 + e^{10}} + \frac{e^{10 - 2x}}{1 + e^{10}} - 1 \right)}$$

2. a) (8b) 1° $4-x^2-y^2 \geq 0$ (1) 2° $y-x>0$

$\Rightarrow x^2+y^2 \leq 4$ $\Rightarrow y>x$

$$\boxed{D_f = \{(x,y)\in \mathbb{R}^2 : x^2+y^2 \leq 4,\ y>x\}}$$



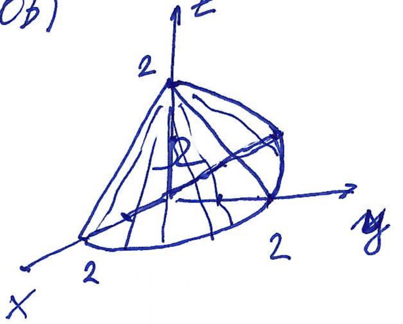
b) (12b)

$$\frac{\partial f}{\partial x} = 2x\cos\frac{x}{y} + (x^2+y^3)\cdot\left(-\sin\frac{x}{y}\right)\cdot\frac{1}{y}$$

$$\frac{\partial f}{\partial y} = 3y^2\cos\frac{x}{y} + (x^2+y^3)\cdot\left(-\sin\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right)$$

$$\frac{\partial f}{\partial x}(0,1)+\frac{\partial f}{\partial y}(0,1) = 0+0+3+0 = \boxed{3}$$

3. (20b)



$$\Omega \,..\, 0 \le \varphi \le \pi$$
$$0 \le \rho \le 2$$
$$0 \le z \le 2-\rho$$

$$m = \iiint_{\Omega} y^3\,dx\,dy\,dz = \int_0^{\pi} d\varphi \int_0^{2} d\rho \int_0^{2-\rho} \rho^3 \sin^3\varphi \cdot \rho\, dz$$

$$= \int_0^{\pi} \sin^3\varphi\, d\varphi \int_0^2 (2\rho^4-\rho^5)\,d\rho = \int_0^{\pi}(1-\cos^2\varphi)\sin\varphi\, d\varphi \cdot \left(\frac{2}{5}\rho^5 - \frac{\rho^6}{6}\right)\Big|_0^2$$

$$= \left| \begin{matrix} t=\cos\varphi & \varphi=0 \Rightarrow t=1 \\ dt=-\sin\varphi\, d\varphi & \varphi=\pi \Rightarrow t=-1 \end{matrix} \right| = \int_{-1}^{1}(1-t^2)\,dt \cdot \left(\frac{64}{5}-\frac{64}{6}\right)$$

$$= 2\left(t-\frac{t^3}{3}\right)\Big|_0^1 \cdot \frac{384-320}{30} = \frac{4}{3}\cdot\frac{64}{30} = \boxed{\frac{128}{45}}$$

4. a) (6b)

$$\operatorname{div}\vec{a} = 2e^{2x} + z\cdot x\cdot(-\sin(xy)) + \frac{z}{y}\cdot\left(-\frac{y}{z^2}\right)$$

$$= 2e^{2x} - xz\sin(xy) - \frac{1}{z}$$

minus

$$\operatorname{div}\vec{a}\Big|_T = 2-0-1 = \boxed{1}$$; NE, NIJE!

b) (14b) $\Gamma \,..\, x(t)=t$, $y(t)=e^t$, $z(t)=\cos t$, $t\in[0,\frac{\pi}{2}]$ $\quad x'(t)=1$, $y'(t)=e^t$, $z'(t)=-\sin t$

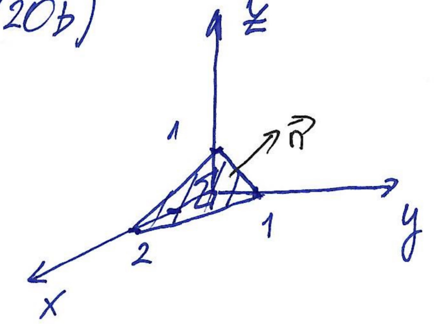
$$\int_{\Gamma}\vec{a}\,d\vec{r} = \int_0^{\frac{\pi}{2}} e^t\cos t\cdot e^t\,dt = \int_0^{\frac{\pi}{2}} e^{2t}\cos t\,dt = \left| \begin{matrix} u=e^{2t} & dv=\cos t\,dt \\ du=2e^{2t}dt & v=\sin t \end{matrix}\right|$$

$$= e^{2t}\sin t\Big|_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} e^{2t}\sin t\,dt = \left| \begin{matrix} u=e^{2t} & dv=\sin t\,dt \\ du=2e^{2t}dt & v=-\cos t \end{matrix}\right|$$

$$= e^{\pi} + \underbrace{2e^{2t}\cos t\Big|_0^{\frac{\pi}{2}}}_{=-2} - 4\int_0^{\frac{\pi}{2}} e^{2t}\cos t\,dt$$

$$\Rightarrow \int_{\Gamma}\vec{a}\,d\vec{r} = \int_0^{\frac{\pi}{2}} e^{2t}\cos t\,dt = \boxed{\frac{1}{5}(e^{\pi}-2)}$$

5. (20b)

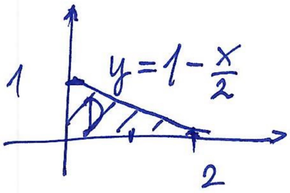


$$\vec{\Sigma}_1 = z = 1 - \frac{x}{2} - y = f(x,y)$$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}, \quad \frac{\partial f}{\partial y} = -1$$

$$\underbrace{\vec{n} = \frac{1}{2}\vec{i} + \vec{j} + \vec{k}}$$

$$\underbrace{\vec{a}\cdot\vec{n}} = y + z - xy = y + 1 - \frac{x}{2} - y - xy = \underbrace{1 - \frac{x}{2} - xy}$$



D.. $0 \le x \le 2$

$0 \le y \le 1 - \frac{x}{2}$

$$\iint_{\vec{\Sigma}} \vec{a}\, d\vec{S} = \int_0^2 dx \int_0^{1-\frac{x}{2}} \left(1 - \frac{x}{2} - xy\right) dy = \int_0^2 \left(\left(1-\frac{x}{2}\right)y - \frac{x}{2}y^2\right)\Big|_0^{1-\frac{x}{2}} dx$$

$$= \int_0^2 \left(\left(1-\frac{x}{2}\right)^2 - \frac{x}{2}\left(1-\frac{x}{2}\right)^2\right) dx = \int_0^2 \left(1-\frac{x}{2}\right)^2\left(1-\frac{x}{2}\right) dx$$

$$= \int_0^2 \left(1-\frac{x}{2}\right)^3 dx = \int_0^2 \left(1 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{x^3}{8}\right) dx$$

$$= \left(x - \frac{3}{4}x^2 + \frac{1}{4}x^3 - \frac{1}{32}x^4\right)\Big|_0^2 = 2 - 3 + 2 - \frac{1}{2} = \boxed{\frac{1}{2}}$$