MATEMATIKA 2, 13.2.2023.

1. Riješite sljedeće diferencijalne jednadžbe:

   a) (8 bodova) $\sqrt{4-x^2}y' = xy^2$,

   b) (12 bodova) $y'' + y' - 2y = \sin x$.

2. a) (12 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \arcsin(x^2+y^2-3) - 2\ln(1-x).$$

   b) (12 bodova) Odredite tangencijalnu ravninu na graf funkcije $f(x,y) = x^2 + 3y^2 - 1$ koja je paralelna ravnini $\pi \ldots 2x - 12y + z - 3 = 0$.

3. (16 bodova) Izračunajte površinu lika koji se nalazi izvan krivulje $r = 2$, a unutar krivulje $r = 2(1+\cos\varphi)$. Skicirajte lik.

4. a) (8 bodova) Izračunajte usmjerenu derivaciju polja

$$f(x,y,z) = xy\sin\frac{x}{z}$$

   u smjeru vektora $\vec{s} = \vec{i} + 2\vec{j} - 2\vec{k}$ u točki $T(\pi, 1, 1)$.

   b) (12 bodova) Izračunajte $\int_{\vec{\Gamma}} y\,dx + z\,dy + x\,dz$, ako je $\vec{\Gamma}$ krivulja određena kao presječnica plohe $x^2 + y^2 = 4$ i ravnine $z = 2y$ u prvom oktantu. Skicirajte krivulju.

5. (20 bodova) Pomoću teorema o divergenciji izračunajte tok vektorskog polja $\vec{a} = x^2y\vec{i} + xz^2\vec{j} + xy^2\vec{k}$ kroz zatvorenu plohu koja je sastavljena od dijela sfere $x^2 + y^2 + z^2 = 1$ u prvom oktantu i dijela koordinatnih ravnina $x = 0$, $y = 0$ i $z = 0$. Ploha je orijentirana u smjeru vanjskih normala. Skicirajte plohu.

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1.a) $\sqrt{4-x^2}\, y' = x y^2$

$\Leftrightarrow \quad \frac{y'}{y^2} = \frac{x}{\sqrt{4-x^2}}$

$\Leftrightarrow \quad \frac{dy}{y^2} = \frac{x\,dx}{\sqrt{4-x^2}} \quad / \int$

$-\frac{1}{y} = -\sqrt{4-x^2} + c$

$y = \frac{1}{-c + \sqrt{4-x^2}}$

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = \left| \begin{array}{l} u = 4-x^2 \\ du = -2x\,dx \end{array} \right|$$

$$= \int \frac{-\frac{1}{2}du}{\sqrt{u}}$$

$$= -\frac{1}{2} \cdot \int u^{-\frac{1}{2}} du$$

$$= -\frac{1}{2} \cdot \frac{1}{\frac{1}{2}} u^{\frac{1}{2}} + c$$

$$= -\sqrt{u} + c$$

$$= -\sqrt{4-x^2} + c$$

1.b) $y'' + y' - 2y = \sin x$

HOMOGENI DIO: KAR. JEDNADŽBA $\lambda^2 + \lambda - 2 = 0$

$(\lambda+2)(\lambda-1) = 0$

$\lambda_1 = 1$

$\lambda_2 = -2$

$\Rightarrow \quad y_H = C_1 \cdot e^{x} + C_2 \cdot e^{-2x}$

PARTIKULARNO:

$y_P = A\sin x + B\cos x$

$y_P' = A\cos x - B\sin x$

$y_P'' = -A\sin x - B\cos x$

$\rightarrow$ Uvrstimo u jednadžbu: $(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2\cdot(A\sin x + B\cos x) = \sin x$

$\Rightarrow \left| \begin{array}{l} -3A - B = 1 \\ A - 3B = 0 \Rightarrow A = 3B \end{array} \right. \rightarrow -3\cdot 3B - B = 1 \Rightarrow \boxed{B = -\frac{1}{10}} \Downarrow \boxed{A = -\frac{3}{10}}$

Konačno rj: $y = y_H + y_P = C_1 \cdot e^{x} + C_2 e^{-2x} - \frac{1}{10}\cos x - \frac{3}{10}\sin x$

2.a) $f(x,y) = \arcsin(x^2+y^2-3) - 2\ln(1-x)$

MORA VRIDEDITI: $\begin{cases} -1 \le x^2+y^2-3 \le 1 \\ 1-x > 0 \end{cases}$

1.) $-1 \le x^2+y^2-3 \le 1$

1.a) $x^2+y^2-3 \ge -1$

$\boxed{x^2+y^2 \ge 2}$ → sve izvan kruga radijusa $\sqrt{2}$

1.b) $x^2+y^2-3 \le 1$

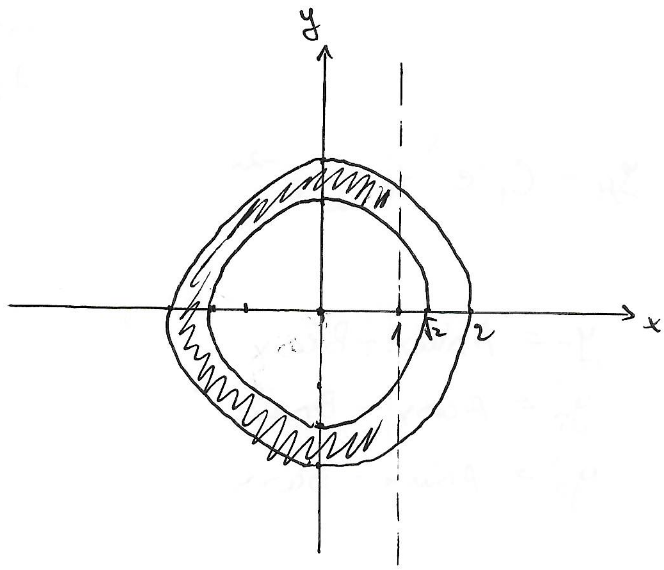
$\boxed{x^2+y^2 \le 4}$ ← sve unutar -//- 2

2.) $1-x>0$

$\boxed{x<1}$

$D_f = \{(x,y) \mid x<1,\ 2 \le x^2+y^2 \le 4\}$

Slica:



2.b) $f(x,y) = x^2 + 3y^2 - 1$

TANG. RAVNINA PARALELNA S $\pi \ldots \; 2x - 12y + z - 3 = 0$

$$\Leftrightarrow \quad z = -2x + 12y + 3$$

TANG. RAVNINA U TOČKI (x_0, y_0) JE

$$z = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + \underbrace{z_0}_{=f(x_0,y_0)}$$

Da bi ravnine bile paralelne mora vrijediti

$$\begin{cases} \frac{\partial f}{\partial x}(x_0,y_0) = -2 \\ \frac{\partial f}{\partial y}(x_0,y_0) = 12 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} 2x_0 = -2 \\ 6y_0 = 12 \end{cases} \Rightarrow \boxed{\begin{matrix} x_0 = -1 \\ y_0 = 2 \end{matrix}}$$

Graf funkcije f ima tang. ravninu paralelnu s π u točki $(-1,2)$.
Ta ravnina je dana s

$$z = -2(x-(-1)) + 12(y-2) + 1^2 + 3 \cdot 2^2 - 1$$

$$\Leftrightarrow \quad \boxed{z = -2x + 12y - 14.}$$

3.1 $r \geq 2$

$r \leq 2(1+\cos\varphi)$

U POLARNIM KOORDINATAMA:

- $r=2 \rightarrow$ kružnica radijusa 2
- $r = 2(1+\cos\varphi)$ $\rightarrow$ kardioida. Primijetimo $2\cdot(1+\cos\varphi) = 2+2\cos\varphi \in [0,4]$

Za $\varphi=0 \rightarrow r=4$

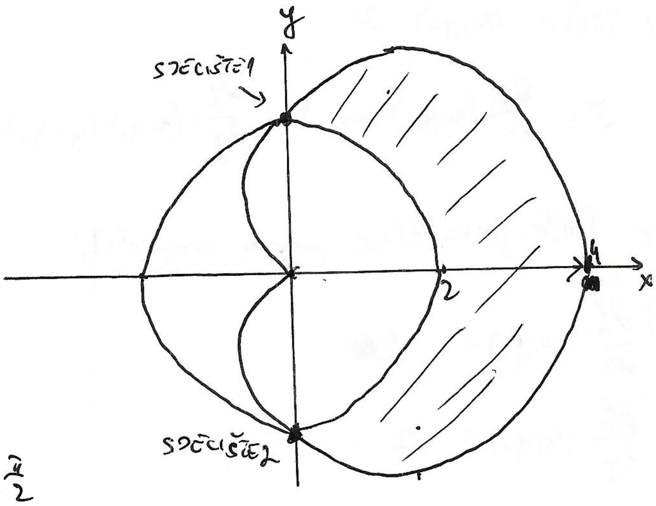
$\varphi=\frac{\pi}{2} \rightarrow r=2$

$\varphi=\pi \rightarrow r=0$

$\varphi=\frac{\pi}{3} \rightarrow r=3$

...

Skica:



SJECIŠTA:

$r=2=2(1+\cos\varphi)$

$\Rightarrow \cos\varphi=0 \Leftrightarrow \varphi=\pm\frac{\pi}{2}$

$$P = \iint_D dx\,dy \overset{\text{POLARNE KOORD.}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_2^{2(1+\cos\varphi)} r\,dr\,d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2}{2}\Big|_2^{2(1+\cos\varphi)} d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cancel{2}(1+2\cos\varphi+\cos^2\varphi) - \cancel{2}\right) d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(4\cos\varphi + \cancel{2}\cdot\frac{1-\cos 2\varphi}{\cancel{2}}\right) d\varphi$$

$$= \left(4\sin\varphi + \varphi - \frac{1}{2}\sin 2\varphi\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4-(-4) + \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) - \frac{1}{2}0 + \frac{1}{2}\cdot 0$$

$$= 8+\pi.$$

4.a) $f(x,y,z) = xy \sin \frac{x}{z}$

$\vec{s} = \vec{i} + 2\vec{j} - 2\vec{k}$

$$\frac{df}{d\vec{s}} = \text{grad} f \cdot \frac{\vec{s}}{|\vec{s}|}$$

$\text{grad} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$

$= \left(y \sin \frac{x}{z} + \frac{xy}{z} \cos \frac{x}{z}\right)\vec{i}$

$+ \left(x \sin \frac{x}{z}\right)\vec{j}$

$+ \left(xy \cos \frac{x}{z} \cdot \left(-\frac{x}{z^2}\right)\right)\vec{k}$

$\frac{\vec{s}}{|\vec{s}|} = \frac{1}{\sqrt{1^2+2^2+2^2}} \vec{s}$

$= \frac{1}{3}\vec{s}$

Evaluiramo grad f u točki T :

$\text{grad} f \,|_{T(\pi,1,1)}$

$$= \left(1 \cdot \sin \frac{\pi}{1} + \frac{\pi \cdot 1}{1} \cos \frac{\pi}{1}\right)\vec{i} + \left(\pi \cdot \sin \frac{\pi}{1}\right)\vec{j} + \left(\pi \cdot 1 \cos \frac{\pi}{1} \cdot \left(-\frac{\pi}{1^2}\right)\right)\vec{k}$$

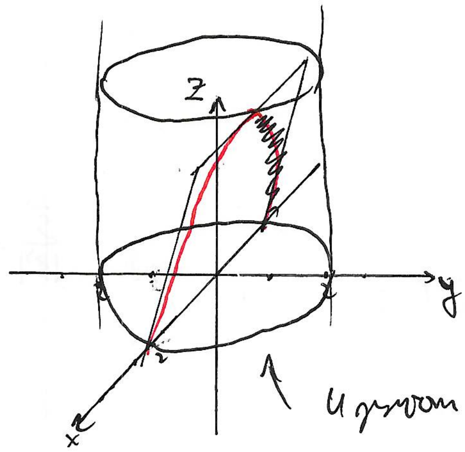
$$= -\pi \vec{i} + \pi^2 \vec{k}$$

$$\Rightarrow \frac{df}{d\vec{s}} = \left(-\pi\vec{i} + \pi^2\vec{k}\right) \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}\right)$$

$$= -\frac{\pi}{3} + 0 - \frac{2}{3}\pi^2 = -\frac{\pi}{3} - \frac{2}{3}\pi^2$$

4.b) $\vec{\Gamma}$ je presjecnica ploha $\begin{cases} x^2+y^2=4 & \text{- valjak čija os je z-os, radijusa 2} \\ z=2y & \text{- ravnina} \end{cases}$

Skica:



u prvom oktantu, to je četvrtina elipse u prostoru.

Parametrizacija

$x=2\cos t$

$y=2\sin t \qquad , \quad t\in[0,\frac{\pi}{2}]$

$z=2y=4\sin t$

(Za sve druge parametre izlazimo iz 1 oktanta)

$$\int_{\vec{\Gamma}} y\,dx+z\,dy+x\,dz = \int_0^{\frac{\pi}{2}}\left(2\sin t\cdot(-2\sin t)+4\sin t\cdot 2\cos t+2\cos t\cdot 4\cos t\right)dt$$

$$=\int_0^{\frac{\pi}{2}}\left(-4\sin^2 t+8\sin t\cos t+8\cos^2 t\right)dt$$

$$=\int_0^{\frac{\pi}{2}}\left(-4\cdot\frac{1-\cos 2t}{2}+4\sin 2t+8\cdot\frac{1+\cos 2t}{2}\right)dt$$

$$=\int_0^{\frac{\pi}{2}}\left(2+6\cos 2t+4\sin 2t\right)dt$$

$$=\left(2t+3\sin 2t-2\cos 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$=2\cdot\left(\frac{\pi}{2}-0\right)+3(0-0)-2\cdot(-1-1)$$

$$=\pi+4.$$

5.) $\vec{a} = x^2 y \,\vec{i} + x z^2 \,\vec{j} + x y^2 \,\vec{k}$

Ploha ... $x^2+y^2+z^2=1$ u 1. oktantu $\Leftrightarrow x, y, z \geq 0$

Koordinatne ravnine $x=0$

$y=0$

$z=0$

TEOREM O DIVERGENCIJI. Za zatvorenu plohu Σ, koja zatvara tijelo V

$$\iiint_V \operatorname{div} \vec{a} \, dx\,dy\,dz = \iint_\Sigma \vec{a} \cdot d\vec{S}$$

TOK V.P.

$$\iint_\Sigma \vec{a} \cdot d\vec{S} = \iiint_V \operatorname{div} \vec{a} \, dx\,dy\,dz$$

$$= \iiint_V (2xy + 0 + 0) \, dx\,dy\,dz$$

$$= \iiint_V 2xy \, dx\,dy\,dz$$

V je $\frac{1}{8}$ kugle u 1. oktantu $\rightarrow$ u sfernim koordinatama $0 \leq r \leq 1$

$0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq \theta \leq \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 2r \sin\theta \cos\varphi \cdot r \sin\theta \sin\varphi \, \underbrace{r^2 \sin\theta}_{=\text{JAKOBIJAN}} \, dr\,d\varphi\,d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin\varphi \cos\varphi \, d\varphi\right)\left(\int_0^1 r^4 \, dr\right)$$

$$= \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{15}.$$

$\hookrightarrow$

- $$\int_0^{\frac{\pi}{2}} \sin^3\varphi \, d\varphi = \int_0^{\pi/2} \sin\varphi(1-\cos^2\varphi)\, d\varphi$$
$$= \int_0^{\pi/2} (\sin\varphi - \cos^2\varphi\sin\varphi)\, d\varphi$$
$$= \int_0^{\pi/2} \sin\varphi \, d\varphi - \int_0^{\pi/2} \cos^2\varphi \sin\varphi \, d\varphi$$
$$= (-\cos\varphi)\Big|_0^{\frac{\pi}{2}} + \frac{\cos^3\varphi}{3}\Big|_0^{\frac{\pi}{2}}$$
$$= 1 - \frac{1}{3} = \frac{2}{3}$$

- $$\int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi \, d\varphi = \frac{\sin^2\varphi}{2}\Big|_0^{\pi/2} = \frac{1}{2}$$

- $$\int_0^1 r^4 \, dr = \frac{r^5}{5}\Big|_0^1 = \frac{1}{5}.$$