MATEMATIKA 3 Kolokvij 12.12.2022. A

1. (20 bodova) Odredite ravnotežni položaj teške homogene žice duljine $l = 3$ i mase $m = 2$, napete na desnom kraju utegom mase $M = 4$, ako je lijevi kraj slobodan.

2. (20 bodova) Riješite problem provođenja topline izoliranog homogenog štapa duljine 2, konstantnog toplinskog kapaciteta $\gamma = 1$ i koeficijenta provođenja $\delta = 4$, uz rubne uvjete

$$u(0,t) = 1,$$
$$u(2,t) = 3,$$

i početni uvjet

$$u(x,0) = 1 + x + \sin(3\pi x).$$

3. (20 bodova) Riješite vanjski Dirichletov problem

$$\left.\begin{array}{ll} \Delta u = 0 & \text{za } 2 < r < \infty \\ u\,|_{r=2} = u(2,\varphi) = 3 + \sin(2\varphi) + \cos(\varphi), & 0 \le \varphi < 2\pi. \end{array}\right\}$$

Tablica derivacija

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
C	0	ctg	$-\frac{1}{\sin^2 x}$
x^α	$\alpha x^{\alpha-1}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\mathrm{arctg}\, x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\mathrm{arcctg}\, x$	$-\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\mathrm{sh}\, x$	$\mathrm{ch}\, x$
$\sin x$	$\cos x$	$\mathrm{ch}\, x$	$\mathrm{sh}\, x$
$\cos x$	$-\sin x$	$\mathrm{th}\, x$	$\frac{1}{\mathrm{ch}^2 x}$
$\mathrm{tg}\, x$	$\frac{1}{\cos^2 x}$	$\mathrm{cth}\, x$	$-\frac{1}{\mathrm{sh}^2 x}$

Tablica integrala

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	$x + C$	$\cos x$	$\sin x + C$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1$	$\frac{1}{\sin^2 x}$	$-\mathrm{ctg}\, x + C$
e^x	$e^x + C$	$\frac{1}{\cos^2 x}$	$\mathrm{tg}\, x + C$
a^x	$\frac{a^x}{\ln a} + C$	$\frac{1}{1+x^2}$	$\mathrm{arctg}\, x + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln\left\lvert x + \sqrt{x^2 \pm 1}\right\rvert + C$

Ortogonalnost trigonometrijskih funkcija: za svaki $l > 0$ vrijedi:

$$\int_{-l}^{l} \sin(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx = \begin{cases} l, & m = n \neq 0 \text{ i } m, n \in \mathbb{Z}, \\ 0, & m = n = 0, \\ 0, & m \neq n \text{ i } m, n \in \mathbb{Z}, \end{cases}$$

$$\int_{-l}^{l} \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l}) dx = \begin{cases} l, & m = n \neq 0 \text{ i } m, n \in \mathbb{Z}, \\ 2l, & m = n = 0, \\ 0, & m \neq n \text{ i } m, n \in \mathbb{Z}, \end{cases}$$

$$\int_{-l}^{l} \cos(\frac{n\pi x}{l}) \sin(\frac{m\pi x}{l}) dx = 0, \ \forall m, n \in \mathbb{Z}.$$

Trigonometrijski identiteti

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ \sin x \cos y &= \frac{1}{2}[\sin(x-y) + \sin(x+y)] \\ \sin x \sin y &= \frac{1}{2}[\cos(x-y) - \cos(x+y)] \\ \cos x \cos y &= \frac{1}{2}[\cos(x-y) + \cos(x+y)] \end{aligned}$$

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(1.) $\ell = 3, \ m = 2$

na desnom kraju napeta utegom mase $M = 4 \ \rightarrow \ u(3) = 0$

lijevi kraj slobodan $\rightarrow \ u'(0) = 0$

$$-\left(p(x)\,u'(x)\right)' = f(x)$$

$$p(x) = Mg = 4g$$

$$f(x) = -\frac{mg}{\ell} = -\frac{2 \cdot g}{3}$$

$$\Rightarrow \ -\left(4\cancel{g}\,u'(x)\right)' = -\frac{2\cancel{g}}{3}$$

$$-4\,u''(x) = -\frac{2}{3} \quad /:(-4)$$

$$u''(x) = \frac{1}{6} \quad / \int dx$$

$$\Rightarrow \ u'(x) = \frac{1}{6}x + C_1 \quad / \int dx$$

$$\Rightarrow \ u(x) = \frac{1}{6}\frac{x^2}{2} + C_1 x + C_2$$

$$0 = u(3) = \frac{9}{12} + 3C_1 + C_2 \ \Rightarrow \ C_2 = -\frac{3}{4}$$

$$0 = u'(0) = C_1$$

$$u(x) = \frac{x^2}{12} - \frac{3}{4}$$

(2.) $\ell = 2, \ \gamma = 1, \ \delta = 4$

$u(0,t) = 1$

$u(2,t) = 3$

$u(x,0) = 1 + x + \sin(3\pi x)$

jednadžba $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

$$c^2 = \frac{\delta}{\gamma} = \frac{4}{1} = 4 \ \Rightarrow \ c = 2$$

nehomogeni rubni uvjeti $\Rightarrow \ u(x,t) = v(x,t) + w(x)$

(1) $w''(x) = 0 \quad / \int dx \ \Rightarrow \ w'(x) = C_1 \quad / \int dx$

$w(0) = 1$

$w(2) = 3$

$$\Rightarrow w(x) = C_1 x + C_2$$

$$1 = w(0) = C_2$$

$$3 = w(2) = 2C_1 + 1 \ \Rightarrow \ C_1 = 1$$

$$w(x) = x + 1$$

$$v(x,t) = \sum_{n=1}^{\infty} E_n e^{-\left(\frac{n\pi c}{\ell}\right)^2 t} \sin\left(\frac{n\pi}{\ell}x\right)$$

$$E_n = \frac{2}{\ell}\int_0^{\ell} v(x,0)\,\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$v(x,0) = u(x,0) - w(x) = \sin(3\pi x)$$

$$\Rightarrow \ E_n = \frac{2}{2}\int_0^2 \sin(3\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = \begin{cases} 1, & n = 6, \\ 0, & n \neq 6 \end{cases}$$

$$\Rightarrow \ v(x,t) = e^{-(6\pi)^2 t} \sin(3\pi x)$$

$$\Rightarrow u(x,t) = e^{-(6\pi)^2 t} \sin(3\pi x) + x + 1$$

3. $$u(r,\varphi) = \sum_{n=0}^{\infty} \frac{1}{r^n}\left(E_{1n}\cos(n\varphi) + E_{2n}\sin(n\varphi)\right)$$

$\alpha(\varphi) = 3 + \sin(2\varphi) + \cos\varphi$

$R = 2$

$$E_{10} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(3 + \sin(2\varphi) + \cos\varphi\right) d\varphi = \frac{1}{2\pi}\left(3\varphi - \frac{\cos(2\varphi)}{2} + \sin\varphi\right)\Bigg|_{-\pi}^{\pi} =$$

$$= \frac{1}{2\pi}\left(3(\pi - (-\pi)) - \underbrace{\frac{\cos(2\pi) - \cos(-2\pi)}{2}}_{=0} + \underset{0}{\sin\pi} - \underset{0}{\sin(-\pi)}\right) = 3$$

$$E_{1n} = \frac{2^n}{\pi}\int_{-\pi}^{\pi}\left(3 + \sin(2\varphi) + \cos\varphi\right)\cos(n\varphi)\, d\varphi = \frac{2^n}{\pi}\, \frac{3\sin(n\varphi)}{n}\Bigg|_{-\pi}^{\pi} + \begin{cases} 2^n, & n=1 \\ 0, & n\neq 1 \end{cases}$$

$$= \frac{2^n}{\pi}\,\underbrace{\frac{3\sin(n\pi) - 3\sin(-n\pi)}{n}}_{=0} + \begin{cases} 2, & n=1, \\ 0, & n\neq 1 \end{cases} = \begin{cases} 2, & n=1 \\ 0, & n\neq 1 \end{cases}$$

$$E_{2n} = \frac{2^n}{\pi}\int_{-\pi}^{\pi}\left(3 + \sin(2\varphi) + \cos\varphi\right)\sin(n\varphi)\, d\varphi = \frac{2^n}{\pi}\, \frac{-3\cos(n\varphi)}{n}\Bigg|_{-\pi}^{\pi} + \begin{cases} 2^n, & n=2 \\ 0, & n\neq 2 \end{cases} =$$

$$= \frac{2^n}{\pi}\,\underbrace{\frac{-3\cos(n\pi) + 3\cos(-n\pi)}{n}}_{=0} + \begin{cases} 4, & n=2 \\ 0, & n\neq 2 \end{cases} = \begin{cases} 4, & n=2 \\ 0, & n\neq 2 \end{cases}$$

$$\Rightarrow u(r,\varphi) = 3 + \frac{1}{r}\cdot 2\cdot\cos(\varphi) + \frac{1}{r^2}\cdot 4\cdot\sin(2\varphi) = 3 + \frac{2\cos\varphi}{r} + \frac{4\sin(2\varphi)}{r^2}$$