Poglavlje 2
Vektori

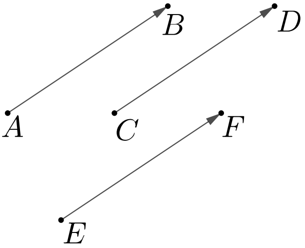
ak. god. 2021./2022.

2.1 Osnovno o vektorima

Skalarne veličine - one veličine određene jednim brojem (npr. duljina, površina, masa, toplina...)

Vektori - za silu, akceleraciju je osim broja potrebno znati i smjer.

Orijentirana (usmjerena) dužina $\overrightarrow{AB}$ je dužina za koju se zna početna točka A i završna točka B.



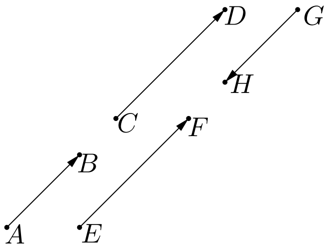
$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF}$ su ekvivalentne usmjerene dužine

Definicija (1.1.)

Skup svih međusobno ekvivalentnih usmjerenih dužina nazivamo **vektorom**.

Geometrijski, vektor je zadan:

1. **duljinom** ili modulom
2. pravcem nosiocem na kojem vektor leži, tj. **smjerom**
3. **orijentacijom**



$|\overrightarrow{AB}| = |AB|$

$\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}, \overrightarrow{GH}$ su vektori istog smjera.
(leže na istom ili na paralelnim pravcima)

$\overrightarrow{AB}, \overrightarrow{CD}$ i $\overrightarrow{EF}$ su iste orijentacije.

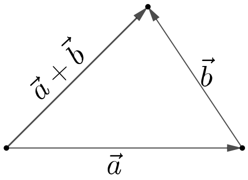
$\overrightarrow{AB}$ i $\overrightarrow{GH}$ su suprotne orijentacije.

Dva su vektora **jednaka** ako imaju istu duljinu, smjer i orijentaciju.

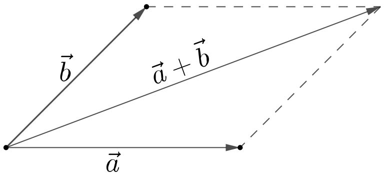
$\overrightarrow{AA} = \overrightarrow{BB} = \overrightarrow{0}$ ... nulvektor (vektor kojemu je početna točka ujedno i krajnja)

Zbrajanje vektora

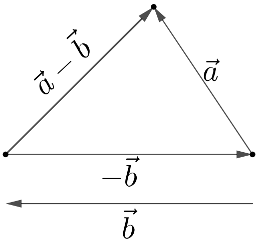
Pravilo trokuta:



Pravilo paralelograma:



Oduzimanje vektora



$$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$$

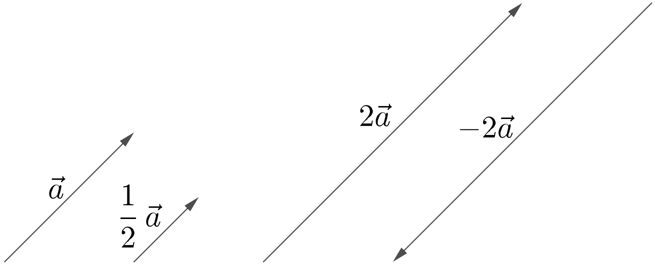
Svojstva zbrajanja i oduzimanja

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{-a}) = \vec{-a} + \vec{a} = \vec{0}$
3. $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$
4. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Množenje vektora skalarom

Neka je $\vec{a}$ vektor i $\lambda \in \mathbb{R}$.

1. $\vec{a}$ i $\lambda\vec{a}$ su kolinearni.
2. $|\lambda\vec{a}| = |\lambda||\vec{a}|$
3. Ako je
    1. $\lambda > 0$, tada su $\vec{a}$ i $\lambda\vec{a}$ iste orijentacije.
    2. $\lambda < 0$, tada su $\vec{a}$ i $\lambda\vec{a}$ suprotne orijentacije.



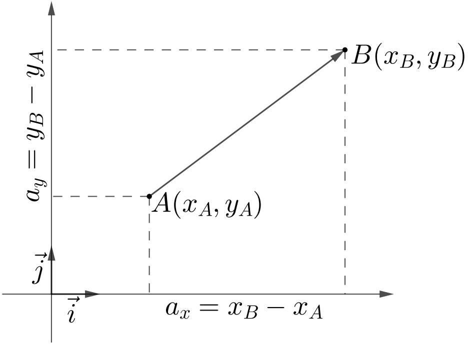
Svojstva množenja vektora skalarom

1. $\lambda(\vec{a}+\vec{b}) = \lambda\vec{a}+\lambda\vec{b}$
2. $(\lambda+\mu)\vec{a} = (\lambda\vec{a}) + \mu\vec{a}$
3. $(\lambda\mu)\vec{a} = \lambda(\mu\vec{a}) = \lambda\mu\vec{a}$
4. $1\cdot\vec{a} = \vec{a}, (-1)\cdot\vec{a} = -\vec{a}, 0\cdot\vec{a} = \vec{0}$

Prikaz vektora u koordinatnom sustavu

$\vec{i}, \vec{j}$...jedinični vektori na koordinatnim osima

$$\begin{aligned} \vec{a} &= \overrightarrow{AB} \\ &= a_x \vec{i} + a_y \vec{j} \\ &= (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} \\ |\vec{a}| &= \sqrt{a_x^2 + a_y^2} \end{aligned}$$

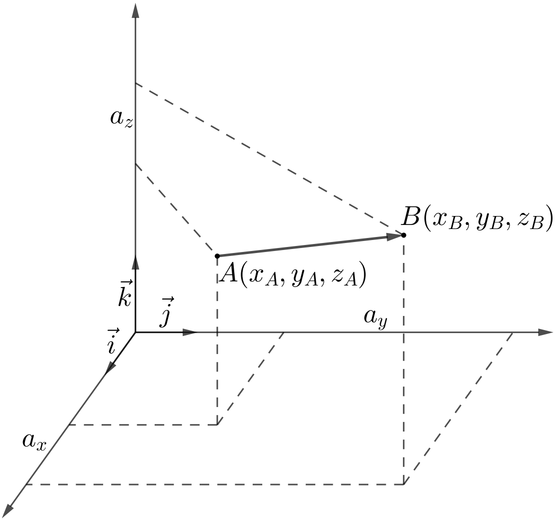


$a_x = x_B - x_A$ i $a_y = y_B - y_A$...skalarne komponente vektora

$\vec{i}, \vec{j}$...vektorske komponente vektora

$$\vec{a} = \overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$



Zadatak (1.1.)

Odredite $\overrightarrow{AB}$ i $|\overrightarrow{AB}|$ ako je:

a) $A(-1,1), B(2,3)$

b) $A(1,1,1), B(4,5,7)$

Rješenje: a)

$$\begin{aligned}\vec{a} &= \overrightarrow{AB} \\ &= (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} \\ &= 3\vec{i} + 2\vec{j}\end{aligned}$$

$$\begin{aligned}|\vec{a}| &= \sqrt{a_x^2 + a_y^2} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13}\end{aligned}$$

Rješenje: b)

$$\begin{aligned}
\vec{a} &= \overrightarrow{AB} \\
&= (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k} \\
&= 3\vec{i} + 4\vec{j} + 6\vec{k}
\end{aligned}$$

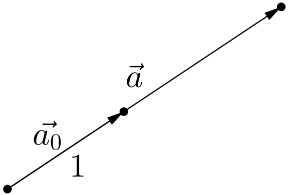
$$\begin{aligned}
|\vec{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\
&= \sqrt{3^2 + 4^2 + 6^2} \\
&= \sqrt{61}
\end{aligned}$$

Jedinični vektor

Jedinični vektor vektora $\vec{a}$ je vektor $\vec{a_0}$

istog smjera i orijentacije kao $\vec{a}$, a duljine 1.

$$\vec{a_0} = \frac{\vec{a}}{|\vec{a}|}$$



Zadatak (1.2.)

Dan je vektor $\vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}$. Odredite $\vec{a_0}$.

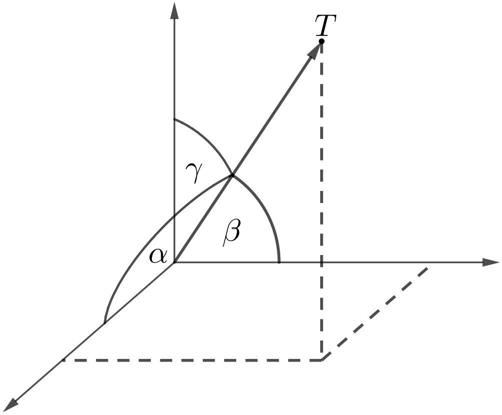
Rješenje:

$$
\begin{aligned}
|\vec{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\
&= \sqrt{2^2 + 2^2 + 1^2} \\
&= \sqrt{9} = 3
\end{aligned}
$$

$$
\begin{aligned}
\vec{a_0} &= \frac{\vec{a}}{|\vec{a}|} \\
&= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}
\end{aligned}
$$

Za vektor $\vec{a_0} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$, skalari uz $\vec{i}, \vec{j}$ i $\vec{k}$ su kosinusi smjera tj.

$$\cos\alpha = \frac{2}{3}, \cos\beta = \frac{2}{3}, \cos\gamma = \frac{1}{3}.$$



Zadatak (1.3.)

Ako je $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}, \vec{b} = \vec{i} - \vec{j} + 2\vec{k}$ izračunajte $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{a}$, $|\vec{a}|$, $\vec{a_0}$, te kosinuse smjera od $\vec{a}$.

$$\begin{aligned}
\vec{a} + \vec{b} &= 2\vec{i} + \vec{j} + 3\vec{k} \\
\vec{a} - \vec{b} &= 3\vec{j} - \vec{k} \\
3\vec{a} &= 3\vec{i} + 6\vec{j} + 3\vec{k} \\
|\vec{a}| &= \sqrt{1+4+1} \\
&= \sqrt{6}
\end{aligned}$$

$$\begin{aligned}
\vec{a_0} &= \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k} \\
&= \frac{\sqrt{6}}{6}\vec{i} + \frac{\sqrt{6}}{3}\vec{j} + \frac{\sqrt{6}}{6}\vec{k}
\end{aligned}$$

Kosinusi smjera: $\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}$ i $\frac{\sqrt{6}}{6}$

Zadatak (1.4.)

Dane su točke $A(2,2,0)$ i $B(0,-2,5)$. Pronađite $\overrightarrow{AB}$, $|\overrightarrow{AB}|$ i $\overrightarrow{AB}_0$.

Rješenje:

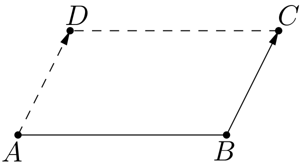
$$\begin{aligned}
\overrightarrow{AB} &= -2\vec{i} - 4\vec{j} + 5\vec{k} \\
|\overrightarrow{AB}| &= \sqrt{4+16+25} \\
&= \sqrt{45} \\
&= 3\sqrt{5}
\end{aligned}$$

$$\overrightarrow{AB}_0 = \frac{-2\sqrt{5}}{15}\vec{i} - \frac{4\sqrt{5}}{15}\vec{j} + \frac{\sqrt{5}}{3}\vec{k}$$

Zadatak (1.5.)

Dana su redom tri uzastopna vrha paralelograma ABCD: $A(1,-2,0)$, $B(2,1,3)$ i $C(-2,0,5)$. Odredite vrh D i duljinu dijagonale $\overrightarrow{BD}$.

Rješenje:



$$D(x,y,z)$$

$$\overrightarrow{AD} = \overrightarrow{BC}$$

$$(x-1)\vec{i} + (y+2)\vec{j} + z\vec{k} = -4\vec{i} - \vec{j} + 2\vec{k}$$

$$\begin{array}{lll} x-1=-4 & y+2=-1 & z=2 \\ x=-3 & y=-3 & \end{array}$$

$$D(-3,-3,2)$$

$$\overrightarrow{BD} = -5\vec{i} - 4\vec{j} - \vec{k}$$

$$|\overrightarrow{BD}| = \sqrt{25+16+1} = \sqrt{42}$$

2.2 Linearna nezavisnost vektora

Neka su $\vec{a_1}, \vec{a_2}, ..., \vec{a_n}$ vektori i $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ skalari.

Za relaciju: $\alpha_1 \vec{a_1} + \alpha_2 \vec{a_2} + ... + \alpha_n \vec{a_n} = \vec{0}$ uvijek postoji trivijalno rješenje $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$.

Ako je to ujedno i jedino rješenje, onda kažemo da su $\vec{a_1}, \vec{a_2}, ..., \vec{a_n}$ linearno nezavisni.

Definicija (1.2.)

Neka su $\vec{a_1}, \vec{a_2}, ..., \vec{a_n}$ vektori i $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ skalari. Ako vrijedi:

$$\alpha_1 \vec{a_1} + \alpha_2 \vec{a_2} + ... + \alpha_n \vec{a_n} = \vec{0} \iff \alpha_1 = \alpha_2 = ... = \alpha_n = 0,$$

onda su vektori $\vec{a_1}, \vec{a_2}, ..., \vec{a_n}$ **linearno nezavisni**.

1. $n = 1$: $\vec{a}$ je linearno nezavisan $\iff$ $\vec{a} \neq \vec{0}$.
2. $n = 2$: $\vec{a}$ i $\vec{b}$ su linearno zavisni $\iff$ $\exists \lambda \in \mathbb{R}, \lambda \neq 0$ tako da je $\vec{a} = \lambda \vec{b}$. $\vec{a}$ i $\vec{b}$ su kolinearni.
3. $n = 3$: $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno zavisni $\iff$ $\exists \alpha, \beta \in \mathbb{R}$, barem jedan $\neq 0$ tako da je $\vec{c} = \alpha \vec{a} + \beta \vec{b}$. $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni.

Geometrijski:

1. 2 vektora su linearno nezavisna akko **ne** leže na istom niti na paralelnim pravcima.
2. 3 vektora su linearno nezavisna akko **ne** leže u istoj ravnini niti u paralelnim ravninama.

Zadatak (1.6.)

Odredite $\lambda \in \mathbb{R}$ takav da vektori $\vec{a} = 2\vec{i} + 3\vec{j}$ i $\vec{b} = \lambda\vec{i} - \vec{j}$ budu kolinearni.

Rješenje: $\vec{a}$ i $\vec{b}$ kolinearni $\Rightarrow$ linearno zavisni $\Rightarrow \exists \alpha \in \mathbb{R}$ tako da je $\vec{a} = \alpha\vec{b}$.

$$2\vec{i} + 3\vec{j} = \alpha\lambda\vec{i} - \alpha\vec{j}$$

$$\alpha = -3, \lambda = \frac{-2}{3}$$

$$\vec{b} = \frac{-2}{3}\vec{i} - \vec{j}$$

$$\vec{a} = -3\vec{b}$$

Zadatak (1.7.)

Ispitajte linearnu (ne)zavisnost vektora:

a) $\vec{a} = 4\vec{i} + 2\vec{j} + 5\vec{k}$
$\vec{b} = -\vec{j} + \vec{k}$
$\vec{c} = 5\vec{k}$

b) $\vec{a} = \vec{i} + \vec{j}$
$\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$
$\vec{c} = 3\vec{j} + \vec{k}$

Rješenje: a)

$$
\begin{aligned}
\vec{0} &= \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} \\
&= \alpha(4\vec{i} + 2\vec{j} + 5\vec{k}) + \beta(-\vec{j} + \vec{k}) + \gamma(5\vec{k}) = \\
&= 4\alpha\vec{i} + (2\alpha - \beta)\vec{j} + (5\alpha + \beta + 5\gamma)\vec{k}
\end{aligned}
$$

$4\alpha = 0 \Rightarrow \alpha = 0$

$2\alpha - \beta = 0 \Rightarrow \beta = 0$

$5\alpha + \beta + 5\gamma = 0 \Rightarrow \gamma = 0$

$\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno nezavisni.

b)

$$\begin{aligned}\vec{0} &= \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} \\ &= (\alpha - \beta)\vec{i} + (\alpha + 2\beta + 3\gamma)\vec{j} + (\beta + \gamma)\vec{k}\end{aligned}$$

$$\begin{aligned}\alpha - \beta = 0 \quad &\Rightarrow \quad \alpha = \beta \\ \alpha + 2\beta + 3\gamma = 0 \\ 3\beta + 3\gamma = 0 \quad &\Rightarrow \quad \beta = -\gamma\end{aligned}$$

$$\alpha = \beta = -\gamma, \quad \gamma \in \mathbb{R}$$

Sustav ima beskonačno mnogo rješenja, pa su vektori linearno zavisni. $(\vec{c} = \vec{a} + \vec{b})$.

Zadatak (1.8.)

Zadani su vektori:

$\vec{a} = \vec{i} + (2\lambda + 1)\vec{j}$

$\vec{b} = 2\vec{i} + \lambda\vec{j} + 2\vec{k}$

$\vec{c} = \vec{i} + \vec{j} + \vec{k}$

Odredite $\lambda \in \mathbb{R}$ tako da zadani vektori budu komplanarni i prikažite vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rješenje:

Vektori su komplanarni $\Rightarrow$ linearno zavisni $\Rightarrow \exists \alpha, \beta \in \mathbb{R}$ tako da je $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

$$
\begin{aligned}
\vec{i}+\vec{j}+\vec{k} &= \alpha\vec{i}+2\alpha\lambda\vec{j}+\alpha\vec{j}+2\beta\vec{i}+\lambda\beta\vec{j}+2\beta\vec{k} \\
&= (\alpha+2\beta)\vec{i}+(2\alpha\lambda+\alpha+\lambda\beta)\vec{j}+2\beta\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
\alpha+2\beta &= 1 &\quad (1)\\
2\alpha\lambda+\alpha+\lambda\beta &= 1 &\quad (2)\\
2\beta &= 1 &\quad (3)
\end{aligned}
$$

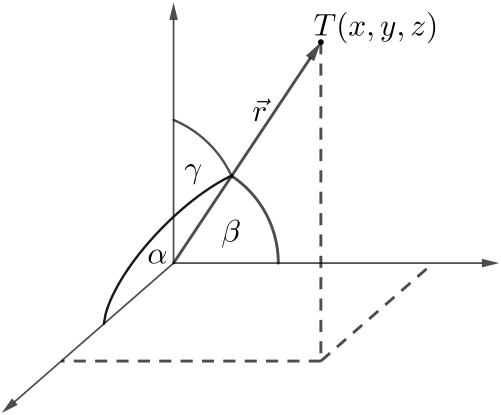
$$
\begin{array}{lcl}
(3) & \Rightarrow & \beta = \frac{1}{2} \\
(1) & \Rightarrow & \alpha = 0 \\
(2) & \Rightarrow & \lambda = 2
\end{array}
\Longrightarrow
\begin{array}{rcl}
\vec{a} &=& \vec{i}+5\vec{j} \\
\vec{b} &=& 2\vec{i}+2\vec{j}+2\vec{k} \\
\vec{c} &=& \frac{1}{2}\vec{b}
\end{array}
$$

Radij-vektor

Radij-vektor je vektor s početnom točkom u ishodištu

$\vec{r} = \overrightarrow{OT} = x\vec{i} + y\vec{j} + z\vec{k}$

$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$...duljina radij-vektora



Kosinusi smjera radijvektora

$\alpha = \angle(\vec{r}, \vec{i}), \beta = \angle(\vec{r}, \vec{j}), \gamma = \angle(\vec{r}, \vec{k})$

$$\cos\alpha = \frac{x}{|\vec{r}|}, \cos\beta = \frac{y}{|\vec{r}|}, \cos\gamma = \frac{z}{|\vec{r}|}$$

Zadatak (1.9.)

Odredite duljinu vektora $\vec{a} = 20\vec{i} + 30\vec{j} - 60\vec{k}$ i kosinuse smjera tog radij-vektora.

Rješenje:

$$
\begin{aligned}
|\vec{a}| &= \sqrt{20^2 + 30^2 + 60^2} \\
&= \sqrt{400 + 900 + 3600} \\
&= \sqrt{4900} \\
&= 70
\end{aligned}
$$

$$\cos\alpha = \frac{2}{7} \Rightarrow \alpha = 73.4^\circ$$

$$\cos\beta = \frac{3}{7} \Rightarrow \beta = 64.6^\circ$$

$$\cos\gamma = \frac{6}{7} \Rightarrow \gamma = 149^\circ$$

Zadatak (1.10.)

Radij-vektor točke M zatvara sa osi y kut od 60°, a sa osi z kut od 45°. Njegova duljina iznosi 8. Odredite koordinate točke M, ako je apscisa negativna.

Rješenje: $\beta = \angle(\overrightarrow{OM}, \vec{j}) = 60^\circ$, $\gamma = \angle(\overrightarrow{OM}, \vec{k}) = 45^\circ$, $|\overrightarrow{OM}| = 8$

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}, x < 0$$

$$\cos 60^\circ = \frac{y}{|\overrightarrow{OM}|}$$

$$\frac{1}{2} = \frac{y}{8} \Rightarrow y = 4$$

$$\cos 45^\circ = \frac{z}{|\overrightarrow{OM}|}$$

$$\frac{\sqrt{2}}{2} = \frac{z}{8} \Rightarrow z = 4\sqrt{2}$$

$$|\overrightarrow{OM}| = \sqrt{x^2 + y^2 + z^2} /^2$$

$$\begin{aligned} x^2 + y^2 + z^2 &= 8^2 \\ x^2 + 4^2 + (4\sqrt{2})^2 &= 64 \end{aligned}$$

$$x^2 = 16 \Longrightarrow x = -4$$

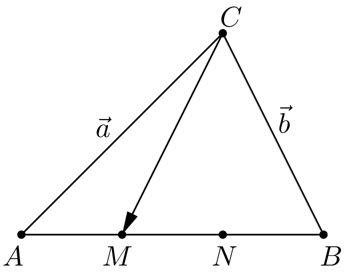
$$\overrightarrow{OM} = -4\vec{i} + 4\vec{j} + 4\sqrt{2}\vec{k}$$

$$M(-4, 4, 4\sqrt{2})$$

Zadatak (1.11.)

U trokutu ABC stranica $\overline{AB}$ je točkama M i N podijeljena na 3 jednaka dijela tako da je $|AM| = |MN| = |NB|$. Odredite vektor $\overrightarrow{CM}$ ako je $\overrightarrow{CA} = \vec{a}$ i $\overrightarrow{CB} = \vec{b}$.

Rješenje:



$$
\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{AC} + \overrightarrow{CB} \\
&= \overrightarrow{CB} + \overrightarrow{AC} \\
&= \overrightarrow{CB} - \overrightarrow{CA} \\
&= \vec{b} - \vec{a}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{AM} &= \frac{1}{3}\overrightarrow{AB} \\
&= \frac{\vec{b} - \vec{a}}{3}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{CM} &= \overrightarrow{CA} + \overrightarrow{AM} \\
&= \vec{a} + \frac{\vec{b} - \vec{a}}{3} \\
&= \frac{2\vec{a} + \vec{b}}{3}
\end{aligned}
$$

2.3 Skalarni produkt vektora

Definicija (1.3.)

Za vektore $\vec{a}$ i $\vec{b}$ definiramo $f : V \times V \to \mathbb{R}$:

$$\vec{a} \cdot \vec{b} := |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi,$$

gdje je $\varphi = \angle(\vec{a}, \vec{b})$.

Svojstva:

1. $\vec{a} \cdot \vec{a} \geqslant 0$, $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}$...nenegativnost
2. $\lambda(\vec{a} \cdot \vec{b}) = (\lambda\vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda\vec{b})$...homogenost
3. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$...komutativnost
4. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$...distributivnost

Uvjet okomitosti: $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$

$$
\begin{aligned}
\vec{i}\cdot\vec{i} &= \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = |1|\cdot|1|\cdot\cos 0^\circ = 1\\
\vec{i}\cdot\vec{j} &= \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = |1|\cdot|1|\cdot\cos 90^\circ = 0
\end{aligned}
$$

$$
\begin{aligned}
\vec{a} &= x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\\
\vec{b} &= x_2\vec{i} + y_2\vec{j} + z_2\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
\vec{a}\cdot\vec{b} &= \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right)\cdot\left(x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\right)\\
&= x_1x_2\,\vec{i}\cdot\vec{i} + x_1y_2\,\vec{i}\cdot\vec{j} + x_1z_2\,\vec{i}\cdot\vec{k} + y_1x_2\,\vec{j}\cdot\vec{i} + y_1y_2\,\vec{j}\cdot\vec{j}\\
&\quad + y_1z_2\,\vec{j}\cdot\vec{k} + z_1x_2\,\vec{k}\cdot\vec{i} + z_1y_2\,\vec{k}\cdot\vec{j} + z_1z_2\,\vec{k}\cdot\vec{k}\\
&= x_1x_2 + y_1y_2 + z_1z_2
\end{aligned}
$$

$$
\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2
$$

$$\vec{a} \cdot \vec{a} = x_1^2 + y_1^2 + z_1^2 = |\vec{a}|^2$$

$$\cos \varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{(x_1^2 + y_1^2 + z_1^2) \cdot (x_2^2 + y_2^2 + z_2^2)}}, \varphi \in [0, \pi]$$

Ortogonalna projekcija vektora $\vec{a}$ na vektor $\vec{b}$

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

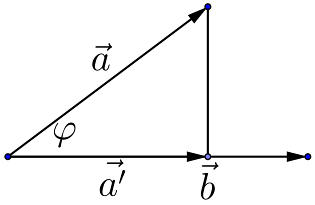
$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

$$|\vec{a'}| = \cos\varphi \cdot |\vec{a}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\cdot|\vec{a}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$\vec{a'} = |\vec{a'}|\cdot\frac{\vec{b}}{|\vec{b}|}$$

$$\mathrm{SkPr}\vec{a}_{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$\mathrm{VekPr}\vec{a}_{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\frac{\vec{b}}{|\vec{b}|}$$



Zadatak (1.12.)

Odredite skalarni produkt vektora $\vec{a} = 3\vec{i} + 4\vec{j} + 7\vec{k}$ i $\vec{b} = 2\vec{i} - 5\vec{j} + 2\vec{k}$.

Rješenje:

$$\vec{a} \cdot \vec{b} = 3 \cdot 2 + 4 \cdot (-5) + 7 \cdot 2 = 6 - 20 + 14 = 0 \quad \Rightarrow \quad \vec{a} \perp \vec{b}.$$

Zadatak (1.13.)

Zadani su vektori $\vec{a} = m\vec{i} + 3\vec{j} + 4\vec{k}$ i $\vec{b} = 4\vec{i} + m\vec{j} - 7\vec{k}$. Odredite konstantu $m \in \mathbb{R}$ tako da vektori $\vec{a}$ i $\vec{b}$ budu okomiti.

Rješenje:

$$0 = \vec{a} \cdot \vec{b} = 4m + 3m - 28 \quad \Rightarrow \quad 7m = 28 \Rightarrow m = 4.$$

Zadatak (1.14.)

Nađite skalarnu i vektorsku projekciju vektora $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ na vektor $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$.

Rješenje:

$$\mathrm{SkPr}\vec{a}_{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2+1-2}{\sqrt{4+1+4}} = \frac{1}{3}$$

$$\mathrm{VekPr}\vec{a}_{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\frac{\vec{b}}{|\vec{b}|} = \frac{1}{3}\cdot\frac{2\vec{i}+\vec{j}-2\vec{k}}{3} = \frac{2}{9}\vec{i} + \frac{1}{9}\vec{j} - \frac{2}{9}\vec{k}$$

Zadatak (1.15.)

Zadani su vektori $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ i $\vec{b} = 6\vec{i} + 4\vec{j} - 2\vec{k}$. Odredite:

a) kosinus kuta između vektora $\vec{a}$ i $\vec{b}$

b) ortogonalnu projekciju vektora $\vec{a}$ na vektor $\vec{b}$.

Rješenje: a)

$$\varphi = \angle(\vec{a}, \vec{b})$$

$$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{36+16+4} = \sqrt{56}$$

$$\begin{aligned}\cos\varphi &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\\ &= \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{(x_1^2+y_1^2+z_1^2)\cdot(x_2^2+y_2^2+z_2^2)}}\\ &= \frac{6+8-6}{\sqrt{14\cdot 56}}\\ &= \frac{2}{7}\end{aligned}$$

Rješenje: b)

$$\begin{aligned}
\vec{a'} &= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\frac{\vec{b}}{|\vec{b}|}\\
&= \frac{8}{\sqrt{56}}\cdot\frac{6\vec{i}+4\vec{j}-2\vec{k}}{\sqrt{56}}\\
&= \frac{6}{7}\vec{i}+\frac{4}{7}\vec{j}-\frac{2}{7}\vec{k}
\end{aligned}$$

Zadatak (1.16.)

Koji kut zatvaraju jedinični vektori $\vec{m}$ i $\vec{n}$ ako su vektori $\vec{p} = \vec{m} + 2\vec{n}$ i $\vec{q} = 5\vec{m} - 4\vec{n}$ okomiti.

Rješenje: $|\vec{m}| = |\vec{n}| = 1$, $\vec{p} \perp \vec{q} \Rightarrow \vec{p} \cdot \vec{q} = 0$

$$
\begin{aligned}
0 &= \vec{p} \cdot \vec{q} \\
&= (\vec{m} + 2\vec{n}) \cdot (5\vec{m} - 4\vec{n}) \\
&= 5|\vec{m}|^2 - 4\vec{m} \cdot \vec{n} + 10\vec{m} \cdot \vec{n} - 8|\vec{n}|^2 \\
&= 6\vec{m} \cdot \vec{n} - 3 \Rightarrow \vec{m} \cdot \vec{n} = \tfrac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\cos\varphi &= \frac{\vec{m} \cdot \vec{n}}{|\vec{m}| \cdot |\vec{n}|} \\
&= \vec{m} \cdot \vec{n} \\
&= \frac{1}{2} \Longrightarrow \varphi = 60^\circ
\end{aligned}
$$

Zadatak (1.17.)

Neka je $|\vec{a}| = 13$, $|\vec{b}| = 19$, $|\vec{a} + \vec{b}| = 24$. Odredite $|\vec{a} - \vec{b}|$.

Rješenje:

$$|\vec{a} + \vec{b}|^2 = 24^2 = 576$$

$$576 = \left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} + \vec{b}\right) = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$2\vec{a} \cdot \vec{b} = 576 - 169 - 361 = 46 \Rightarrow \vec{a} \cdot \vec{b} = 23$$

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a} - \vec{b}|^2} = \sqrt{\left(\vec{a} - \vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right)} = \sqrt{|\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2}$$

$$|\vec{a} - \vec{b}| = \sqrt{169 - 46 + 361} = \sqrt{484} = 22$$

Zadatak (1.18.)

Dokažite da su $\vec{a}$ i $\vec{b}$ ortogonalni(okomiti) ako vrijedi: $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$.

Rješenje:

$$
\begin{aligned}
|\vec{a}+\vec{b}| &= |\vec{a}-\vec{b}| /^2 \\
|\vec{a}+\vec{b}|^2 &= |\vec{a}-\vec{b}|^2 \\
\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right) &= \left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right) \\
|\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 &= |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \\
4\vec{a}\cdot\vec{b} &= 0 \\
\vec{a}\cdot\vec{b} &= 0 \Longrightarrow \vec{a}\perp\vec{b}
\end{aligned}
$$

Zadatak (1.19.)

Odredite jedinični vektor $\vec{n}_0$ komplanaran s vektorima $\vec{p}$ i $\vec{q}$ ako je $|\vec{p}| = 2$, $|\vec{q}| = 3$, $\angle(\vec{p}, \vec{q}) = \frac{\pi}{3}$, $\vec{n} \cdot \vec{p} = 7$ i $\vec{n} \cdot \vec{q} = 3$.

Rješenje:

$\vec{n}$ komplanaran s $\vec{p}$ i $\vec{q} \Rightarrow \exists \alpha, \beta \in \mathbb{R}$ tako da je $\vec{n} = \alpha\vec{p} + \beta\vec{q}$

$\vec{p} \cdot \vec{q} = |\vec{p}| \cdot |\vec{q}| \cdot \cos\frac{\pi}{3} = 3$

$$\begin{aligned}
\vec{n} \cdot \vec{p} &= 7 \\
(\alpha\vec{p} + \beta\vec{q}) \cdot \vec{p} &= 7 \\
\alpha|\vec{p}|^2 + \beta\vec{p} \cdot \vec{q} &= 7 \\
4\alpha + 3\beta &= 7
\end{aligned}$$

$$\begin{aligned}
\vec{n} \cdot \vec{q} &= 3 \\
(\alpha\vec{p} + \beta\vec{q}) \cdot \vec{q} &= 3 \\
\alpha\vec{p} \cdot \vec{q} + \beta|\vec{q}|^2 &= 3 \\
3\alpha + 9\beta &= 3 / : 3 \\
\alpha + 3\beta &= 1
\end{aligned}$$

$$
\begin{array}{rcll}
4\alpha + 3\beta & = & 7 & \\
\alpha + 3\beta & = & 1 & \quad / \cdot (-4) \\
\hline
4\alpha + 3\beta & = & 7 & \\
-4\alpha - 12\beta & = & -4 & \\
\hline
-9\beta & = & 3 & \\
\beta & = & -\dfrac{1}{3} & \\
\alpha & = & 1 - 3\beta & \\
\alpha & = & 1 - 3\left(-\dfrac{1}{3}\right) & \\
\alpha & = & 2 &
\end{array}
$$

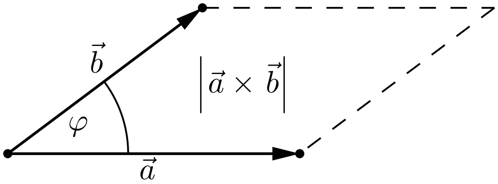
$$
\begin{aligned}
\vec{n} &= 2\vec{p} - \frac{1}{3}\vec{q} \\
|\vec{n}| &= \sqrt{4|\vec{p}|^2 - \tfrac{4}{3}\vec{p}\cdot\vec{q} + \tfrac{1}{9}|\vec{q}|^2} \\
&= \sqrt{16 - 4 + 1} \\
&= \sqrt{13} \\
\vec{n_0} &= \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{13}} \cdot \left(2\vec{p} - \frac{1}{3}\vec{q}\right)
\end{aligned}
$$

2.4 Vektorski produkt vektora

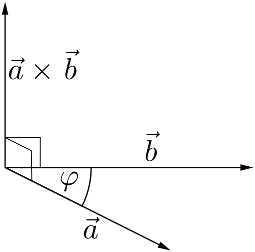
$f : V \times V \to V$...rezultat je vektor
$\left(\vec{a}, \vec{b}\right) \longmapsto \vec{a} \times \vec{b}$

Duljina: Za $\vec{a} \times \vec{b}$ vrijedi $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$

Geometrijski, duljina vektorskog produkta jednaka je površini paralelograma što ga razapinju vektori $\vec{a}$ i $\vec{b}$.



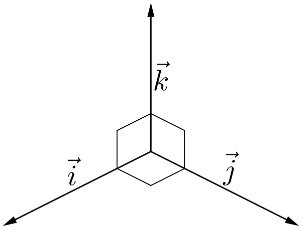
Smjer: Za nekolinearne vektore $\vec{a}$ i $\vec{b}$ vrijedi: $\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$



Ako su $\vec{a}$ i $\vec{b}$ kolinearni $\Rightarrow \vec{a} \times \vec{b} = \vec{0}$

Ako je $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a}$ i $\vec{b}$ su kolinearni ili je jedan od njih $\vec{0}$.

Orijentacija: $\left(\vec{a}, \vec{b}, \vec{a}\times\vec{b}\right)$ čine desnu trojku, tj. gledano iz vrha vektora $\vec{a}\times\vec{b}$ rotacija iz $\vec{a}$ u $\vec{b}$ suprotna je gibanju kazaljke na satu.



$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

Svojstva:

1. $\vec{a} \times \vec{a} = \vec{0}$
2. $\lambda\left(\vec{a} \times \vec{b}\right) = \lambda\vec{a} \times \vec{b} = \vec{a} \times \lambda\vec{b}$...homogenost
3. $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$...antikomutativnost
4. $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$...distributivnost

Napomena: asocijativnost ne vrijedi, tj. $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\
&= (y_1 z_2 - y_2 z_1)\,\vec{i} - (x_1 z_2 - x_2 z_1)\,\vec{j} + (x_1 y_2 - x_2 y_1)\,\vec{k}
\end{aligned}$$

Zadatak (1.20.)

Odredite vektorski produkt vektora $\vec{a} = 2\vec{i} + 3\vec{j} + 5\vec{k}$ i $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$ i površinu paralelograma određenog tim vektorima.

Rješenje:

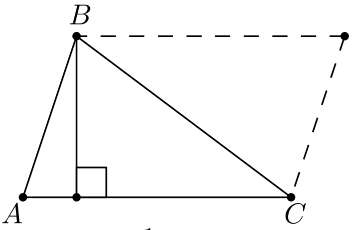
$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 5 \\ 1 & 2 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\
&= (3 \cdot 1 - 2 \cdot 5) \cdot \vec{i} - (2 \cdot 1 - 1 \cdot 5) \cdot \vec{j} + (2 \cdot 2 - 1 \cdot 3) \cdot \vec{k} \\
&= -7\vec{i} + 3\vec{j} + \vec{k}
\end{aligned}$$

$$P = |\vec{a} \times \vec{b}| = \sqrt{49 + 9 + 1} = \sqrt{59}$$

Zadatak (1.21.)

Točke $A(1,1,-1)$, $B(2,0,1)$ i $C(0,-1,1)$ su vrhovi trokuta. Korištenjem vektorskog računa izračunajte visinu trokuta spuštenog iz vrha B.

Rješenje:



$$P_\Delta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$P_\Delta = \frac{1}{2} v \cdot |\overrightarrow{AC}|$$

$$\overrightarrow{AB} = \vec{i} - \vec{j} + 2\vec{k}$$

$$\overrightarrow{AC} = -\vec{i} - 2\vec{j} + 2\vec{k}$$

$$\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ -1 & -2 & 2 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} \\
&= (-2-(-4)) \cdot \vec{i} - (2-(-2)) \cdot \vec{j} + (-2-1)) \cdot \vec{k} \\
&= 2\vec{i} - 4\vec{j} - 3\vec{k}
\end{aligned}$$

$$\begin{aligned}
|\overrightarrow{AB} \times \overrightarrow{AC}| &= \sqrt{4+16+9} \\
&= \sqrt{29}
\end{aligned}$$

$$\begin{aligned}
|\overrightarrow{AC}| &= \sqrt{1+4+4} \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| &= \frac{1}{2}v \cdot |\overrightarrow{AC}| \\
v &= \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AC}|} \\
&= \frac{\sqrt{29}}{3}
\end{aligned}$$

Zadatak (1.22.)

Odredite površinu paralelograma određenog vektorima $\vec{a}+3\vec{b}$ i $3\vec{a}+\vec{b}$, ako je $|\vec{a}|=|\vec{b}|=1$ i $\angle\left(\vec{a},\vec{b}\right)=30^\circ$

Rješenje:

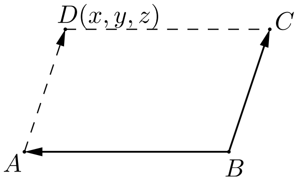
$$\begin{aligned}
\left(\vec{a}+3\vec{b}\right)\times\left(3\vec{a}+\vec{b}\right) &= 3\underbrace{\vec{a}\times\vec{a}}_{\vec{0}}+\vec{a}\times\vec{b}+\underbrace{9\vec{b}\times\vec{a}}_{-9\vec{a}\times\vec{b}}+3\underbrace{\vec{b}\times\vec{b}}_{\vec{0}}\\
&= \vec{a}\times\vec{b}-9\vec{a}\times\vec{b}\\
&= -8\vec{a}\times\vec{b}
\end{aligned}$$

$$\begin{aligned}
P &= |-8\vec{a}\times\vec{b}|\\
&= 8|\vec{a}\times\vec{b}|\\
&= 8\cdot\underbrace{|\vec{a}|}_{1}\cdot\underbrace{|\vec{b}|}_{1}\cdot\sin\angle\left(\vec{a},\vec{b}\right)\\
&= 8\cdot\sin 30^\circ\\
&= 8\cdot\tfrac{1}{2}\\
&= 4
\end{aligned}$$

Zadatak (1.23.)

Neka su $A(1,-2,3)$, $B(3,2,1)$ i $C(6,4,4)$ vrhovi paralelograma $ABCD$. Odredite površinu P paralelograma i koordinate vrha D.

Rješenje:



$$\overrightarrow{BA} = -2\vec{i} - 4\vec{j} + 2\vec{k}$$
$$\overrightarrow{BC} = 3\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\begin{aligned}
\overrightarrow{BA} \times \overrightarrow{BC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -4 & 2 \\ 3 & 2 & 3 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -4 & 2 \\ 2 & 3 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} -2 & 2 \\ 3 & 3 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} -2 & -4 \\ 3 & 2 \end{vmatrix} \\
&= -16\vec{i} + 12\vec{j} + 8\vec{k}
\end{aligned}$$

$$
\begin{aligned}
P &= |\overrightarrow{BA} \times \overrightarrow{BC}| \\
&= \sqrt{16^2 + 12^2 + 8^2} \\
&= \sqrt{464} = 4\sqrt{29}
\end{aligned}
$$

$\overrightarrow{AD} = \overrightarrow{BC}$

$$(x-1)\,\vec{i} + (y+2)\,\vec{j} + (z-3)\,\vec{k} = 3\vec{i} + 2\vec{j} + 3\vec{k}$$

$$
\begin{array}{lll}
x - 1 = 3 & y + 2 = 2 & z - 3 = 3 \\
x = 4 & y = 0 & z = 6
\end{array}
$$

Imamo da je $D(4,0,6)$.

2.5 Mješoviti produkt vektora

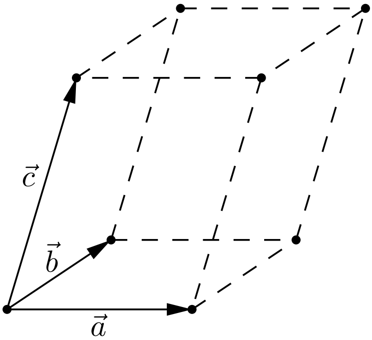
$$\left(\vec{a},\vec{b},\vec{c}\right) \longmapsto \underbrace{\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}}_{Oznaka:\left[\vec{a},\vec{b},\vec{c}\right] ili \left(\vec{a},\vec{b},\vec{c}\right)}$$

$f : V \times V \times V \to \mathbb{R}$...rezultat je skalar

Svojstva:

1. $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \left(\vec{b}\times\vec{c}\right)\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b} = \left(\vec{a},\vec{b},\vec{c}\right)$
2. $\left(\vec{b}\times\vec{a}\right)\cdot\vec{c} = (\vec{a}\times\vec{c})\cdot\vec{b} = \left(\vec{c}\times\vec{b}\right)\cdot\vec{a} = -\left(\vec{a},\vec{b},\vec{c}\right)$

Geometrijska interpretacija:



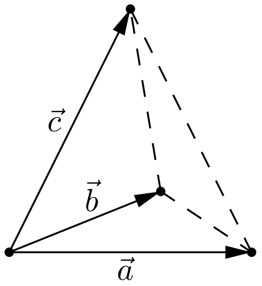
$$V = \left| \left(\vec{a} \times \vec{b} \right) \cdot \vec{c} \right|$$

Apsolutna vrijednost mješovitog produkta triju nekomplanarnih vektora je jednaka obujmu paralelepipeda što ga razapinju ti vektori.

$$\left.\begin{aligned}\vec{a} &= x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\\ \vec{b} &= x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\\ \vec{c} &= x_3\vec{i} + y_3\vec{j} + z_3\vec{k}\end{aligned}\right\} \Rightarrow \left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= x_1 \cdot \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - y_1 \cdot \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1 \cdot \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = |\vec{a}| \cdot |\vec{b} \times \vec{c}| \cdot \cos\varphi, \varphi = \angle\left(\vec{a}, \vec{b} \times \vec{c}\right)$$



Obujam tetraedra: $V_T = \dfrac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\,\right|$

Uvjet komplanarnosti:

Za vektore $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$ vrijedi:

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = 0 \iff \vec{a}, \vec{b}, \vec{c} \text{ komplanarni.}$$

Zadatak (1.24.)

Odredi mješoviti produkt vektora

$$\vec{a} = 2\vec{i} - \vec{j} - \vec{k}$$
$$\vec{b} = \vec{i} + 3\vec{j} - \vec{k}$$
$$\vec{c} = \vec{i} + \vec{j} + 4\vec{k}.$$

Rješenje:

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} &= \begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & 4 \end{vmatrix} \\
&= 2 \cdot \begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\
&= 2 \cdot 13 + 1 \cdot 5 - 1 \cdot (-2) \\
&= 26 + 5 + 2 \\
&= 33
\end{aligned}$$

Zadatak (1.25.)

Dokažite da su vektori

$$\vec{a} = 2\vec{i} + 5\vec{j} + 7\vec{k}$$
$$\vec{b} = \vec{i} + \vec{j} - \vec{k}$$
$$\vec{c} = \vec{i} + 2\vec{j} + 2\vec{k}$$

komplanarni.

Rješenje:

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} &= \begin{vmatrix} 2 & 5 & 7 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} \\
&= 2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} - 5 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 7 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\
&= 2 \cdot 4 - 5 \cdot 3 + 7 \cdot 1 \\
&= 8 - 15 + 7 \\
&= 0
\end{aligned}$$

$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = 0 \Longrightarrow \vec{a}, \vec{b}, \vec{c}$ su komplanarni

Zadatak (1.26.)

Odredite volumen tetraedra čiji su vrhovi točke $A(2,2,2)$, $B(4,3,3)$, $C(4,5,4)$ i $D(5,5,6)$.

Rješenje: Tetraedar je razapet vektorima $\overrightarrow{AB}$,$\overrightarrow{AC}$ i $\overrightarrow{AD}$

$$\overrightarrow{AB} = 2\vec{i} + \vec{j} + \vec{k}$$
$$\overrightarrow{AC} = 2\vec{i} + 3\vec{j} + 2\vec{k}$$
$$\overrightarrow{AD} = 3\vec{i} + 3\vec{j} + 4\vec{k}$$

$$\begin{aligned} \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD} &= \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} \\ &= 2 \cdot \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} \\ &= 2 \cdot 6 - 2 - 3 \\ &= 7 \end{aligned}$$

$$V_T = \frac{1}{6} \left| \left(\vec{a} \times \vec{b}\right) \cdot \vec{c} \right| = \frac{7}{6}$$

Zadatak (1.27.)

Odredite $\alpha \in \mathbb{R}$ tako da obujam tetraedra razapet vektorima $\vec{a}$, $\vec{b}$ i $\alpha\vec{c}$ iznosi $\frac{2}{3}$, gdje je $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$ i $\vec{c} = \vec{i} - \frac{1}{3}\vec{k}$.

Rješenje:

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) \cdot \alpha\vec{c} &= \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \\ \alpha & 0 & -\frac{\alpha}{3} \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} 1 & -1 \\ 0 & -\frac{\alpha}{3} \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ \alpha & -\frac{\alpha}{3} \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 1 \\ \alpha & 0 \end{vmatrix} \\
&= -\frac{\alpha}{3} - \left(-\frac{2\alpha}{3} + \alpha\right) + 2\alpha \\
&= -\frac{\alpha}{3} - \frac{\alpha}{3} + 2\alpha \\
&= \frac{4}{3}\alpha
\end{aligned}$$

$$\begin{aligned} V_T &= \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\alpha\vec{c}\right| \\ \frac{2}{3} &= \frac{1}{6}\cdot\left|\frac{4}{3}\alpha\right| \\ 4 &= \left|\frac{4}{3}\alpha\right| \end{aligned}$$

$|\alpha| = 3 \Rightarrow \alpha = 3$ ili $\alpha = -3$

Zadatak (1.28.)

Pokažite da ako za vektore $\vec{a}$, $\vec{b}$ i $\vec{c}$ vrijedi

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0},$$

onda su $\vec{a}$, $\vec{b}$ i $\vec{c}$ komplanarni, a $\vec{d} = \vec{a} - \vec{c}$ i $\vec{e} = \vec{b} - \vec{a}$ kolinearni.

Rješenje:

$$\begin{aligned}
\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} &= \vec{0} \,/ \cdot \vec{c} \\
(\vec{a} \times \vec{b}) \cdot \vec{c} + \underbrace{(\vec{b} \times \vec{c}) \cdot \vec{c}}_{0} + \underbrace{(\vec{c} \times \vec{a}) \cdot \vec{c}}_{0} &= 0 \ (\vec{b} \times \vec{c} \perp \vec{c}, \vec{c} \times \vec{a} \perp \vec{c}) \\
(\vec{a} \times \vec{b}) \cdot \vec{c} &= 0 \Longrightarrow \text{komplanarni su}
\end{aligned}$$

$$\begin{aligned}
\vec{d} \times \vec{e} &= (\vec{a} - \vec{c}) \times \left(\vec{b} - \vec{a}\right) \\
&= \vec{a} \times \vec{b} - \underbrace{\vec{a} \times \vec{a}}_{\vec{0}} - \underbrace{\vec{c} \times \vec{b}}_{+\vec{b} \times \vec{c}} + \vec{c} \times \vec{a} \\
&= \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0} \Longrightarrow \text{kolinearni su}
\end{aligned}$$

Zadatak (1.29.)

Odredite parametar $t \in \mathbb{R}$ takav da vektori

$$\vec{a} = 3\vec{i} + 2\vec{j} + t\vec{k}$$
$$\vec{b} = -\vec{j}$$
$$\vec{c} = 4\vec{i} + \vec{j}.$$

budu komplanarni.

Rješenje:

$$\begin{aligned} 0 &= \left(\vec{a} \times \vec{b}\right) \cdot \vec{c} \\ &= \begin{vmatrix} 3 & 2 & t \\ -1 & 0 & 0 \\ 4 & 1 & 0 \end{vmatrix} \\ &= 3 \cdot \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} -1 & 0 \\ 4 & 0 \end{vmatrix} + t \cdot \begin{vmatrix} -1 & 0 \\ 4 & 1 \end{vmatrix} \\ &= 3 \cdot 0 - 2 \cdot 0 - t \\ &= -t \implies t = 0 \end{aligned}$$