Matematika 1 - pismeni ispit

27.8.2025.

Ime i prezime: ____

1.	2.	3.	4.	5.

$\sum$

Ocjena pismenog ispita: ____

1. (20 bodova) Gaussovom metodom riješite sustav

$$\begin{cases} -x_1 + 2x_2 - 3x_3 - 2x_4 = -1 \\ x_1 - x_2 + 2x_3 + x_4 = 1 \\ 2x_1 - x_2 + 3x_3 + x_4 = 2 \\ 2x_1 - 2x_2 + 4x_3 + 2x_4 = 2. \end{cases}$$

2. a) (5 bodova) Odredite nepoznati parametar $\lambda \in \mathbb{R}$ tako da vektori

$$\vec{a} = 2\vec{i} + \lambda\vec{j} - \lambda\vec{k} \text{ i } \vec{b} = 2\vec{i} + 2\vec{j} + 3\vec{k}$$

budu okomiti.

b) (15 bodova) Odredite točku M simetričnu točki $N(1,-1,3)$ s obzirom na ravninu

$$\pi \ldots x + y + z = 0.$$

3. (20 bodova) Odredite prirodnu domenu, nultočke, točke ekstrema, intervale rasta i pada, asimptote te skicirajte graf funkcije

$$f(x) = \frac{x^3}{x^2 - 4}.$$

4. (15 bodova) Izračunajte

$$\int x^3 \ln\left(1 + x^2\right) dx.$$

5. a) (10 bodova) Odredite površinu lika omeđenog krivuljama $y = -x^2$ i $y = -x - 2$. Skicirajte.

b) (15 bodova) Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y = \sin x$ i osi x, na segmentu $[0, \pi]$, oko osi x. Skicirajte.

1.) $\begin{cases} -x_1 + 2x_2 - 3x_3 - 2x_4 = -1 \\ x_1 - x_2 + 2x_3 + x_4 = 1 \\ 2x_1 - x_2 + 3x_3 + x_4 = 2 \\ 2x_1 - 2x_2 + 4x_3 + 2x_4 = 2 \end{cases}$

$$\begin{array}{c} \\ +1 \\ +2 \\ +2 \end{array}\left[\begin{array}{cccc|c} -1 & 2 & -3 & -2 & -1 \\ 1 & -1 & 2 & 1 & 1 \\ 2 & -1 & 3 & 1 & 2 \\ 2 & -2 & 4 & 2 & 2 \end{array}\right] \sim \begin{array}{c} \\ \\ -3 \\ -2 \end{array}\left[\begin{array}{cccc|c} -1 & 2 & -3 & -2 & -1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 3 & -3 & -3 & 0 \\ 0 & 2 & -2 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} -1 & 2 & -3 & -2 & -1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} -x_1 + 2x_2 - 3x_3 - 2x_4 = -1 \\ x_2 - x_3 - x_4 = 0 \end{cases} \rightsquigarrow$$

x_3, x_4 su parametri

$$\boxed{\begin{matrix} x_3 = t \\ x_4 = s \end{matrix}}$$

$\Rightarrow$ $x_2 = x_3 + x_4$

$$\boxed{x_2 = s + t}$$

$x_1 = 2x_2 - 3x_3 - 2x_4 + 1$

$= 2(s+t) - 3t - 2s + 1$

$$\boxed{x_1 = -t + 1.}$$

(VEKTORSKI ZAPIS)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t+1 \\ s+t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

2.a) $\vec{a} = 2\vec{i} + \lambda\vec{j} - \lambda\vec{k}$

$\vec{b} = 2\vec{i} + 2\vec{j} + 3\vec{k}$

UVJET OKOMITOSTI: $\vec{a} \cdot \vec{b} = 0$

$4 + 2\lambda - 3\lambda = 0$

$\boxed{\lambda = 4}$

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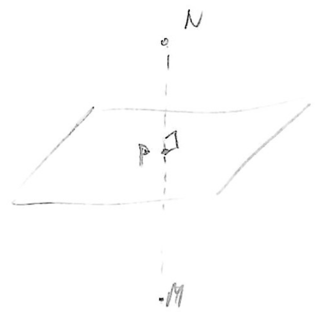
b) $\pi \ldots x + y + z = 0$

$N(1, -1, 3)$.

Idejna slika



1.) Naći P.

2.) P je polovište $\Rightarrow$ računamo M.

1.) Tražimo okomicu na π: vektor normale je $\vec{n} = \vec{i} + \vec{j} + \vec{k}$.

normala je $n \ldots \dfrac{x-1}{1} = \dfrac{y+1}{1} = \dfrac{z-3}{1} = t \Leftrightarrow n \ldots \begin{cases} x = t+1 \\ y = t-1 \\ z = t+3 \end{cases}$

P je presjek $n \cap \pi$: $\Rightarrow$ $(t+1) + (t-1) + (t+3) = 0$

$3t + 3 = 0$

$\boxed{t = -1} \Rightarrow x_P = 0,\ y_P = -2,\ z_P = 2$

$\Rightarrow$ Točka presjeka je $P(0, -2, 2)$.

2.) $N(1, -1, 3)$

$\underline{P(0, -2, 2)}$

$M(-1, -3, 1)$.

(Riješi se napamet ili postupkom izjednačavajući $\vec{NP} = \vec{PM}$)

#3. $f(x) = \dfrac{x^3}{x^2-4}$

DOMENA: NAZIVNIK $\neq 0 \Leftrightarrow x \neq \pm 2 \Leftrightarrow D_f = \mathbb{R} \setminus \{\pm 2\}$

NULTOČKE: $f(x) = 0 \Leftrightarrow \dfrac{x^3}{x^2-4} = 0 \,/ \cdot (x^2-4)$

$$x^3 = 0 \Rightarrow \boxed{x = 0}$$

EKSTREMI: $f'(x) = 0$

$$f'(x) = \left(\frac{x^3}{x^2-4}\right)' = \frac{3x^2(x^2-4) - x^3 \cdot 2x}{(x^2-4)^2} = \frac{x^4 - 12x^2}{(x^2-4)^2} = 0$$

$$\Rightarrow x^2(x^2-12) = 0$$

$\Rightarrow x_{1,2} = 0$

STAC. TOČKE $\Rightarrow$ $x_3 = \sqrt{12} = 2\sqrt{3}$

$x_4 = -2\sqrt{3}$

INTERVALI RASTA/PADA:

$$f'(x) = \frac{x^2(x^2-12)}{(x^2-4)^2}$$

KRITIČNE TOČKE:

$-2\sqrt{3}, -2, 0, 2, 2\sqrt{3}$

	$-\infty$	$-2\sqrt{3}$	-2	0	2	$2\sqrt{3}$	$+\infty$
f'	+	−	−	−	−	+	
f	↗	↘	↘	↘	↘	↗	
		M				m	

$\Rightarrow$ $T_1(0,0)$ je točka infleksije

$T_2(2\sqrt{3}, 3\sqrt{3})$ je minimum

$T_3(-2\sqrt{3}, -3\sqrt{3})$ je maksimum

$f(2\sqrt{3}) = \dfrac{\cancel{8} \cdot 3\sqrt{3}}{\cancel{8}} = 3\sqrt{3}$

ASIMPTOTE:

$$\lim_{x\to 2} \frac{x^3}{x^2-4} = \text{"}\frac{2^3}{0}\text{"} = \infty$$

$$\lim_{x\to -2} \frac{x^3}{x^2-4} = \text{"}\frac{(-2)^3}{0}\text{"} = \infty$$

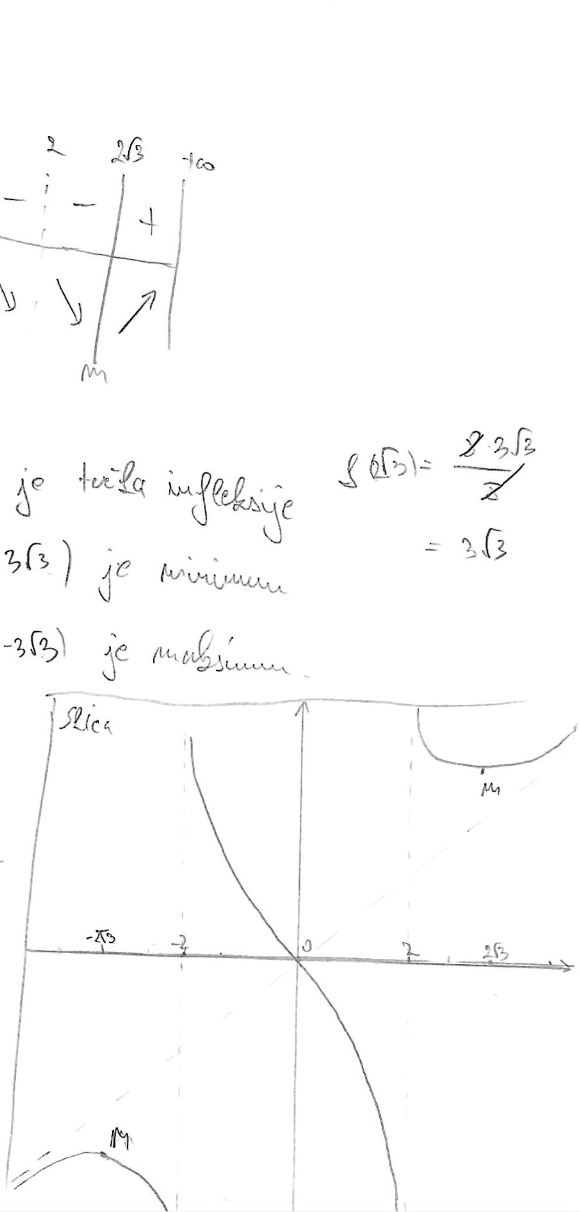
$\Rightarrow$ VERTIKALNE ASIMPTOTE

KOSA ASIMPTOTA:

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3}{x(x^2-4)} = 1$$

$$l = \lim_{x\to\infty} (f(x) - kx) = \lim_{x\to\infty} \frac{x^3 - (x^3-4x)}{x^2-4} = 0.$$

je $y = x$



4. $\int x^3 \ln(1+x^2)\,dx \overset{P.I.}{=} \left| \begin{matrix} u = \ln(1+x^2) & du = \frac{2x}{1+x^2}dx \\ dv = x^3 dx & v = \frac{x^4}{4} \end{matrix} \right|$

$= \frac{x^4}{4}\ln(1+x^2) - \int \frac{x^4}{4}\frac{2x}{1+x^2}\,dx$

$= \frac{x^4}{4}\ln(1+x^2) - \frac{1}{2}\int \frac{x^5}{1+x^2}\,dx = \frac{x^4}{4}\ln(1+x^2) - \frac{1}{2}\int \frac{x^5+x^3-x^3-x+x}{1+x^2}\,dx$

↑ Treba dijeliti polinome.

$= \frac{x^4}{4}\ln(1+x^2) - \frac{1}{2}\int \left(x^3 - x + \frac{x}{1+x^2}\right)dx$

$= \frac{x^4}{4}\ln(1+x^2) - \frac{1}{2}\cdot\frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^2}{2} - \frac{1}{2}\cdot\frac{1}{2}\ln(1+x^2) + C$

$= \frac{x^4-1}{4}\ln(1+x^2) - \frac{x^4-2x^2}{8} + C.$

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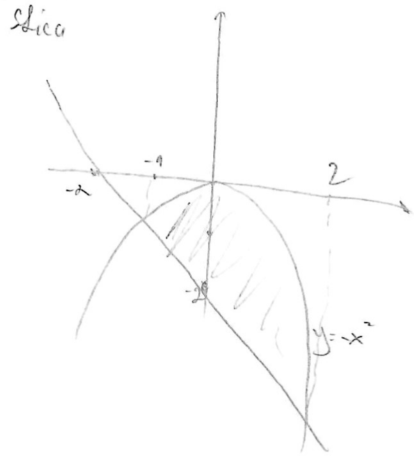
5. a) $y = -x^2$
$y = -x-2$

Skica



Sjecišta:

$-x^2 = -x-2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x_1 = -1$

$x_2 = 2$

$\Rightarrow \quad P = \int_{-1}^{2} \left(-x^2 - (-x-2)\right)dx$

$= \int_{-1}^{2} (-x^2 + x + 2)\,dx = \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Big|_{-1}^{2} = -\frac{8}{3} + 2 + 4 - \frac{1}{3} - \frac{1}{2} + 2 = \frac{9}{2}.$

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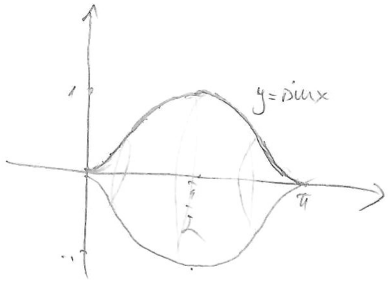
5. b).

5. b) $y = \sin x$ na $[0, \pi]$

Skica:



$$V = \pi \int_0^{\pi} f(x)^2 \, dx = \pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1-\cos 2x}{2} \, dx$$

$$= \pi \left(\frac{1}{2} x - \frac{1}{4} \sin 2x \right) \Big|_0^{\pi}$$

$$= \pi \cdot \frac{1}{2} \pi = \frac{\pi^2}{2}.$$