Matematika 1 - pismeni ispit

3. 4. 2025.

Ime i prezime: ____

1.	2.	3.	4.	5.

$\sum$

Ocjena pismenog ispita: ____

1. a) (10 bodova) Odredite za koje vrijednosti $\lambda \in \mathbb{R}$ su vektori $\vec{a} = \lambda\vec{i} + 3\vec{j} + 5\vec{k}$, $\vec{b} = \lambda\vec{i} + (\lambda + 2)\vec{j}$, $\vec{c} = (\lambda - 1)\vec{j} - 5\vec{k}$ komplanarni.

   b) (15 bodova) Pravac p je zadan točkom pravca $A(1, 3, 9)$ i vektorom smjera $\vec{c} = \vec{i} + 4\vec{k}$. Nađite projekciju točke $T(-8, 4, 7)$ na pravac p.

2. (15 bodova) Gaussovom metodom riješite sustav

$$\begin{cases} x_1 + x_2 + 2x_3 - x_4 = 1 \\ x_1 + 2x_2 - x_3 + x_4 = 5 \\ 2x_1 + 3x_2 - x_3 + 2x_4 = 12 \\ 3x_1 - 2x_2 + 3x_3 - x_4 = 7. \end{cases}$$

3. a) (10 bodova) Odredite točke krivulje $f(x) = \frac{x^2}{2} + x - 2\ln(x^2 + 1)$ u kojima tangenta na krivulju zatvara kut od $45°$ s x-osi.

   b) (10 bodova) Odredite limes

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + x} - \sqrt{x^2 - x}}{x}.$$

4. (15 bodova) Izračunajte

$$\int \frac{x^5}{(x^3 + 3)\sqrt{x^3 - 1}} dx.$$

5. a) (12 bodova) Odredite površinu lika omeđenog krivuljama $y = x^2 - x - 2$ i $y = x + 1$.

   b) (13 bodova) Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y = x \ln x$ na segmentu $[1, e]$, oko osi x.

1.a) $\vec{a} = \lambda\vec{i} + 3\vec{j} + 5\vec{k}$

$\vec{b} = \lambda\vec{i} + (\lambda+2)\vec{j}$

$\vec{c} = (\lambda-1)\vec{j} - 5\vec{k}$

Vektori komplanarni $\Leftrightarrow$ u istoj su ravnini

$\Leftrightarrow$ Volumen paralelepipeda kojeg razapinju je 0.

$\Leftrightarrow$ Mješoviti produkt je 0.

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} \lambda & 3 & 5 \\ \lambda & \lambda+2 & 0 \\ 0 & \lambda-1 & -5 \end{vmatrix} = \lambda\begin{vmatrix} \lambda+2 & 0 \\ \lambda-1 & -5 \end{vmatrix} - \lambda\begin{vmatrix} 3 & 5 \\ \lambda-1 & -5 \end{vmatrix}$$

$\uparrow$
Razvoj po 1 stupcu

$$= \lambda(-5\lambda-10) - \lambda(-15 - 5\lambda + 5)$$

$$= -5\lambda^2 - 10\lambda + 15\lambda + 5\lambda^2 - 5\lambda$$

$$= 0$$

Dakle, <u>neovisno</u> o λ, produkt je 0.

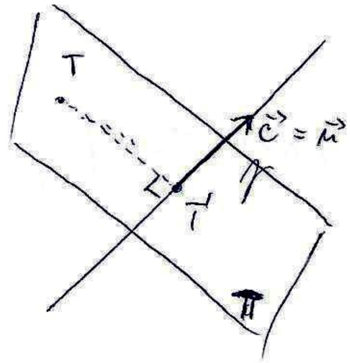
$\Rightarrow$ Vektori su komplanarni za sve $\lambda \in \mathbb{R}$.

1. b) p zadan točkom $A(1,3,9)$

smjerom $\vec{c} = \vec{i} + 4\vec{k}$ $\Rightarrow p \ldots \quad \dfrac{x-1}{1} = \dfrac{y-3}{0} = \dfrac{z-9}{4}$

$$\Leftrightarrow \quad p \ldots \begin{cases} x = 1 + t \\ y = 3 \\ z = 9 + 4t \end{cases}$$

Ideja – da nađemo okomiti pravac iz T na p, prvo nađemo okomitu ravninu kroz T na p.



Okomita ravnina kao vektor normale ima vektor smjera pravca $\vec{n} = \vec{c}$

$$= \vec{i} + 4\vec{k}.$$

$\Rightarrow$ Jednadžba okomite ravnine na p kroz T je

$$\pi \ldots \quad 1\cdot(x+8) + 0\cdot(y-4) + 4\cdot(z-7) = 0$$

$$x + 4z = 20.$$

Presjek s p:

$$(1+t) + 4\cdot(9+4t) = 20$$

$$17t = -17$$

$$\boxed{t = -1}$$

Dakle, točka T' koja je projekcija od T je:

$$x_0 = 1 + (-1) = 0$$

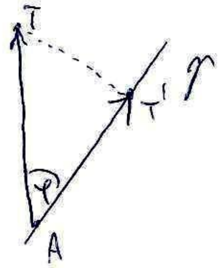
$$y_0 = 3$$

$$z_0 = 9 + 4\cdot(-1) = 5$$

$$\Rightarrow T'(0,3,5).$$

1.b.) 2. NAČIN

Zadatak je moguće riješiti i ovako:



Primijetimo da je vektor $\overrightarrow{AT'}$ vektorska projekcija od $\overrightarrow{AT}$ u smjeru pravca p.

To je vektor duljine $|AT|\cdot\cos\varphi$, u smjeru $\frac{\vec{c}}{|\vec{c}|}$, dakle:

$$\overrightarrow{AT'} = |AT|\cdot\cos\varphi\cdot\frac{\vec{c}}{|\vec{c}|}$$

$$= \left(\frac{\overrightarrow{AT}\cdot\vec{c}}{|\vec{c}|}\right)\frac{\vec{c}}{|\vec{c}|} = \frac{\overrightarrow{AT}\cdot\vec{c}}{|\vec{c}|^2}\,\vec{c}$$

$$\left(\begin{aligned}\overrightarrow{AT} &= (-8-1)\vec{i} + (4-3)\vec{j} + (7-9)\vec{k}\\ &= -9\vec{i} + \vec{j} - 2\vec{k}\end{aligned}\right)$$

$$\Rightarrow \quad \overrightarrow{AT'} = \frac{(-9\vec{i}+\vec{j}-2\vec{k})\cdot(\vec{i}+4\vec{k})}{\left(\sqrt{1^2+0^2+4^2}\right)^2}\cdot(\vec{i}+4\vec{k})$$

$$= \frac{-9-8}{17}\cdot(\vec{i}+4\vec{k}) = -\vec{i}-4\vec{k}.$$

Dakle, da dobijemo T', od A se trebamo pomaknuti za $-\vec{i}-4\vec{k}$

$$\begin{array}{c} A\,(1,3,9)\\ \overrightarrow{AT'} = -\vec{i}-4\vec{k}\\ \hline T' = (0,3,5)\end{array}.$$

2.

$$\begin{cases} x_1 + x_2 + 2x_3 - x_4 = 1 \\ x_1 + 2x_2 - x_3 + x_4 = 5 \\ 2x_1 + 3x_2 - x_3 + 2x_4 = 12 \\ 3x_1 - 2x_2 + 3x_3 - x_4 = 7 \end{cases}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 1 & 2 & -1 & 1 & 5 \\ 2 & 3 & -1 & 2 & 12 \\ 3 & -2 & 3 & -1 & 7 \end{array}\right] \begin{array}{l} \\ \cdot(-1) \\ \cdot(-2) \\ \cdot(-3) \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & 2 & 4 \\ 0 & 1 & -5 & 4 & 10 \\ 0 & -5 & -3 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \\ \cdot(-1) \\ +5 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & 2 & 4 \\ 0 & 0 & -2 & 2 & 6 \\ 0 & 0 & -18 & 12 & 24 \end{array}\right] \begin{array}{l} \\ \\ \\ \cdot(-9) \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & 2 & 4 \\ 0 & 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & -6 & -30 \end{array}\right]$$

$\Rightarrow$ $-6x_4 = -30$

$\boxed{x_4 = 5}$

$-2x_3 + 2x_4 = 6$

$-2x_3 + 10 = 6$

$-2x_3 = -4$

$\boxed{x_3 = 2}$

$x_2 - 3x_3 + 2x_4 = 4$

$x_2 - 6 + 10 = 4$

$\boxed{x_2 = 0}$

$x_1 + x_2 + 2x_3 - x_4 = 1$

$x_1 + 0 + 4 - 5 = 1$

$\boxed{x_1 = 2}$

3. a) $f(x) = \frac{x^2}{2} + x - 2\ln(x^2+1)$.

Jednadžba tangente u točki x_0: $y - f(x_0) = f'(x_0)(x - x_0)$

$\downarrow$

koef. smjera tangente

= tangens kuta s x-osi.

Dakle, tražimo točke x t.d. $f'(x) = \tan 45^\circ = 1$.

$f'(x) = x + 1 - 2 \cdot \frac{1}{1+x^2} \cdot 2x$ treba biti jednako 1

$\Rightarrow \quad x + \cancel{1} - 2 \cdot \frac{2x}{1+x^2} = \cancel{1}$

$\frac{x + x^3 - 4x}{1+x^2} = 0$

$\frac{x^3 - 3x}{1+x^2} = 0$

$\frac{x(x^2-3)}{1+x^2} = 0 \Rightarrow$ $x = 0$ ili $x^2 - 3 = 0$ $\Rightarrow x \in \{-\sqrt{3}, 0, \sqrt{3}\}$.

Za ove x, tangenta u $(x, f(x))$ na $f(x)$ ima kut 45° s x-osi.

3.b)

$$\lim_{x\to+\infty} \frac{\sqrt{2x^2+x}-\sqrt{x^2-x}}{x} \begin{matrix} /:x \\ /:x \end{matrix} = \lim_{x\to+\infty} \frac{\sqrt{\frac{2x^2+x}{x^2}}-\sqrt{\frac{x^2-x}{x^2}}}{1}$$

$$= \lim_{x\to+\infty}\left(\sqrt{2+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}\right)$$

$$= \sqrt{2+\lim_{x\to\infty}\frac{1}{x}}-\sqrt{1-\lim_{x\to\infty}\frac{1}{x}} = \sqrt{2}-\sqrt{1}$$

$$= \sqrt{2}-1$$

(NAPOMENA: Ovdje nije bila potrebna standarna tehnika $\sqrt{a}-\sqrt{b}=\frac{a-b}{\sqrt{a}+\sqrt{b}}$.)

4.

$$\int \frac{x^5}{(x^3+3)\sqrt{x^3-1}}\,dx = \left| \begin{matrix} t=x^3-1 \to x^3=t+1 \\ dt=3x^2dx \end{matrix} \right| = \int \frac{(t+1)\cdot\frac{dt}{3}}{(t+4)\sqrt{t}} = \left| \begin{matrix} t=u^2 \\ dt=2u\,du \end{matrix} \right|$$

$$= \frac{1}{3}\int \frac{(u^2+1)\cdot 2\cancel{u}}{(u^2+4)\cancel{u}}\,du$$

$$= \frac{2}{3}\int \frac{u^2+1}{u^2+4}\,du$$

NAPOMENA :

Mogla je odmah supstitucija

$$\left| \begin{matrix} u^2=x^3-1 \\ 2u\,du=3x^2dx \end{matrix} \right|$$

$$= \frac{2}{3}\int \frac{u^2+4-3}{u^2+4}\,du = \frac{2}{3}\left(\int\left(1-\frac{3}{u^2+4}\right)du\right)$$

$$= \frac{2}{3}\left(u-3\cdot\frac{1}{2}\operatorname{arctg}\frac{u}{2}\right)+C$$

$$= \frac{2}{3}\sqrt{x^3-1} - \operatorname{arctg}\frac{\sqrt{x^3-1}}{2} + C .$$

5.a) Površina između $\begin{cases} y = x^2 - x - 2 = (x-2)(x+1) \\ y = x+1 \end{cases}$

PRESJEK:

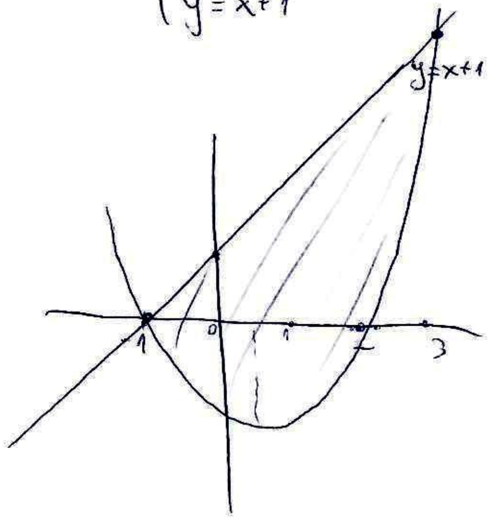
$x^2 - x - 2 = x + 1$

$x^2 - 2x - 3 = 0$

$x_{1,2} = \frac{2 \pm \sqrt{4+12}}{2}$

$\boxed{\begin{matrix} x_1 = 3 \\ x_2 = -1 \end{matrix}}$

Skica:



Površina je $\int_a^b (\text{gornja} - \text{donja}) = \int_{-1}^{3} \left((x+1) - (x^2 - x - 2)\right) dx$

$$= \int_{-1}^{3} (-x^2 + 2x + 3)\, dx = \left(-\frac{x^3}{3} + x^2 + 3x\right)\Big|_{-1}^{3}$$

$$= \left(-\frac{27}{3} + 9 + 3 \cdot 3\right) - \left(-\frac{(-1)^3}{3} + (-1)^2 + 3 \cdot (-1)\right)$$

$$= 9 - \left(\frac{1}{3} + 1 - 3\right) = 9 - \left(-\frac{5}{3}\right) = \frac{32}{3}.$$

5.b) $y = x \cdot \ln x \quad , \quad x \in [1, e]$

Rotiramo oko x-osi: $V = \pi \int_a^b f(x)^2 \, dx$

$$V = \pi \cdot \int_1^e (x \cdot \ln x)^2 \, dx = \left| \begin{array}{ll} u = (\ln x)^2 & dv = x^2 dx \\ du = 2\ln x \cdot \frac{1}{x} & v = \frac{x^3}{3} \end{array} \right|$$

$$= \pi \cdot \left(\frac{x^3}{3} \ln^2 x \Big|_1^e - \int_1^e \frac{\cancel{x^3}^{\,2}}{3} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} dx \right)$$

$$= \pi \cdot \left(\frac{e^3}{3} - \frac{2}{3} \int_1^e x^2 \ln x \, dx \right) = \left| \begin{array}{ll} u = \ln x & dv = x^2 dx \\ du = \frac{1}{x} dx & v = \frac{x^3}{3} \end{array} \right|$$

$$= \pi \cdot \left(\frac{e^3}{3} - \frac{2}{3} \frac{x^3}{3} \ln x \Big|_1^e + \frac{2}{3} \int_1^e \frac{x^3}{3} \cdot \frac{1}{x} dx \right)$$

$$= \pi \cdot \left(\frac{e^3}{3} - \frac{2}{9} \cdot e^3 + \frac{2}{9} \cdot \frac{x^3}{3} \Big|_1^e \right)$$

$$= \pi \cdot \left(\frac{e^3}{9} + \frac{2}{27} \cdot (e^3 - 1) \right) = \pi \cdot \left(\frac{5}{27} e^3 - \frac{2}{27} \right).$$