Matematika 1 - pismeni ispit

24. 2. 2025.

Ime i prezime: ______________________________

1.	2.	3.	4.	5.

$\sum$

Ocjena pismenog ispita: ______________________________

1. Zadane su ravnine $\pi_1 ... 2x - y + z - 1 = 0$ i $\pi_2 ... x - y + 1 = 0$.
   a) (9 bodova) Odredite kut između ravnina π_1 i π_2.
   b) (11 bodova) Odredite presjek ravnina π_1 i π_2.

2. (20 bodova) Odredite svojstvene vrijednosti i svojstveni vektor koji pripada najvećoj svojstvenoj vrijednosti matrice
$$A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 0 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

3. (20 bodova) Odredite domenu, nultočke, točke ekstrema, intervale rasta i pada, asimptote te skicirajte graf funkcije
$$f(x) = \frac{2x^2}{4x - 1}.$$

4. (15 bodova) Izračunajte
$$\int \frac{1}{\sqrt{e^x - 4} - 1} dx.$$

5. a) (12 bodova) Odredite površinu lika omeđenog krivuljama $y = (x-1)(x-2)$, $y = (x-2)(x-5)$ i $y = (x-1)(x-5)$. Skicirajte lik.
   b) (13 bodova) Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y = \ln x$ i osi x na segmentu $[1, e]$, oko osi x. Skicirajte lik.

1. Zadano je $\pi_1 \ldots \; 2x - y + z - 1 = 0 \;\rightsquigarrow$

$\pi_2 \ldots \; x - y + 1 = 0 \;\longrightarrow$

VEKTORI NORMALA:

$\vec{n}_1 = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{n}_2 = \vec{i} - \vec{j}$

a) Kut između ravnina

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{2\cdot 1 + (-1)\cdot(-1) + 1\cdot 0}{\sqrt{2^2+(-1)^2+1^2}\,\sqrt{1^2+(-1)^2+0^2}}$$

$$= \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\varphi = \frac{\pi}{6}}$$

b) Presjek ravnina je pravac okomit na $\vec{n}_1, \vec{n}_2$

$\Rightarrow$ Vektor smjera pravca je dan s

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= (0+1)\vec{i} + (1-0)\vec{j} + (-2+1)\vec{k}$$

$$= \vec{i} + \vec{j} - \vec{k}$$

Preostaje naći neku točku u presjeku $\pi_1 \cap \pi_2$.

Uzmimo $\boxed{x = 0} \Rightarrow -y + 1 = 0 \Rightarrow \boxed{y = 1}$

$\Rightarrow 2\cdot 0 - 1 + z - 1 = 0$

$\boxed{z = 2}$

Presjek ravnina je pravac $\dfrac{x}{1} = \dfrac{y-1}{1} = \dfrac{z-2}{-1}$.

2.1

$$A=\begin{bmatrix}2&2&3\\3&0&0\\2&2&3\end{bmatrix}$$

$$k_A(\lambda)=\det(\lambda I-A)$$

Laplace $\rightarrow$

$$=\begin{vmatrix}\lambda-2&-2&-3\\-3&\lambda&0\\-2&-2&\lambda-3\end{vmatrix}=-(-3)\cdot\begin{vmatrix}-2&-3\\-2&\lambda-3\end{vmatrix}+\lambda\begin{vmatrix}\lambda-2&-3\\-2&\lambda-3\end{vmatrix}+0\cdot\begin{vmatrix}&\\&\end{vmatrix}$$

$$=3\cdot(-2\lambda+6-6)+\lambda\cdot(\lambda^2-5\lambda+6-6)$$

$$=-6\lambda+\lambda\cdot(\lambda^2-5\lambda)$$

$$=\lambda(\lambda^2-5\lambda-6)=0 \Rightarrow \underline{\lambda_1=0},\qquad \lambda_{2,3}=\frac{5\pm\sqrt{25-4\cdot(-6)}}{2}$$

$$\underline{\lambda_2=6}$$
$$\underline{\lambda_3=-1}$$

Najveća svojstvena vrijednost je $\underline{\lambda=6}$.

Svojstveni vektori zadovoljavaju $(\lambda I-A)v=0$, $v\neq\vec{0}$

$\Rightarrow$ Tražimo rješenja sustava:

$$\left[\begin{array}{ccc|c}4&-2&-3&0\\-3&6&0&0\\-2&-2&3&0\end{array}\right]\sim\left[\begin{array}{ccc|c}4&-2&-3&0\\1&-2&0&0\\-2&-2&3&0\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&-2&0&0\\4&-2&-3&0\\-2&-2&3&0\end{array}\right]\begin{array}{c}\cdot(-4)\\+2\end{array}\sim\left[\begin{array}{ccc|c}1&-2&0&0\\0&6&-3&0\\0&-6&3&0\end{array}\right]\cdot 1$$

$$\sim\left[\begin{array}{ccc|c}1&-2&0&0\\0&6&-3&0\\0&0&0&\end{array}\right]\Rightarrow\begin{array}{l}x-2y=0\\6y-3z=0\end{array}\Rightarrow\boxed{\begin{array}{l}x=2y\\z=2y\end{array}}$$

y param...

$y=t$

$\Rightarrow$ Svojstveni vektori su oblika

$t\cdot(2\vec{i}+\vec{j}+2\vec{k}),\ t\neq 0.$

3.1 $f(x)=\dfrac{x^2}{4x-1}$

DOMENA: Nazivnik $\neq 0 \Rightarrow$ $4x-1=0$

$x=\frac{1}{4}$ treba isključiti iz domene.

$\Rightarrow D_f = \mathbb{R} \setminus \{\frac{1}{4}\}$

NULTOČKE:

$f(x)=0 \iff \dfrac{2x^2}{4x-1}=0 \Rightarrow$ $x=0$ je jedina nultočka

TOČKE EKSTREMA: $f'(x)=\dfrac{2x\cdot(4x-1)-x^2\cdot 4}{(4x-1)^2}$

$=\dfrac{4x^2-2x}{(4x-1)^2}=\underline{2x(2x-1)}$

$f'(x)=0 \Rightarrow 4x^2-2x=0$

$2x(2x-1)=0 \Rightarrow \boxed{x_1=0,\ x_2=\frac{1}{2}}$ Stac. točke

INTERVALI RASTA/PADA.

	0		$\frac{1}{4}$		$\frac{1}{2}$		$+\infty$
f'	+		−	−		+	
f	↗		↘	↘		↗	
	M				m		

Iz tablice vidimo da je $(0,0)$ lok. maksimum, a $(\frac{1}{2}, 1)$ lok. minimum.

ASIMPTOTE: Vertikalne mogu biti samo u $x=\frac{1}{4}$.

$\lim\limits_{x\to\frac{1}{4}} \dfrac{x^2}{4x-1} = \text{"}\dfrac{\frac{1}{4}}{0}\text{"} = \infty \Rightarrow$ $x=\frac{1}{4}$ jest vertikalna asimptota.

Kose: $k=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\pm\infty}\dfrac{2x^2\ /:x^2}{x\cdot(4x-1)\ /:x^2}=\lim\limits_{x\to\pm\infty}\dfrac{2}{4-\frac{1}{x}}=\dfrac{2}{4}=\dfrac{1}{2}$

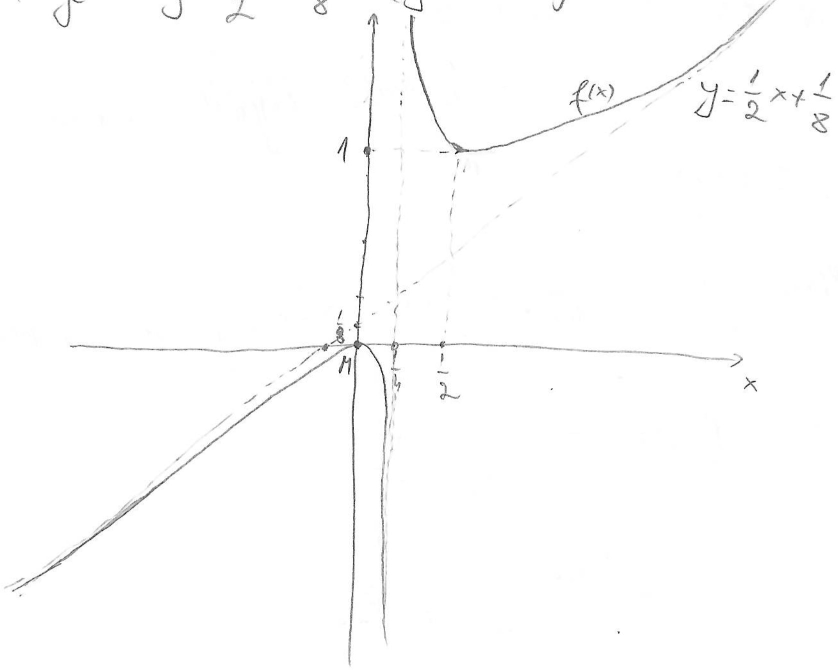
$l=\lim\limits_{x\to\pm\infty}(f(x)-kx)=\lim\limits_{x\to\pm\infty}\left(\dfrac{2x^2}{4x-1}-\dfrac{1}{2}x\right)$

$=\lim\limits_{x\to\pm\infty}\dfrac{4x^2-4x^2+x}{2(4x-1)}=\lim\limits_{x\to\pm\infty}\dfrac{x\ /:x}{2\cdot(4x-1)\ /:x}=\dfrac{1}{8}$

.) nastavak

Kosa asimptota je $y = \frac{1}{2}x + \frac{1}{8}$ (ujedno i lijeva i desna)

<u>Slika</u>:



4. $\int \frac{1}{\sqrt{e^x-4}-1}\,dx = \left| \begin{array}{l} t=\sqrt{e^x-4} \Rightarrow t^2=e^x-4 \Rightarrow e^x=t^2+4 \\ \Rightarrow 2t\,dt = e^x\,dx \\ \Rightarrow 2t\,dt=(t^2+4)\,dx \Rightarrow dx = \frac{2t\,dt}{t^2+4}=dx \end{array} \right|$

$$= \int \frac{1}{t-1}\cdot\frac{2t\,dt}{t^2+4} = \int \frac{2t\,dt}{(t-1)(t^2+4)} = (*)$$

PARC. RAZLOMCI: $\frac{2t}{(t-1)(t^2+4)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+4} \quad / \cdot (t-1)(t^2+4)$

$$2t = A(t^2+4)+(Bt+C)(t-1)$$

$$2t = t^2(A+B)+t(-B+C)+4A-C$$

Izjednači koeficijente $\Rightarrow$ $\left| \begin{array}{l} A+B=0 \Rightarrow B=-A \\ -B+C=2 \\ 4A-C=0 \Rightarrow C=4A \end{array} \right. \Rightarrow A+4A=2$

$\boxed{A=\frac{2}{5}} \Rightarrow \boxed{B=-\frac{2}{5}}$

$\boxed{C=\frac{8}{5}}$

$$(*) = \int \frac{\frac{2}{5}}{t-1}\,dt + \int \frac{-\frac{2}{5}t+\frac{8}{5}}{t^2+4}\,dt$$

$$= \frac{2}{5}\ln|t-1| + -\frac{2}{5}\int \frac{t}{t^2+4}\,dt + \frac{8}{5}\int\frac{dt}{t^2+4}$$

$\hookrightarrow$ supstituimo $u=t^2+4$

$du=2t\,dt$

$$= \frac{2}{5}\ln|t-1| - \frac{1}{5}\ln(t^2+4) + \frac{8}{5}\cdot\frac{1}{2}\,\mathrm{arctg}\,\frac{t}{2}+C \quad \text{gdje } t=\sqrt{e^x-4}$$

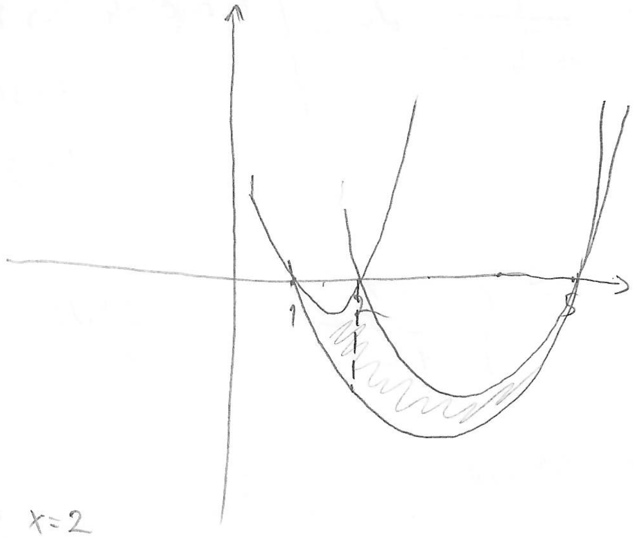
$$\left(\text{ili } = \frac{2}{5}\ln\left|\sqrt{e^x-4}-1\right| - \frac{1}{5}x + \frac{4}{5}\,\mathrm{arctg}\,\frac{\sqrt{e^x-4}}{2}+C\right).$$

5.a) I. $y=(x-1)(x-2)$

II. $y=(x-2)(x-5)$

III. $y=(x-1)(x-5)$

Skica



Tražimo sjecišta:

I. ∩ II. : $(x-1)\cancel{(x-2)} = \cancel{(x-2)}(x-5) \Rightarrow x=2$

$\cancel{x}-1=\cancel{x}-5$ ↯

$\downarrow$ jedino sjecište

slično II. ∩ III. , jedino sjecište je za $x=5$

I. ∩ III. , jedino sjecište je za $x=2$.

Površina lika je dana s:

$$P = \int_1^2 \big((x-1)(x-2)-(x-1)(x-5)\big)dx + \int_2^5 \big((x-2)(x-5)-(x-1)(x-5)\big)dx$$

$$= \int_1^2 \big((x^2-3x+2)-(x^2-6x+5)\big)dx + \int_2^5 \big((x^2-7x+10)-(x^2-6x+5)\big)dx$$

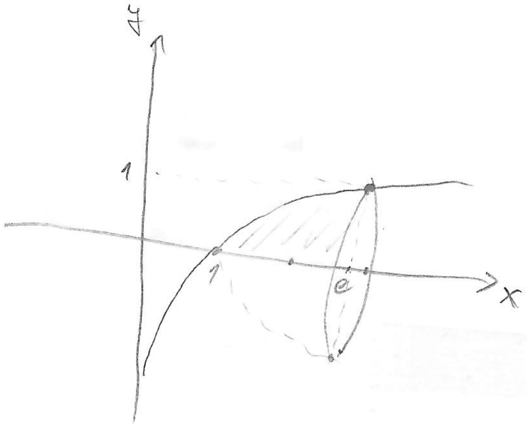
$$= \int_1^2 (3x-3)dx + \int_2^5 (-x+5)dx$$

$$= \left(\frac{3x^2}{2}-3x\right)\Big|_1^2 + \left(-\frac{x^2}{2}+5x\right)\Big|_2^5$$

$$= \left(\cancel{\frac{3\cdot4}{2}-3\cdot2}-\frac{3}{2}+3\right) + \left(-\frac{25}{2}+25+\frac{2^2}{2}-2\cdot5\right)$$

$$= \frac{3}{2}+\frac{9}{2}=6$$

5. b) $y = \ln x$ Skica:



$$V = \pi \int_a^b f(x)^2 \, dx = \pi \int_1^e (\ln x)^2 \, dx$$

$$\overset{\text{PARC. INT.}}{=} \pi \left(x \ln^2 x \Big|_1^e - \int_1^e \cancel{x} \cdot 2 \ln x \cdot \frac{1}{\cancel{x}} \, dx \right)$$

$$= \pi \left(e \cdot 1^2 - 1 \cdot 0 - 2 \int_1^e \ln x \, dx \right)$$

$$= \pi \left(e - 2 \left(x \ln x \Big|_1^e - \int_1^e \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx \right) \right)$$

$$= \pi \cdot \left(e - 2 \cdot \left(e - 0 - (e - 1) \right) \right)$$

$$= \pi \cdot (e - 2) \,//$$