Poglavlje 4
Nizovi i redovi

ak. god. 2021./2022.

4.1 Nizovi

Definicija

Svaka funkcija $f : \mathbb{N} \to A$, gdje je A neki skup, naziva se **niz**.

- $f : \mathbb{N} \to \mathbb{N}$
- $f : \mathbb{N} \to \mathbb{R}$
- $f : \mathbb{N} \to \mathbb{Q}$
- $a : \mathbb{N} \to \mathbb{R}$

Općenito, niz $a : \mathbb{N} \to \mathbb{R}$ označavat ćemo s $(a_n)_{n\in\mathbb{N}}$.

Definicija

Za svaki niz realnih brojeva $(a_n)_{n\in\mathbb{N}}$ kažemo da je **ograničen** ako:
$\exists M \in \mathbb{R}, M > 0$ takav da je $|a_n| \leq M, \forall n \in \mathbb{N}$.

Zadatak (4.1.)

Ispišite prvih nekoliko članova niza i odredite je li ograničen, ako je

a) $a_n = \dfrac{n^2}{2n+1}$

b) $b_n = (-1)^n$

c) $c_n = \dfrac{n}{n+1}$

d) $d_n = (-1)^n \dfrac{1}{n}$

e) $e_n = \dfrac{1}{n}$ (sami)

f) $f_n = \dfrac{3n+1}{n^2+1}$ (sami)

g) $g_n = 3$ (sami)

Rješenje:

a) $\frac{1}{3}, \frac{4}{5}, \frac{9}{7}, \frac{16}{9}, \ldots$ Niz nije ograničen.

b) $-1, 1, -1, 1, \ldots$

Imamo npr. $M = 2$ jer je $|b_n| \leq 2, \forall n \in \mathbb{N}$.

c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

$|c_n| = \frac{n}{n+1} \leq 1, \forall n \in \mathbb{N}$.

d) $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$

$|d_n| = \frac{1}{n} \leq 1, \forall n \in \mathbb{N}$.

e) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$|e_n| = \frac{1}{n} \leq 1, \forall n \in \mathbb{N}.$

f) $2, \frac{7}{5}, 1, \frac{13}{17}, \frac{16}{26}, \ldots$

$|f_n| = \frac{3n+1}{n^2+1} < \frac{3n^2+3}{n^2+1} = 3, \forall n \in \mathbb{N}.$

g) $3, 3, 3, 3, \ldots$

$|g_n| = 3 \leq 4, \forall n \in \mathbb{N}.$

Definicija

Za niz $(a_n)_{n\in\mathbb{N}}$ kažemo da je

- **padajući** ako je $a_{n+1} \leq a_n, \forall n \in \mathbb{N}$
- **strogo padajući** ako je $a_{n+1} < a_n, \forall n \in \mathbb{N}$
- **rastući** ako je $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$
- **strogo rastući** ako je $a_n < a_{n+1}, \forall n \in \mathbb{N}$
- **monoton** ako je rastući ili padajući
- **strogo monoton** ako je strogo rastući ili strogo padajući

Zadatak (4.2)

Provjerite je li niz (strogo) rastući ili (strogo) padajući:

a) $a_n = \dfrac{n^2}{2n+1}$

b) $b_n = \dfrac{n}{n+1}$

c) $c_n = \dfrac{1}{n}$

d) $d_n = \dfrac{3n+1}{n^2+1}$ (sami)

Rješenje:

a) $a_n = \dfrac{n^2}{2n+1}$

$$\begin{aligned}
a_n &\overset{?}{<} a_{n+1}, \forall n \in \mathbb{N} \\
\frac{n^2}{2n+1} &< \frac{(n+1)^2}{2(n+1)+1} \\
\frac{n^2}{2n+1} &< \frac{n^2+2n+1}{2n+3} \quad / \cdot (2n+1)(2n+3) > 0 \\
n^2(2n+3) &< (2n+1)(n^2+2n+1) \\
2n^3+3n^2 &< 2n^3+4n^2+2n+n^2+2n+1 \\
0 &< 2n^2+4n+1, \forall n \in \mathbb{N}
\end{aligned}$$

b) $b_n = \dfrac{n}{n+1}$

$$
\begin{aligned}
b_n &\overset{?}{<} b_{n+1}, \forall n \in \mathbb{N} \\
\frac{n}{n+1} &< \frac{n+1}{n+2} \quad / \cdot (n+1)(n+2) > 0 \\
n(n+2) &< (n+1)(n+1) \\
n^2 + 2n &< n^2 + 2n + 1 \\
0 &< 1, \forall n \in \mathbb{N}
\end{aligned}
$$

c) $c_n = \frac{1}{n}$

$$\begin{aligned} c_n &\overset{?}{>} c_{n+1}, \forall n \in \mathbb{N} \\ \frac{1}{n} &> \frac{1}{n+1} \quad / \cdot n(n+1) > 0 \\ n+1 &> n \\ 1 &> 0, \forall n \in \mathbb{N} \end{aligned}$$

d) $d_n = \dfrac{3n+1}{n^2+1}$

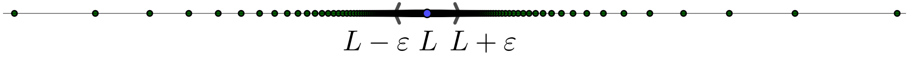
$$\begin{aligned}
d_n &\overset{?}{>} d_{n+1}, \forall n \in \mathbb{N} \\
\frac{3n+1}{n^2+1} &> \frac{3(n+1)+1}{(n+1)^2+1} \\
\frac{3n+1}{n^2+1} &> \frac{3n+4}{n^2+2n+2} \Big/ \cdot (n^2+1)(n^2+2n+2) \\
(3n+1)(n^2+2n+2) &> (3n+4)(n^2+1) \\
3n^3+6n^2+6n+n^2+2n+2 &> 3n^3+3n+4n^2+4 \\
3n^2+5n-2 &> 0, \forall n \in \mathbb{N}
\end{aligned}$$

Definicija

Neka je $(a_n)_{n\in\mathbb{N}}$ niz realnih brojeva. Ako postoji $L \in \mathbb{R}$ tako da vrijedi:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \Rightarrow |a_n - L| < \varepsilon,$$

kažemo da je L **limes niza** i označavamo sa $L = \lim\limits_{n\to+\infty} a_n$. Za niz koji ima limes kažemo da je **konvergentan**. Inače kažemo da je **divergentan**.



$$|a_n - L| < \varepsilon \Rightarrow -\varepsilon < a_n - L < \varepsilon \Rightarrow L - \varepsilon < a_n < L + \varepsilon$$

Limes niza je broj, takav da svaka okolina tog broja, koliko mala ona bila, sadrži beskonačno mnogo članova, a istovremeno je izvan te okoline najviše konačno mnogo članova niza.

Svojstva konvergentnih nizova: Neka su $(a_n)_{n\in\mathbb{N}}$ i $(b_n)_{n\in\mathbb{N}}$ dva konvergentna niza te $A = \lim\limits_{n\to+\infty} a_n$ i $B = \lim\limits_{n\to+\infty} b_n$. Tada vrijedi:

i) $(a_n + b_n)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}(a_n + b_n) = A + B$

ii) Za svaki $c \in \mathbb{R}$ niz $(c \cdot a_n)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}(c \cdot a_n) = c \cdot A$

iii) $(a_n \cdot b_n)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}(a_n \cdot b_n) = A \cdot B$

iv) Ako je $b_n \neq 0, \forall n \in \mathbb{N}$ i $B \neq 0$, $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$

v) $\left(a_n^{b_n}\right)_{n\in\mathbb{N}}$ je konvergentan i $\lim\limits_{n\to+\infty}\left(a_n^{b_n}\right) = A^B$

- npr. za $b_n = \frac{1}{2}, \forall n \in \mathbb{N} \Longrightarrow \lim\limits_{n\to+\infty}\sqrt{a_n} = \sqrt{A}$

Primjeri nekih limesa:

i) $a_n = A, \forall n \in \mathbb{N}$ je konstantni niz i $\lim\limits_{n\to+\infty} a_n = A$

ii) $a_n = \dfrac{1}{n}$ i $\lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$ (Za bilo koji $\varepsilon > 0$ odaberemo $n_0 \in \mathbb{N}$ takav da je $\dfrac{1}{n_0} < \varepsilon$ pa za $n > n_0$ imamo $|a_n - 0| = \dfrac{1}{n} \leq \dfrac{1}{n_0} < \varepsilon$)

iii) $a_n = q^n$, za neki $q \in \mathbb{R}$

- $|q| < 1$ imamo da je $\lim\limits_{n\to+\infty} q^n = 0$
- $q = 1$ imamo da je $\lim\limits_{n\to+\infty} q^n = 1$
- $|q| > 1$ niz divergira
- $q = -1$ imamo niz $-1, 1, -1, 1, ..$ pa divergira

iv) $a_n = \sqrt[n]{n}$ i $\lim\limits_{n \to +\infty} \sqrt[n]{n} = 1$

v) $a_n = \sqrt[n]{a}$, za neki $a > 0$ i $\lim\limits_{n \to +\infty} \sqrt[n]{a} = 1$

vi) $a_n = \left(1 + \frac{1}{n}\right)^n$ i $\lim\limits_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$

Teorem

Svaki ograničen i monoton niz je konvergentan.

Teorem

Neka su $(a_n)_{n\in\mathbb{N}}$ i $(b_n)_{n\in\mathbb{N}}$ konvergentni i neka je $a_n \leq b_n, \forall n \in \mathbb{N}$. Tada je $\lim\limits_{n\to+\infty} a_n \leq \lim\limits_{n\to+\infty} b_n$.

Zadatak (4.3)

Izračunajte limese sljedećih nizova:

a) $a_n = \dfrac{n}{n+1}$, b) $b_n = \dfrac{n+1}{n^2+2}$,

c) $c_n = \dfrac{n^2+n-1}{n^2+2}$, d) $d_n = \dfrac{n^3+n^2-1}{2n^2+3}$,

e) $e_n = \left[\left(1+\dfrac{3}{n}\right)\left(2-\dfrac{4}{n}\right)^2\left(\dfrac{5}{n^2}-1\right)\right]$,

f) $f_n = (\sqrt{n+1}-\sqrt{n})$, g) $g_n = \left(\dfrac{1+2+\cdots+n}{n+2}-\dfrac{n}{2}\right)$,

h) $h_n = \left(\dfrac{n+1}{n-1}\right)^n$, i) $i_n = \dfrac{\sin n}{n}$

Rješenje:

a) $$\begin{aligned} \lim_{n\to+\infty} a_n &= \lim_{n\to+\infty} \frac{n}{n+1} \\ &= \lim_{n\to+\infty} \frac{n}{n+1} : \frac{n}{n} \\ &= \lim_{n\to+\infty} \frac{1}{1+\frac{1}{n}} \\ &= \frac{1}{1+0} \\ &= 1 \end{aligned}$$

b)
$$
\begin{aligned}
\lim_{n\to+\infty} b_n &= \lim_{n\to+\infty} \frac{n+1}{n^2+2} \\
&= \lim_{n\to+\infty} \frac{n+1}{n^2+2} : \frac{n^2}{n^2} \\
&= \lim_{n\to+\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{2}{n^2}} \qquad \left(\lim_{n\to+\infty} \frac{1}{n^2} = 0\right) \\
&= \frac{0+0}{1+0} \\
&= 0
\end{aligned}
$$

c) (sami) $\lim_{n\to+\infty} c_n$

$$\begin{aligned}
\lim_{n\to+\infty} c_n &= \lim_{n\to+\infty} \frac{n^2+n-1}{n^2+2} \\
&= \lim_{n\to+\infty} \frac{n^2+n-1}{n^2+2} : \frac{n^2}{n^2} \\
&= \lim_{n\to+\infty} \frac{1+\frac{1}{n}-\frac{1}{n^2}}{1+\frac{2}{n^2}} \\
&= \frac{1+0-0}{1+0} \\
&= 1
\end{aligned}$$

$$
\begin{aligned}
\text{d) } \lim_{n\to+\infty} d_n &= \lim_{n\to+\infty} \frac{n^3+n^2-1}{2n^2+3} \\
&= \lim_{n\to+\infty} \frac{n^3+n^2-1}{2n^2+3} : \frac{n^3}{n^3} \\
&= \lim_{n\to+\infty} \frac{1+\frac{1}{n}-\frac{1}{n^3}}{\frac{2}{n}+\frac{3}{n^3}} \\
&= \frac{1}{0} \\
&= +\infty
\end{aligned}
$$

$$
\begin{aligned}
\text{e) } \lim_{n\to+\infty} e_n &= \lim_{n\to+\infty}\left[\left(1+\frac{3}{n}\right)\left(2-\frac{4}{n}\right)^2\left(\frac{5}{n^2}-1\right)\right] \\
&= (1+0)\,(2-0)^2\,(0-1) \\
&= -4
\end{aligned}
$$

$$
\begin{aligned}
\text{f) } \lim_{n\to+\infty} f_n &= \lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = (\infty-\infty) \\
&= \lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\
&= \lim_{n\to+\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\
&= \lim_{n\to+\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} : \frac{\sqrt{n}}{\sqrt{n}} \\
&= \lim_{n\to+\infty}\frac{\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}+1} \\
&= \frac{0}{\sqrt{1+0}+1} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{g) } \lim_{n\to+\infty} g_n &= \lim_{n\to+\infty}\left(\frac{1+2+\cdots+n}{n+2}-\frac{n}{2}\right) \quad \left(1+\cdots+n=\frac{n(n+1)}{2}\right)\\
&= \lim_{n\to+\infty}\left(\frac{n(n+1)}{2(n+2)}-\frac{n}{2}\right)=\\
&= \lim_{n\to+\infty}\frac{n(n+1)-n(n+2)}{2(n+2)}\\
&= \lim_{n\to+\infty}\frac{n^2+n-n^2-2n}{2n+4}\\
&= \lim_{n\to+\infty}\frac{-n}{2n+4}:\frac{n}{n}\\
&= \lim_{n\to+\infty}\frac{-1}{2-\frac{4}{n}}\\
&= \frac{-1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{h)} \lim_{n\to+\infty} h_n &= \lim_{n\to+\infty} \left(\frac{n+1}{n-1}\right)^n = (1^\infty) \\
&= \lim_{n\to+\infty} \left(\frac{n-1}{n-1} + \frac{2}{n-1}\right)^n \\
&= \lim_{n\to+\infty} \left(1 + \frac{2}{n-1}\right)^n \\
&= \lim_{n\to+\infty} \left(1 + \frac{1}{\frac{n-1}{2}}\right)^n
\end{aligned}
$$

$$= \lim_{n\to+\infty}\left(\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}}\right)^{\frac{2n}{n-1}}$$

$$= \left(\lim_{n\to+\infty}\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}}\right)^{\lim\limits_{n\to+\infty}\frac{2n}{n-1}} \qquad \left(n\to+\infty \Longrightarrow \frac{n-1}{2}\to+\infty\right)$$

$$= e^{\lim\limits_{n\to+\infty}\frac{2}{1-\frac{1}{n}}}$$

$$= e^2$$

i) $\lim\limits_{n\to+\infty} i_n \quad = \quad \lim\limits_{n\to+\infty} \frac{\sin n}{n}$

$$-1 \leq \quad \sin n \quad \leq 1 \quad / : n$$

$$\frac{-1}{n} \leq \quad \frac{\sin n}{n} \quad \leq \frac{1}{n} \quad / \lim_{n\to+\infty}$$

$$\underbrace{\lim_{n\to+\infty} \frac{-1}{n}}_{0} \leq \quad \lim_{n\to+\infty} \frac{\sin n}{n} \quad \leq \underbrace{\lim_{n\to+\infty} \frac{1}{n}}_{0}$$

$$\lim_{n\to+\infty} \frac{\sin n}{n} \quad = 0$$

Napomena: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$

Zadatak (4.4.)

Za niz a_n odredite limes L te $n_0 \in \mathbb{N}$ takav da vrijedi:
$n \geq n_0 \Rightarrow |a_n - L| < \varepsilon$ ako je

a) $a_n = \dfrac{(-1)^n}{n} + 1$ i $\varepsilon = 0.02$,

b) $a_n = \dfrac{3n^2 - 1}{n^2}$ i $\varepsilon = 0.001$. (sami)

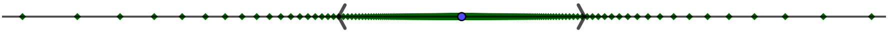
Rješenje:

a)
$$\begin{aligned}\lim_{n\to+\infty} a_n &= \lim_{n\to+\infty}\left(\frac{(-1)^n}{n}+1\right)\\ &= \lim_{n\to+\infty}\frac{(-1)^n}{n}+1\\ &= 0+1\\ &= 1\end{aligned}$$

Odredimo sada n_0 takav da je $|a_n - 1| < 0.02$.

$$
\begin{aligned}
\left|\frac{(-1)^n}{n} + 1 - 1\right| &< 0.02 \\
\frac{1}{n} &< 0.02 \quad / \cdot n \\
1 &< 0.02 \cdot n \quad / : 0.02 \\
50 &< n \\
n_0 &= 51
\end{aligned}
$$

Članovi niza nakon 51. se nalaze u okolini $\langle 1 - 0.02, 1 + 0.02 \rangle$

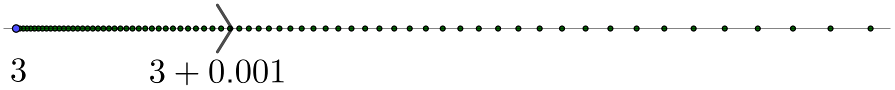


$$\begin{aligned} \text{b) } \lim_{n\to+\infty} a_n &= \lim_{n\to+\infty} \frac{3n^2+n}{n^2} \\ &= \lim_{n\to+\infty} \frac{3n^2+n}{n^2} : \frac{n^2}{n^2} \\ &= \lim_{n\to+\infty} \frac{3+\frac{1}{n}}{1} \\ &= \frac{3+0}{1} \\ &= 3 \end{aligned}$$

Odredimo sada n_0 takav da je $|a_n - 3| < 0.001$.

$$
\begin{aligned}
\left|\frac{3n^2-n}{n^2}-3\right| &< 0.001\\
\left|\frac{3n^2-n-3n^2}{n^2}\right| &< 0.001\\
\left|\frac{-1}{n}\right| &< 0.001\\
\frac{1}{n} &< 0.001 \quad /\cdot n\\
1 &< 0.001\cdot n \quad /:0.001\\
1000 &< n\\
n_0 &= 1001
\end{aligned}
$$

Članovi niza nakon 1001. se nalaze u okolini $\langle 3-0.001, 3+0.001\rangle$.



4.2 Redovi

Definicija

Neka je $(a_n)_{n\in\mathbb{N}}$ niz realnih brojeva. $S_n = a_1 + a_2 + \cdots + a_n = \sum\limits_{k=1}^{n} a_k$ kažemo da je $n-$**ta parcijalna suma** niza $(a_n)_{n\in\mathbb{N}}$.

Definicija

Uređeni par (a_n, S_n), gdje je a_n opći član niza, a S_n niz pripadajućih parcijalnih suma od a_n naziva se **red**.

Zadatak (4.5.)

Odredite opći član reda:

a) $\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots$

b) $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\cdots$

Rješenje:

a) $a_n=\frac{1}{2n+2}$

b) $a_n=\frac{n+2}{(n+1)^2}$

Zadatak (4.6.)

Odredite 3. i 4. parcijalnu sumu redova:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Rješenje:

a)
$$S_3 = \sum_{n=1}^{3} \frac{n}{2^{n-1}} = \frac{1}{2^{1-1}} + \frac{2}{2^{2-1}} + \frac{3}{2^{3-1}} = \frac{1}{1} + \frac{2}{2} + \frac{3}{4} = \frac{11}{4}$$

$$S_4 = \sum_{n=1}^{4} \frac{n}{2^{n-1}} = S_3 + \frac{4}{2^{4-1}} = \frac{11}{4} + \frac{4}{8} = \frac{13}{4}$$

$$
\begin{aligned}
\text{b) } S_3 &= \sum_{n=1}^{3} \frac{1}{n} \\
S_3 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \\
S_3 &= \frac{11}{6}
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= \sum_{n=1}^{4} \frac{1}{n} \\
&= S_3 + \frac{1}{4} \\
&= \frac{11}{6} + \frac{1}{4} \\
&= \frac{25}{12}
\end{aligned}
$$

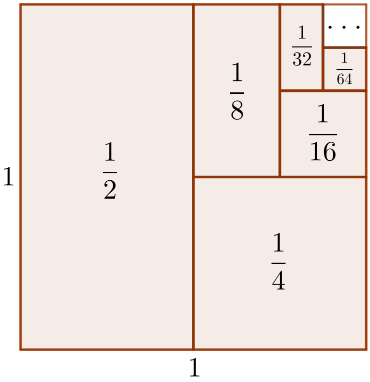
Definicija

Za red (a_n, S_n) kažemo da je **konvergentan** ako je S_n konvergentan. Ako postoji limes $s = \lim_{n\to\infty} S_n$, kažemo da je s **suma** reda (a_n, S_n) i pišemo

$$s = \sum_{n=1}^{\infty} a_n.$$

Primjer:

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ konvergira i $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$



Kriteriji konvergencije:

Kriteriji uspoređivanja:

1) Neka su $\sum\limits_{n=1}^{\infty} a_n$ i $\sum\limits_{n=1}^{\infty} b_n$ redovi i neka postoji $K > 0$ takav da je $0 \leq a_n < K \cdot b_n, \forall n \in \mathbb{N}$. Tada vrijedi:

- Ako red $\sum\limits_{n=1}^{\infty} b_n$ konvergira, konvergira i red $\sum\limits_{n=1}^{\infty} a_n$.
  $\left(\sum\limits_{n=1}^{\infty} a_n \leq K \sum\limits_{n=1}^{\infty} b_n \right)$
- Ako red $\sum\limits_{n=1}^{\infty} a_n$ divergira, divergira i red $\sum\limits_{n=1}^{\infty} b_n$.

2) Ako postoji $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} \neq 0$, onda oba reda $\sum\limits_{n=1}^{\infty} a_n$ i $\sum\limits_{n=1}^{\infty} b_n$ ili konvergiraju ili divergiraju.

D'Alembertov kriterij: Neka je $\sum_{n=1}^{\infty} a_n$ red s pozitivnim članovima i postoji $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$. Ako je:

- $q < 1$, onda red $\sum_{n=1}^{\infty} a_n$ konvergira.
- $q > 1$, onda red $\sum_{n=1}^{\infty} a_n$ divergira.
- $q = 1$, onda ne možemo donijeti zaključak.

Cauchyjev kriterij: Neka je $\sum_{n=1}^{\infty} a_n$ red s pozitivnim članovima i postoji $\lim_{n\to\infty} \sqrt[n]{a_n} = q$. Ako je:

- $q < 1$, onda red $\sum_{n=1}^{\infty} a_n$ konvergira.
- $q > 1$, onda red $\sum_{n=1}^{\infty} a_n$ divergira.
- $q = 1$, onda ne možemo donijeti zaključak.

Leibnitzov kriterij: Neka je $\sum_{n=1}^{\infty}(-1)^n a_n$ alternirajući red. Ako vrijedi:

- $a_1 \geq a_2 \geq a_3 \geq \cdots$, tj. niz (a_n) je padajući i
- $\lim_{n\to\infty} a_n = 0$

onda red $\sum_{n=1}^{\infty}(-1)^n a_n$ konvergira.

Napomena:

i) Ako red $\sum_{n=1}^{\infty} a_n$ konvergira, onda je $\lim_{n\to\infty} a_n = 0$. Obrat ne vrijedi.

ii) Ako red $\sum_{n=1}^{\infty} |a_n|$ konvergira, onda konvergira i red $\sum_{n=1}^{\infty} a_n$. Kažemo da apsolutna konvergencija povlači običnu konvergenciju.

Primjeri redova:

i) Geometrijski red $\sum_{n=1}^{\infty} q^n$

- konvergira, ako je $|q| < 1$,
- divergira, ako je $|q| \geq 1$.

ii) Harmonijski red $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira. Red $\sum_{n=1}^{\infty} \frac{1}{n^k}$ konvergira ako je $k > 1$.

Npr. red $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira.

Zadatak (4.7.)

Ispitajte konvergenciju redova:

a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

b) $\sum_{n=1}^{\infty} \frac{n+1}{n \cdot 2^n}$ (sami)

c) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

d) $\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$

e) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+2n^2+5}$

f) $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$ (sami)

g) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

h) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

i) $\sum_{n=1}^{\infty} n \cdot \left(\frac{4}{3}\right)^{n+1}$ (sami)

j) $\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \left(1+\frac{1}{n}\right)^{n^2}$ (sami)

k) $\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$

l) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Rješenje:

a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Red divergira jer opći član ne teži prema 0.

b) $\sum_{n=1}^{\infty} \frac{n+1}{n \cdot 2^n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{n+2}{(n+1)\cdot 2^{n+1}}}{\frac{n+1}{n\cdot 2^n}} \\
&= \lim_{n\to\infty} \frac{n(n+2)}{2(n+1)^2} \\
&= \lim_{n\to\infty} \frac{n^2+2n}{2n^2+4n+2} : \frac{n^2}{n^2} \\
&= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{2+\frac{4}{n}+\frac{2}{n^2}} \\
&= \tfrac{1}{2} < 1
\end{aligned}$$

Prema D'Alembertovom kriteriju, red konvergira.

c) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

$$0 \leq \frac{1}{(2n+1)^2} \leq \frac{1}{n^2}$$

Kako $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira, konvergira i $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ prema 1. kriteriju uspoređivanja.

d) $\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$

$$0 \leq \frac{|\sin n|}{1+n^2} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira, pa konvergira i $\sum_{n=1}^{\infty} \frac{|\sin n|}{1+n^2}$, pa naposlijetku konvergira i $\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$ jer apsolutna konvergencija povlači i konvergenciju reda.

e) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n^3+2n^2+5}$

Znamo da $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ divergira i vrijedi:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n^2}{n^3+2n^2+5}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n^3}{n^3+2n^2+5} \\
&= \lim_{n\to\infty} \frac{n^3}{n^3+2n^2+5} : \frac{n^3}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{5}{n^3}} \\
&= \frac{1}{1+0+0} \\
&= 1 \neq 0
\end{aligned}$$

Prema 2. kriteriju uspoređivanja, red $\sum_{n=1}^{\infty} \frac{n^2}{n^3+2n^2+5}$ divergira.

f) $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{2(n+1)-1}{2^{n+1}}}{\frac{2n-1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2n+1}{2(2n-1)} \\
&= \lim_{n\to\infty} \frac{2n+1}{4n-2} : \frac{n}{n} \\
&= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{4-\frac{2}{n}} \\
&= \frac{1}{2} < 1
\end{aligned}$$

Prema D'Alembertovom kriteriju, red konvergira.

g) $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$

$$
\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{a_n} &= \lim_{n\to\infty} a_n^{\frac{1}{n}} \\
&= \lim_{n\to\infty}\left(\left(\frac{n}{2n+1}\right)^n\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty}\frac{n}{2n+1} : \frac{n}{n} \\
&= \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}} \\
&= \frac{1}{2} < 1
\end{aligned}
$$

Prema Cauchyjevom kriteriju, red konvergira.

h) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} \\
&= \lim_{n\to\infty} \frac{\frac{n!\cdot(n+1)}{3\cdot 3^n}}{\frac{n!}{3^n}} \\
&= \lim_{n\to\infty} \frac{n+1}{3} \\
&= \infty
\end{aligned}$$

Prema D'Alembertovom kriteriju, red divergira.

i) $\sum_{n=1}^{\infty} n \cdot \left(\frac{4}{3}\right)^{n+1}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)\cdot\left(\frac{4}{3}\right)^{n+2}}{n\cdot\left(\frac{4}{3}\right)^{n+1}} \\
&= \frac{4}{3} \lim_{n\to\infty} \frac{n+1}{n} \\
&= \frac{4}{3} > 1
\end{aligned}$$

Prema D'Alembertovom kriteriju, red divergira.

j) $\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \left(1 + \frac{1}{n}\right)^{n^2}$

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^n} \cdot \left(1 + \frac{1}{n}\right)^{n^2}} \\ &= \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} \\ &= \frac{1}{2} e > 1\end{aligned}$$

Prema Cauchyjevom kriteriju, red divergira.

k) $\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^{10}}}{\frac{2^n}{n^{10}}} \\
&= 2 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{10} \\
&= 2 \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^{10} \\
&= 2 \cdot 1^{10} \\
&= 2 > 1
\end{aligned}$$

Prema D'Alembertovom kriteriju, red divergira.

l) $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

$a_1 \geq a_2 \geq a_3 \cdots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Prema Leibnitzovom kriteriju, red konvergira.

Zadatak (4.8.)

Odredite opći član i ispitajte konvergenciju reda:

a) $\frac{3\cdot 1}{\sqrt[3]{1}} + \frac{3\cdot 2}{\sqrt[3]{2}} + \frac{3\cdot 4}{\sqrt[3]{3}} + \frac{3\cdot 8}{\sqrt[3]{4}} + \cdots$

b) $\frac{2}{2\cdot 3} + \frac{4}{4\cdot 9} + \frac{6}{8\cdot 27} + \frac{8}{16\cdot 81} + \cdots$ (sami)

c) $\frac{1}{3!} + \frac{\sqrt{3}}{5!} + \frac{\sqrt{5}}{7!} + \frac{\sqrt{7}}{9!} + \cdots$

Rješenje: a) $a_n = \dfrac{3 \cdot 2^{n-1}}{\sqrt[3]{n}}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\dfrac{3 \cdot 2^n}{\sqrt[3]{n+1}}}{\dfrac{3 \cdot 2^{n-1}}{\sqrt[3]{n}}} \\
&= \lim_{n\to\infty} 2 \cdot \sqrt[3]{\frac{n}{n+1}} \\
&= 2 \cdot \sqrt[3]{\lim_{n\to\infty} \frac{n}{n+1} : \frac{n}{n}} \\
&= 2 \cdot \sqrt[3]{\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}} \\
&= 2 \cdot \sqrt[3]{\frac{1}{1+0}} \\
&= 2 > 1 \qquad \text{Divergira, prema D'Alembertovom kriteriju.}
\end{aligned}$$

b) $a_n = \dfrac{2n}{2^n \cdot 3^n}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\dfrac{2(n+1)}{2^{n+1} \cdot 3^{n+1}}}{\dfrac{2n}{2^n \cdot 3^n}} \\
&= \lim_{n\to\infty} \frac{n+1}{6n} \\
&= \frac{1}{6} \underbrace{\lim_{n\to\infty} \frac{n+1}{n}}_{=1} \\
&= \frac{1}{6} < 1
\end{aligned}
$$

Prema D'Alembertovom kriteriju, red konvergira.

c) $a_n = \dfrac{\sqrt{2n-1}}{(2n+1)!}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{\sqrt{2(n+1)-1}}{(2(n+1)+1)!}}{\frac{\sqrt{2n-1}}{(2n+1)!}} \\
&= \lim_{n\to\infty} \frac{\frac{\sqrt{2n+1}}{(2n+3)!}}{\frac{\sqrt{2n-1}}{(2n+1)!}} \\
&= \lim_{n\to\infty} \left(\frac{\sqrt{2n+1}}{\sqrt{2n-1}} \cdot \frac{1}{(2n+2)(2n+3)} \right) \\
&= \lim_{n\to\infty} \sqrt{\frac{2n+1}{2n-1}} \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\lim_{n\to\infty} \frac{2n+1}{2n-1} : \frac{n}{n}} \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} \\
&= \sqrt{\lim_{n\to\infty} \frac{2+\frac{1}{n}}{2-\frac{1}{n}}} \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} \\
&= \sqrt{\frac{2+0}{2-0}} \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} \\
&= 1 \cdot 0 \\
&= 0 < 1
\end{aligned}
$$

Prema D'Alembertovom kriteriju, red konvergira.