1. Riješite diferencijalne jednadžbe:

   (a) (8 bodova) $2(1-x^2)y' = xy$,

   (b) (12 bodova) $y'' - 2y' = -4x + 2$.

2. (a) (10 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \sqrt{x^2 - 2x + y^2} + \ln(1 - y^2).$$

   (b) (10 bodova) Odredite ekstreme funkcije $f(x,y) = x^4 + y^2 - 2xy + 1$.

3. (20 bodova) Izračunajte volumen tijela koje je omeđeno odozdo sa $z = \sqrt{x^2+y^2}$, a odozgo sa $z = \sqrt{2 - x^2 - y^2}$. Skicirajte tijelo.

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4. (a) (10 bodova)Zadano je polje

$$\vec{a} = (2x + x^2 z)e^{xz}\vec{i} + 2y\vec{j} + x^3 e^{xz}\vec{k}.$$

   Je li dano polje potencijalno? Je li solenoidalno?

   (b) (15 bodova) Neka je $\vec{\Gamma}$ pozitivno orijentirana krivulja sastavljena od dijela parabole $y = 2 - 2x^2$, $x \in [-1,1]$ i polukružnice $y = -\sqrt{1-x^2}$, $x \in [-1,1]$. Skicirajte krivulju i označite joj orijentaciju. Izračunajte

$$\int_{\vec{\Gamma}} \frac{\operatorname{arctg} y}{x}\, dx + (x^2 y + \frac{\ln x}{1+y^2})\, dy.$$

   (Uputa: Koristite Greenov teorem.)

5. (15 bodova) Stožac Σ je zadan sa $z = 1 - \sqrt{(x-1)^2 + y^2}$, $z \geq 0$. Izračunajte

$$\iint_{\Sigma} (x^2 + y^2)\, dS.$$

Prvi dio čine prva tri zadatka. **Drugi dio** čine 4. i 5. zadatak.

Za polaganje ispita treba skupiti 50 bodova (od tog barem 30 bodova iz prvog dijela i barem 16 bodova iz drugog dijela).

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
C	0	$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$
x^α	$\alpha x^{\alpha-1}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{sh} x$	$\operatorname{ch} x$
$\sin x$	$\cos x$	$\operatorname{ch} x$	$\operatorname{sh} x$
$\cos x$	$-\sin x$	$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	$x + C$	$\cos x$	$\sin x + C$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + C$
e^x	$e^x + C$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + C$
a^x	$\frac{a^x}{\ln a} + C$	$\frac{1}{1+x^2}$	$\operatorname{arctg} x + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln\left\lvert x + \sqrt{x^2 \pm 1}\right\rvert + C$

MATEMATIKA 2 – ROK 6. 2. 2023.

1.a) $2(1-x^2)y' = xy$

(8 bodova)

Separiramo varijable $\rightarrow$ $\frac{2y'}{y} = \frac{x}{1-x^2}$, $y' = \frac{dy}{dx}$

$\rightarrow \frac{2dy}{y} = \frac{x}{1-x^2}dx \quad / \int$

$2\ln|y| = -\frac{1}{2}\ln|1-x^2| + C \quad / e^{\cdot}$

$y^2 = \frac{A}{\sqrt{1-x^2}}$

$\Leftrightarrow \quad (1-x^2)y^4 = A \quad , \quad A > 0$

Integral $\int \frac{x}{1-x^2}dx$ se može izračunati preko parcijalnih razlomaka ili supstitucijom

$\int \frac{x}{1-x^2}dx = \begin{vmatrix} u = 1-x^2 \\ du = -2x\,dx \end{vmatrix} = \int -\frac{1}{2}\frac{du}{u} = -\frac{1}{2}\ln|u| + c.$

$= -\frac{1}{2}\ln|1-x^2| + c$

1.b) $y'' - 2y' = -4x + 2$

(12 bodova)

$\rightarrow$ KAR. JEDNADŽBA : $\lambda^2 - 2\lambda = 0 \Rightarrow \lambda_1 = 0, \ \lambda_2 = 2 \Rightarrow$

HOMOGENO RJEŠENJE: $y_H = C_1 + C_2 e^{2x}$

$\rightarrow$ PARTIKULARNO RJEŠENJE: Tražimo u obliku $y_P = x\cdot(Ax+B)\cdot e^{0\cdot x} = Ax^2 + Bx$

↳ Moramo pomnožiti s x, jer $e^{0x} = 1$ tj. konstante, automatski zadovoljavaju homogeni dio.

Uvstimo y_P u jednadžbu:

$y_P'' - 2y_P' = -4x + 2$

$\Leftrightarrow \quad 2A - 4Ax - 2B = -4x + 2$

$\Rightarrow \quad -4A = -4, \quad 2A - 2B = 2 \quad \Rightarrow \quad \boxed{A = 1, \ B = 0}$

$\Rightarrow \quad y = y_H + y_P = C_1 + C_2 e^{2x} + x^2$

2. a) (10b) $f(x,y) = \sqrt{x^2-2x+y^2} + \ln(1-y^2)$

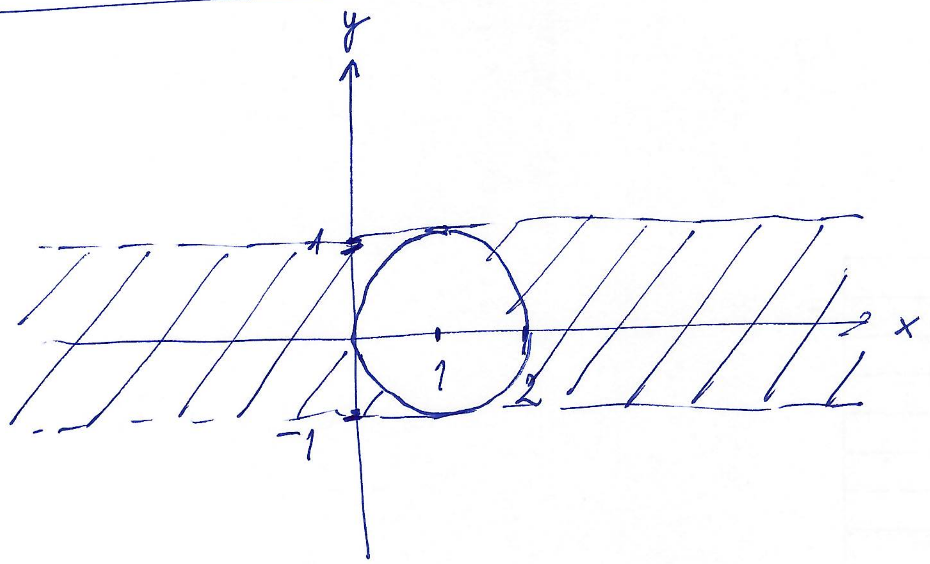
$x^2+y^2-2x \geq 0$ i $1-y^2>0$

$\boxed{y^2<1}$

$x^2-2x+1+y^2 \geq 1$

$(x-1)^2+y^2 \geq 1$

$$\boxed{D_f = \{(x,y)\in\mathbb{R}^2 : (x-1)^2+y^2 \geq 1,\ y^2<1\}}$$



b) (10b) $f(x,y) = x^4+y^2-2xy+1$

$\frac{\partial f}{\partial x} = 4x^3-2y = 0 \Rightarrow 4x^3-2x=0 \Rightarrow 2x(x^2-1)=0$

$x_1 = 0$
$x_2 = -1$
$x_3 = 1$

$\frac{\partial f}{\partial y} = 2y-2x=0 \Rightarrow x=y$

$T_1(0,0)$, $T_2(-1,-1)$, $T_3(1,1)$ stacionarne točke

$\frac{\partial^2 f}{\partial x^2} = 12x^2$, $\frac{\partial^2 f}{\partial x\partial y} = -2$, $\frac{\partial^2 f}{\partial y^2} = 2$

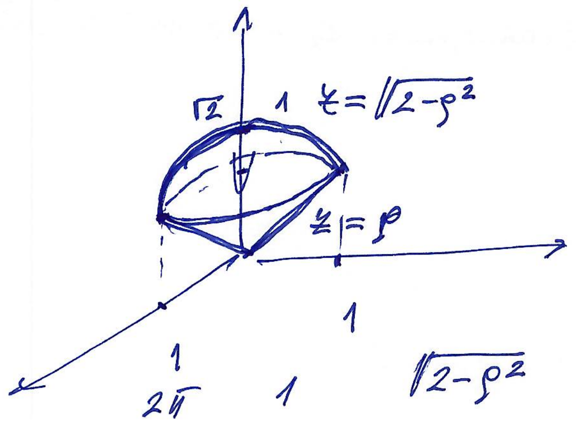
$T_1(0,0) \Rightarrow AC-B^2 = 0-4=-4<0 \Rightarrow T_1$ je sedlasta točka

$T_2(-1,-1) \Rightarrow AC-B^2 = 12\cdot 2-4 = 20>0$, $A=12>0$
$\Rightarrow T_2$ je lok. minimum

$T_3(1,1) \Rightarrow AC-B^2 = 12\cdot 2-4=20>0$, $A=12>0$
$\Rightarrow T_3$ je lok minimum

3. (20b) Presjek ploha: $\sqrt{x^2+y^2} = \sqrt{2-x^2-y^2} \,\Big|^2$, $r^2 = x^2+y^2$

$r^2 = 2 - r^2 \Rightarrow 2r^2 \cancel{+r-2=0} = 2 \Rightarrow \cancel{r_{1,2} = \frac{-1\pm\sqrt{1+8}}{2}} \Rightarrow \boxed{r=1}$



$V\,..\quad 0 \le \varphi \le 2\pi$

$0 \le \rho \le 1$

$\rho \le z \le \sqrt{2-\rho^2}$

$$V = \int_0^{2\pi} d\varphi \int_0^1 d\rho \int_\rho^{\sqrt{2-\rho^2}} \rho\, dz = 2\pi \int_0^1 \rho\left(\sqrt{2-\rho^2} - \rho\right) d\rho$$

$$= 2\pi\left(\int_0^1 \rho\sqrt{2-\rho^2}\, d\rho - \frac{\rho^3}{3}\Big|_0^1\right) = \left| \begin{matrix} t = 2-\rho^2 & \rho=0 \Rightarrow t=2 \\ dt = -2\rho\, d\rho & \rho=1 \Rightarrow t=1 \end{matrix} \right|$$

$$= 2\pi\left(\frac{1}{2}\int_1^2 \sqrt{t}\, dt - \frac{1}{3}\right) = 2\pi\left(\frac{1}{2}\cdot\frac{2}{3} t\sqrt{t}\,\Big|_1^2 - \frac{1}{3}\right)$$

$$= \boxed{\frac{2\pi}{3}\left(2\sqrt{2} - 1\right)}$$

4.a) (10b)

$$\operatorname{rot}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+x^2z)e^{xz} & 2y & x^3e^{xz} \end{vmatrix}$$

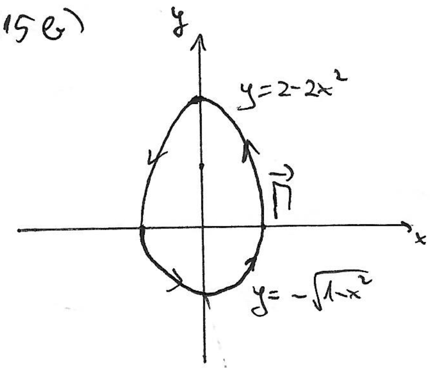
$$= (0-0)\,\vec{i} + \left(\left(x^2e^{xz} + (2x+x^2z)xe^{xz}\right) - 3x^2e^{xz} - x^3ze^{xz}\right)\vec{j} + (0-0)\,\vec{k}$$

$$= \vec{0} \Rightarrow \vec{a} \text{ je potencijalno polje}$$

$$\operatorname{div}\vec{a} = (2+2xz)e^{xz} + (2x+x^2z)ze^{xz} + 2 + x^3\cdot xe^{xz}$$

$$= e^{xz}(2+4xz+x^2z^2+x^4) + 2 \neq 0 \Rightarrow \text{nije solenoidalno}$$

4.b) (15b)



$$M(x,y) = \frac{\operatorname{arctg} y}{x}$$

$$N(x,y) = x^2y + \frac{\ln x}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x(1+y^2)}$$

$$\frac{\partial N}{\partial x} = 2xy + \frac{1}{x(1+y^2)}$$

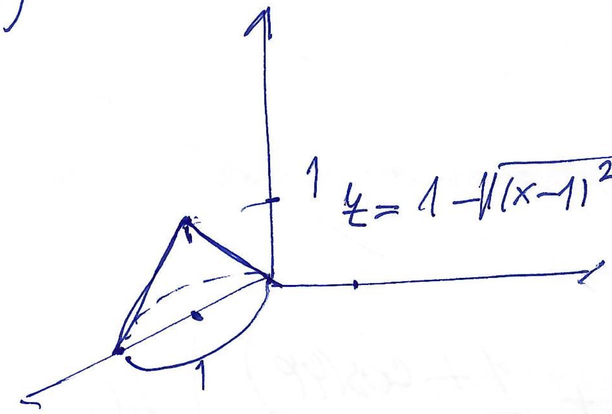
$$\Rightarrow \oint_{\vec{\Gamma}} M\,dx + N\,dy \overset{\text{GREEN}}{=} \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_D 2xy\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{2-2x^2} 2xy\,dy\,dx$$

$$= \int_{-1}^{1} x\cdot y^2\Big|_{-\sqrt{1-x^2}}^{2-2x^2} dx = \int_{-1}^{1} x\left((2-2x^2)^2 - (1-x^2)\right)dx$$

$$= \int_{-1}^{1} x\left((4-8x^2+4x^4) - 1 + x^2\right)dx = \int_{-1}^{1}(3x - 7x^3 + 4x^5)\,dx = 0.$$

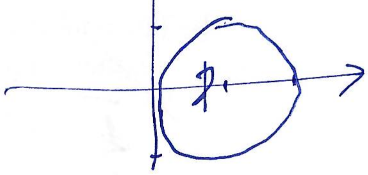
5. (15b)



$z = 1 - \sqrt{(x-1)^2+y^2} = f(x,y)$

$D: -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$0 \le r \le 2\cos\varphi$



$$\frac{\partial f}{\partial x} = -\frac{2(x-1)}{2\sqrt{(x-1)^2+y^2}} \qquad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{(x-1)^2+y^2}}$$

$$\iint_{\Sigma} (x^2+y^2)\,dS = \iint_D \cancel{xy}\,(x^2+y^2)\sqrt{1+\frac{(x-1)^2+y^2}{(x-1)^2+y^2}}\,dx\,dy$$

$$= \sqrt{2}\iint_D \cancel{xy}\,(x^2+y^2)\,dx\,dy = \cancel{\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r\cos\varphi \cdot r\sin\varphi \cdot r\,dr}$$

$$\cancel{= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\varphi\cos\varphi \,\frac{r^4}{4}\Big|_0^{2\cos\varphi} = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^5\varphi\sin\varphi\,d\varphi}$$

$$\cancel{= \left| \begin{matrix} t = \cos\varphi & \varphi = -\frac{\pi}{2} \Rightarrow t = 0 \\ dt = -\sin\varphi\,d\varphi & \varphi = \frac{\pi}{2} \Rightarrow t = 0 \end{matrix} \right|}$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \cdot r\,dr = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_0^{2\cos\varphi}$$

$$= 4\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi\,d\varphi = 4\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1+\cos(2\varphi)\right)^2 d\varphi$$

$\Rightarrow$