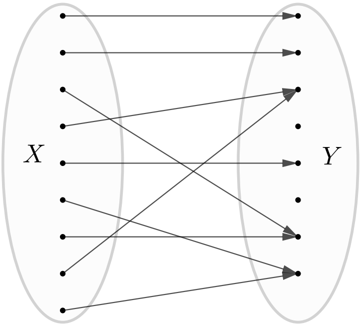
Poglavlje 5
Funkcije

ak. god. 2021./2022.

5.1 Osnovni pojmovi

Definicija

Neka su X i Y skupovi, f pravilo preslikavanja elemenata skupa X u Y. Ako $\forall x \in X, \exists! y \in Y$ takav da je $f(x) = y$ onda uređenu trojku (X, f, Y) nazivamo **funkcija** i pišemo $f : X \to Y$.



Definicija

Neka je (X, f, Y) funkcija. Skup X nazivamo domena, a skup Y kodomena funkcije f.

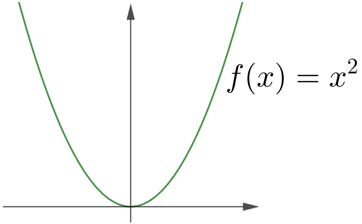
Definicija

Skup $\mathrm{Im} f = \{y \in Y : \exists x \in X, f(x) = y\}$ nazivamo **slika funkcije**.

Definicija

Skup $\Gamma_f = \{(x, f(x)), x \in X\} \subset X \times Y$ nazivamo **graf funkcije**.

Primjer: $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$
$\mathrm{Im} f = \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$



Definicija

Za dvije funkcije kažemo da su jednake ako imaju isto domenu, istu kodomenu i isto pravilo.

Primjer:

$$\left.\begin{array}{l} f_1 : \mathbb{R} \to \mathbb{R}, \quad f_1(x) = x^2 \\ f_2 : \mathbb{R} \to \mathbb{R}^+, f_2(x) = x^2 \end{array}\right\} f_1 \neq f_2$$

Definicija

Neka je $f : X \to Y$ funkcija. Za funkciju $f_1 : A \subseteq X \to Y$ kažemo da je restrikcija funkcije f na A ako je $f_1(x) = f(x), \forall x \in A$ i označavamo $f_1 = f|_A$. Za funkciju $f_2 : A \supseteq X \to Y$ kažemo da je proširenje funkcije f na A ako je $f_2(x) = f(x), \forall x \in X$.

Definicija

Za funkciju $f : X \to Y$ kažemo da je **injekcija** ako $\forall x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ ili ekvivalentno, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Definicija

Za funkciju $f : X \to Y$ kažemo da je **surjekcija** ako $\forall y \in Y, \exists x \in X, f(x) = y$, odnosno, $\mathrm{Im} f = Y$.

Definicija

Za funkciju $f : X \to Y$ kažemo da je **bijekcija** ako je i injekcija i surjekcija.

5.2 Realna funkcija realne varijable

$f : D \to \mathbb{R}, D \subseteq \mathbb{R}$

Prirodna domena funkcije (u oznaci $D(f)$ ili D_f) je skup svih realnih brojeva za koje je funkcija dobro definirana.

Zadatak (5.1.)

Odredite prirodnu domenu funkcije:

a) $f(x) = \dfrac{x}{(x-2)(x-5)}$

b) $f(x) = \sqrt{x^2 - 5x + 6}$ (sami)

c) $f(x) = \ln(2x-4)$ (sami)

d) $f(x) = \dfrac{x}{\sqrt{-x^2 + 3x - 2}}$

Rješenje: a) $f(x) = \dfrac{x}{(x-2)(x-5)}$

$x \neq 2$ i $x \neq 5 \Longrightarrow D_f = \langle -\infty, 2\rangle \bigcup \langle 2, 5\rangle \bigcup \langle 5, +\infty\rangle = \mathbb{R} \setminus \{2, 5\}$

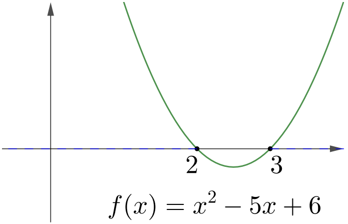
b) $f(x) = \sqrt{x^2 - 5x + 6}$

$$x^2 - 5x + 6 \geq 0$$

$$x_{1,2} = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6 \cdot 1}}{2 \cdot 1}$$

$$x_{1,2} = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$x_1 = 2 \quad , \quad x_2 = 3$$



$D_f = \langle -\infty, 2] \bigcup [3, +\infty\rangle$

c) $f(x) = \ln(2x - 4)$

$2x - 4 > 0 \Longrightarrow x > 2$

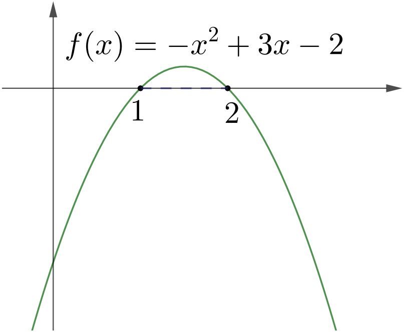
$D_f = \langle 2, +\infty \rangle$

d) $f(x) = \dfrac{x}{\sqrt{-x^2 + 3x - 2}}$

$$-x^2 + 3x - 2 > 0$$

$$x_{1,2} = \frac{-3 \pm \sqrt{1}}{-2}$$

$$x_1 = 1 \quad , \quad x_2 = 2$$



$D_f = \langle 1, 2 \rangle$

Definicija

Neka je $f : D \subseteq \mathbb{R} \to \mathbb{R}$ funkcija. Kažemo da je f:

- **(strogo) rastuća** ako $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) (f(x_1) < f(x_2))$
- **(strogo) padajuća** ako $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) (f(x_1) > f(x_2))$
- **parna** ako je $f(-x) = f(x), \forall x \in D$
- **neparna** ako je $f(-x) = -f(x), \forall x \in D$
- **periodična** ako $\exists a \in \mathbb{R}$ takav da je $f(x+a) = f(x), \forall x \in D$
- **omeđena** ako $\exists M \in \mathbb{R}, M > 0$ takav da je $|f(x)| \leq M, \forall x \in D$, tj. $-M \leq f(x) \leq M, \forall x \in D$

Definicija

Neka su $f : X \subseteq \mathbb{R} \to \mathbb{R}$ i $g : Y \supseteq \mathrm{Im} f \to \mathbb{R}$ funkcije. **Kompozicija funkcija** f i g je funkcija $h : X \to \mathbb{R}$, definirana pravilom $h(x) = g(f(x))$. Kompoziciju funkcija f i g označavamo: $h = g \circ f$, odnosno $(g \circ f)(x) = g(f(x))$.

Zadatak (5.2.)

Odredite $f \circ g$ i $g \circ f$ ako je:

a) $f(x) = 3x$ i $g(x) = \sqrt{x-1}$ (sami)

b) $f(x) = \dfrac{x+2}{x-1}$ i $g(x) = \dfrac{x}{x-1}$

c) $f(x) = \sqrt{x}$ i $g(x) = \dfrac{1}{1+x^2}$

Rješenje: a)

$$\begin{aligned}(f\circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x-1}) \\ &= 3\sqrt{x-1}\end{aligned}\qquad \begin{aligned}(g\circ f)(x) &= g(f(x)) \\ &= g(3x) \\ &= \sqrt{3x-1}\end{aligned}$$

b)

$$\begin{aligned}(f\circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x}{x-1}\right) \\ &= \frac{\frac{x}{x-1}+2}{\frac{x}{x-1}-1} \\ &= \frac{\frac{x+2x-2}{x-1}}{\frac{x-x+1}{x-1}} \\ &= \frac{3x-2}{1} \\ &= 3x-2\end{aligned}\qquad \begin{aligned}(g\circ f)(x) &= g(f(x)) \\ &= g\left(\frac{x+2}{x-1}\right) \\ &= \frac{\frac{x+2}{x-1}}{\frac{x+2}{x-1}-1} \\ &= \frac{\frac{x+2}{x-1}}{\frac{x+2-x+1}{x-1}} \\ &= \frac{x+2}{3}\end{aligned}$$

c)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{1+x^2}\right) \\ &= \sqrt{\frac{1}{1+x^2}}\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(\sqrt{x}\right) \\ &= \frac{1}{1+\left(\sqrt{x}\right)^2} \\ &= \frac{1}{1+x}\end{aligned}$$

Napomena: $\sqrt{x^2} = |x|$, $\left(\sqrt{x}\right)^2 = x$

Definicija

Neka je $f : X \subseteq \mathbb{R} \to \mathrm{Im} f$ funkcija. Funkcija f ima **inverz** ako postoji funkcija $f^{-1} : \mathrm{Im} f \to X$, takva da vrijedi $(f \circ f^{-1})(y) = y$, $y \in \mathrm{Im} f$ i $(f^{-1} \circ f)(x) = x$, $x \in X$.

Zadatak (5.3.)

Nađite inverz funkcija:

a) $f(x) = x - 5$,

b) $f(x) = \dfrac{x}{x+1}$,

c) $f(x) = \log_2 x$.

Rješenje: a) $y = x - 5$,
$x = y + 5$,
$f^{-1}(y) = y + 5$.
b) $y = \dfrac{x}{x+1} \,|\cdot (x+1)$,
$xy + y = x$, tj. $x(1-y) = y$,
$x = \dfrac{y}{1-y}$,
$f^{-1}(y) = \dfrac{y}{1-y}$.
c) $y = \log_2 x \quad |2^{\cdot}$,
$2^y = 2^{\log_2 x} = x$,
$f^{-1}(y) = 2^y$.

5.2.1 Elementarne funkcije

Definicija

Neka su $a_0, a_1, a_2, \cdots, a_n \in \mathbb{R}$ i $a_n \neq 0$. Funkciju $f : \mathbb{R} \to \mathbb{R}$ oblika $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ nazivamo **polinomom** *n*-tog stupnja.

Svaki polinom $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ može se zapisati u obliku $f(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n)$ gdje su $x_1, x_2, ..., x_n \in \mathbb{C}$ **nultočke** polinoma.

Ako je $\alpha + \beta i$ nultočka polinoma f, onda je i $\alpha - \beta i$ također nultočka tog polinoma, pa imamo:

$$\begin{aligned} (x - (\alpha + \beta i))(x - (\alpha - \beta i)) &= (x - \alpha - \beta i)(x - \alpha + \beta i) \\ &= (x - \alpha)^2 - (\beta i)^2 \\ &= x^2 - 2x\alpha + \alpha^2 + \beta^2 \end{aligned},$$

odnosno, polinom možemo rastaviti na linearne i kvadratne članove s realnim koeficijentima.

Definicija

Neka su $P_m(x)$ i $Q_n(x)$ polinomi m-tog, odnosno n-tog stupnja. Funkciju oblika $f(x) = \frac{P_m(x)}{Q_n(x)}$ nazivamo **racionalna funkcija**.

Ako je $m < n$ kažemo da je racionalna funkcija **prava**. Nepravu racionalnu funkciju $f(x) = \frac{P_m(x)}{Q_n(x)}, m \geq n$ možemo zapisati kao

$f(x) = R_{m-n}(x) + \frac{S_t(x)}{Q_n(x)}, t < n$

Rastav prave racionalne funkcije $f(x) = \dfrac{P_m(x)}{Q_n(x)}$ na parcijalne razlomke:

$Q_n(x)$ faktoriziramo na linearne i kvadratne faktore:

- Ako se u rastavu pojavi $(x - x_0)^s$, onda u rastavu na parcijalne razlomke imamo

$$\frac{A_1}{x - x_0} + \frac{A_2}{(x - x_0)^2} + \cdots + \frac{A_s}{(x - x_0)^s}$$

- Ako se u rastavu pojavi $(x^2 + ax + b)^t$, onda u rastavu na parcijalne razlomke imamo

$$\frac{A_1 x + B_1}{x^2 + ax + b} + \frac{A_2 x + B_2}{(x^2 + ax + b)^2} + \cdots + \frac{A_t x + B_t}{(x^2 + ax + b)^t}$$

Zadatak (5.4.)

Rastavite na parcijalne razlomke racionalnu funkciju

a) $Q(x) = \dfrac{1}{x^4 - 16}$

b) $Q(x) = \dfrac{2x^2 - 3x + 5}{(x+2)(x-1)(x-3)}$

c) $Q(x) = \dfrac{1}{(x-2)^2(x^2+1)}$ (sami)

.

Rješenje: a)

$$\frac{1}{x^4-16}=\frac{1}{(x^2-4)(x^2+4)}=\frac{1}{(x-2)(x+2)(x^2+4)}$$

$$\frac{1}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4} \quad / \cdot (x-2)(x+2)(x^2+4)$$

$$\begin{aligned}
1 &= A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x+2)(x-2)\\
1 &= A(x^3+4x+2x^2+8)+B(x^3+4x-2x^2-8)+(Cx+D)(x^2-4)\\
1 &= Ax^3+4Ax+2Ax^2+8A+Bx^3+4Bx-2Bx^2-8B+Cx^3-4Cx+Dx^2-4D\\
1 &= x^3(A+B+C)+x^2(2A-2B+D)+x(4A+4B-4C)+(8A-8B-4D)
\end{aligned}$$

Dva su polinoma jednaka ako su koeficijenti uz odgovarajuće potencije jednaki.

$$\begin{aligned} A+B+C &= 0 \\ 2A-2B+D &= 0 \\ 4A+4B-4C &= 0 \\ 8A-8B-4D &= 1 \end{aligned}$$

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ 4 & 4 & -4 & 0 & 0 \\ 8 & -8 & 0 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \\ /:4 \\ \\ \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 8 & -8 & 0 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \text{II}-2\text{I} \\ \text{III}-\text{I} \\ \text{IV}-8\text{I} \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -16 & -8 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \text{II}-\frac{1}{4}\text{IV} \\ :/(-2) \\ \\ \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -16 & -8 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \text{IV}\leftrightarrow\text{II} \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & -16 & -8 & -4 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -\frac{1}{4} \end{array}\right] \begin{array}{l} \\ \text{II}+8\text{III} \\ \text{II}+2\text{IV} \\ \\ \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 0 \\ 0 & -16 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -\frac{1}{4} \end{array}\right]$$

$$D = -\frac{1}{8}$$

$$C = 0$$

$$B = -\frac{1}{32}$$

$$A + B + C = 0$$

$$A = \frac{1}{32}$$

$$\frac{1}{(x-2)(x+2)(x^2+4)} = \frac{\frac{1}{32}}{x-2} - \frac{\frac{1}{32}}{x+2} - \frac{\frac{1}{8}}{x^2+4}$$

b)

$$\frac{2x^2-3x+5}{(x+2)(x-1)(x-3)} = \frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{x-3} \quad / \cdot (x+2)(x-1)(x-3)$$

$$\begin{aligned}
2x^2-3x+5 &= A(x-1)(x-3)+B(x+2)(x-3)+C(x+2)(x-1)\\
2x^2-3x+5 &= A(x^2-4x+3)+B(x^2-x-6)+C(x^2+x-2)\\
2x^2-3x+5 &= Ax^2-4Ax+3A+Bx^2-Bx-6B+Cx^2+Cx-2C\\
2x^2-3x+5 &= x^2(A+B+C)+x(-4A-B+C)+(3A-6B-2C)
\end{aligned}$$

$$\begin{aligned} A + B + C &= 2 \\ -4A - B + C &= -3 \\ 3A - 6B - 2C &= 5 \end{aligned}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ -4 & -1 & 1 & -3 \\ 3 & -6 & -2 & 5 \end{array}\right] \begin{array}{l} \\ \text{II}+4\text{I} \\ \text{III}-3\text{I} \end{array} \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & 3 & 5 & 5 \\ 0 & -9 & -5 & -1 \end{array}\right] \begin{array}{l} \\ \\ \text{III}+3\text{II} \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & 3 & 5 & 5 \\ 0 & 0 & 10 & 14 \end{array}\right]$$

$$D = \frac{7}{5}$$

$$\begin{aligned}
3B + 5C &= 5 \\
3B &= 5 - 5 \cdot \frac{7}{5} \\
3B &= -2 \\
B &= -\frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
A + B + C &= 2 \\
A &= 2 + \frac{2}{3} - \frac{7}{5} \\
A &= \frac{19}{15}
\end{aligned}$$

c)

$$\frac{1}{(x-2)^2(x^2+1)} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+1} \quad / \cdot (x-2)^2(x^2+1)$$

$$1 = A(x-2)(x^2+1)+B(x^2+1)+(Cx+D)(x-2)^2$$

$$1 = A(x^3-2x^2+x-2)+Bx^2+B+(Cx+D)(x^2-4x+4)$$

$$1 = Ax^3-2Ax^2+Ax-2A+Bx^2+B+Cx^3+Dx^2-4Cx^2-4Dx+4Cx+4D$$

$$1 = x^3(A+C)+x^2(-2A+B-4C+D)+x(A+4C-4D)+(-2A+B+4D)$$

$$\begin{aligned} A + C &= 0 \\ -2A + B - 4C + D &= 0 \\ A + 4C - 4D &= 0 \\ -2A + B + 4D &= 1 \end{aligned}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 0 \\ -2 & 1 & -4 & 1 & 0 \\ 1 & 0 & 4 & -4 & 0 \\ -2 & 1 & 0 & 4 & 1 \end{array}\right] \begin{array}{l} \\ \text{II}+2\text{I} \\ \text{III}-\text{I} \\ \text{IV}+2\text{I} \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 4 & 1 \end{array}\right] \text{IV}-\text{II}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & 3 & 1 \end{array}\right] \text{IV}-\tfrac{4}{3}\text{III} \sim \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & \frac{25}{3} & 1 \end{array}\right]$$

$$\begin{aligned} D &= \frac{3}{25} \\ 3C - 4D &= 0 \\ 3C &= 4 \cdot \frac{3}{25} \\ C &= \frac{4}{25} \end{aligned}$$

$$\begin{aligned} B - 2C + D &= 0 \\ B &= 2C - D \\ B &= 2 \cdot \frac{4}{25} - \frac{3}{25} \\ B &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} A + C &= 0 \\ A &= -C \\ A &= -\frac{4}{25} \end{aligned}$$

Definicija

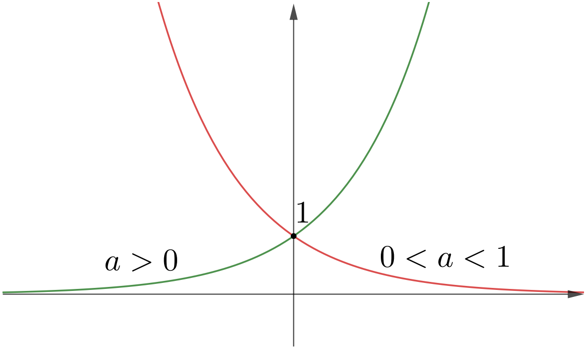
Neka je $a \in \mathbb{R}, a \neq 1, a > 0$. Funkcija $f : \mathbb{R} \to \mathbb{R}^+$, definirana sa $f(x) = a^x$ naziva se **eksponencijalna** funkcija.

$$a^0 = 1, \forall a \in \mathbb{R}$$

$$a^{x_1 + x_2} = a^{x_1} \cdot a^{x_2}$$

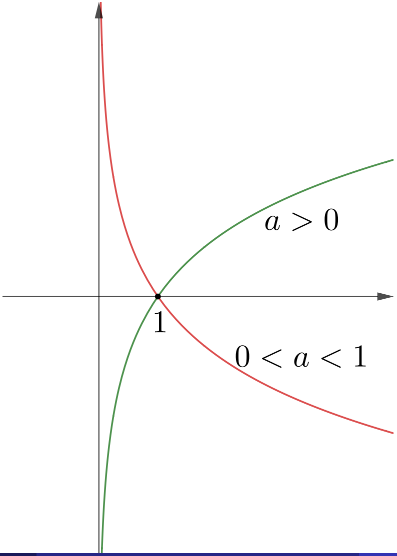
$$a^{x_1 - x_2} = \frac{a^{x_1}}{a^{x_2}}$$

$$a^{x_1 \cdot x_2} = (a^{x_1})^{x_2}$$



Definicija

Neka je $a \in \mathbb{R}, a \neq 1, a > 0$. Funkcija $f : \mathbb{R}^+ \to \mathbb{R}$, definirana sa $f(x) = \log_a x$ naziva se **logaritamska** funkcija.



$$\log_a x = b \Longleftrightarrow a^b = x$$

$$\log_a(x_1 \cdot x_2) = \log_a x_1 + \log_a x_2$$

$$\log_a \left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$\log_a b = \frac{1}{\log_b a}$$

$$a^{\log_a x} = \log_a a^x = x$$

$$\log_a x^n = n \cdot \log_a x, \forall n > 0$$

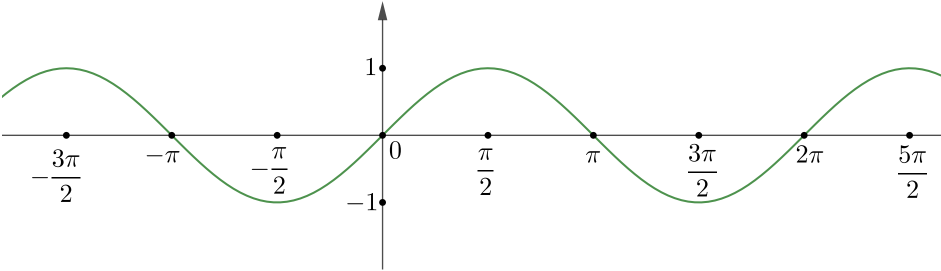
$$\log_e x = \ln x, e \approx 2.71$$

Sinus

$\sin : \mathbb{R} \to [-1, 1]\,, f(x) = \sin x$

Funkcija $f(x) = \sin x$ je

- **periodična**: $\sin x = \sin(x + 2k\pi), k \in \mathbb{Z}$
- **neparna**: $\sin(-x) = -\sin x$

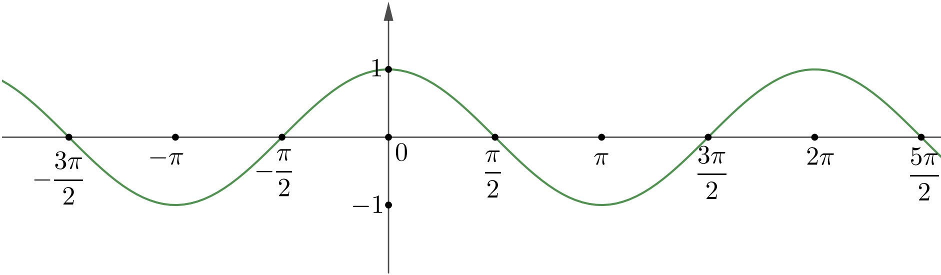


Kosinus

$\cos : \mathbb{R} \to [-1, 1]\,, f(x) = \cos x$

Funkcija $f(x) = \cos x$ je

- **periodična**: $\cos x = \cos(x + 2k\pi), k \in \mathbb{Z}$
- **parna**: $\cos(-x) = \cos x$

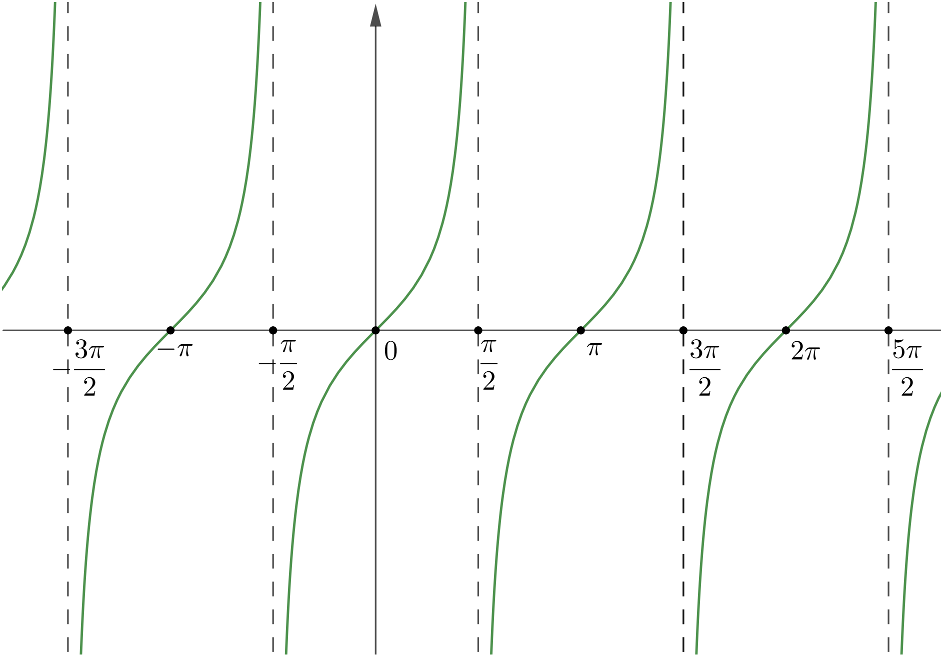


Tangens

tg: $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\} \to \mathbb{R}, f(x) = \text{tg}x$

Funkcija $f(x) = \text{tg}x = \dfrac{\sin x}{\cos x}$ je

- **periodična**: $\text{tg}x = \text{tg}(x + k\pi), k \in \mathbb{Z}$
- **neparna**: $\text{tg}(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x} = -\text{tg}x$

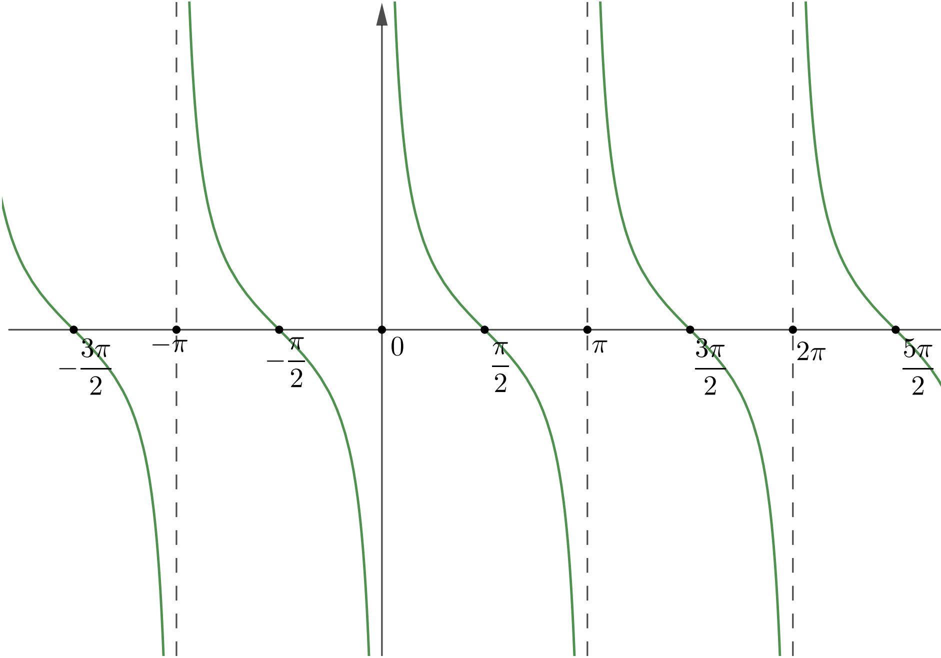

$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$

Kotangens

ctg: $\mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\} \to \mathbb{R}, f(x) =ctgx$

Funkcija $f(x) =$ctg$x = \dfrac{\cos x}{\sin x}$ je

- **periodična**: ctg$x =$ctg$(x + k\pi), k \in \mathbb{Z}$
- **neparna**: ctg$(-x) = \dfrac{\cos(-x)}{\sin(-x)} = \dfrac{\cos x}{-\sin x} = -ctgx$

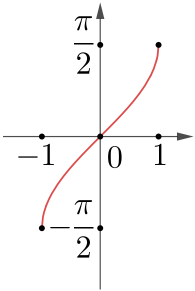

$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$

Arkus Sinus

Funkcija $\sin : \mathbb{R} \to [-1, 1]\,, f(x) = \sin x$ nije injekcija, pa nije ni bijekcija, no restrikcija $\sin\Big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \to [-1, 1]$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju $\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\arcsin(\sin x) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin(\arcsin x) = x, \forall x \in [-1, 1]$

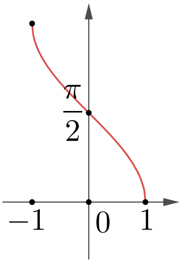


Arkus Kosinus

Funkcija $\cos : \mathbb{R} \to [-1, 1]$ nije injekcija, pa nije ni bijekcija, no restrikcija $\cos\Big|_{[0,\pi]} \to [-1, 1]$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju $\arccos : [-1, 1] \to [0, \pi]$.

$\arccos(\cos x) = x, \forall x \in [0, \pi]$

$\cos(\arccos x) = x, \forall x \in [-1, 1]$

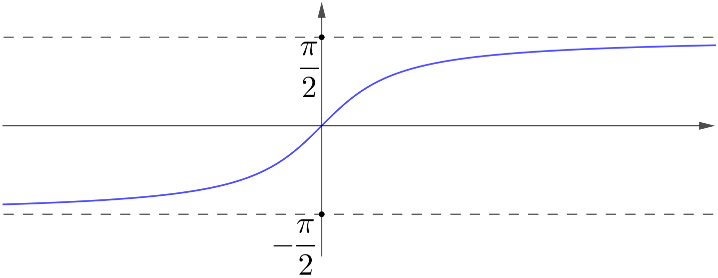


Arkus Tangens

Funkcija tg: $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\} \to \mathbb{R}$ nije injekcija, pa nije ni bijekcija, no restrikcija $\text{tg}\Big|_{\left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle} \to \mathbb{R}$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju arctg: $\mathbb{R} \to \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$.

$\text{arctg}(\text{tg}x) = x, \forall x \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$

$\text{tg}(\text{arctg}x) = x, \forall x \in \mathbb{R}$

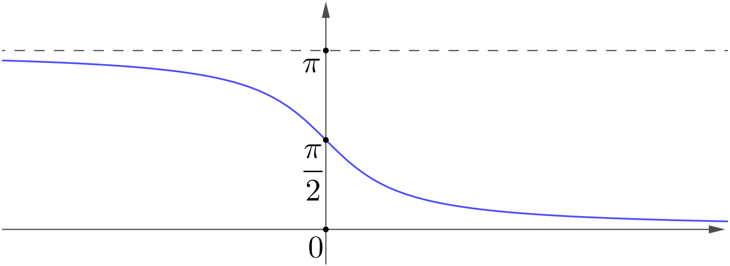


Arkus Kotangens

Funkcija ctg: $\mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\} \to \mathbb{R}$ nije injekcija, pa nije ni bijekcija, no restrikcija $\text{ctg}\Big|_{\langle 0,\pi\rangle} \to \mathbb{R}$ je bijekcija. Za tako definiranu funkciju definiramo inverznu funkciju arcctg: $\mathbb{R} \to \langle 0, \pi\rangle$.

$\text{arcctg}(\text{ctg}x) = x, \forall x \in \langle 0, \pi\rangle$

$\text{ctg}(\text{arcctg}x) = x, \forall x \in \mathbb{R}$



Sinus hiperbolni

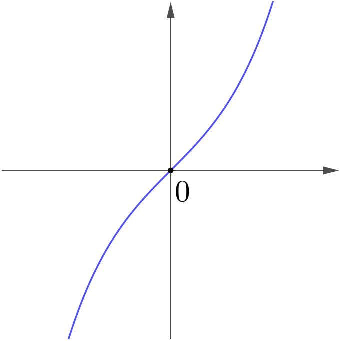
Funkcija sh: $\mathbb{R} \to \mathbb{R}$ definirana sa

$$\text{sh}x = \frac{e^x - e^{-x}}{2}$$

naziva se sinus hiperbolni.

Funkcija $f(x) =$sh$x$ je **neparna**:

$$\text{sh}(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\text{sh}x$$



Inverz funkcije:

$$\begin{aligned}
f(x) &= \frac{e^x - e^{-x}}{2} \\
y &= \frac{e^x - e^{-x}}{2} \quad / \cdot 2 \\
2y &= e^x - e^{-x} \quad / \cdot e^x \\
0 &= e^{2x} - 2ye^x - 1, \qquad t = e^x \\
0 &= t^2 - 2yt - 1
\end{aligned}$$

$$t_{1,2} = \frac{2y \pm \sqrt{(2y)^2 - 4 \cdot 1 \cdot (-1)}}{2}$$

$$t_{1,2} = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$

$$t_{1,2} = \frac{2y \pm \sqrt{4\left(y^2 + 1\right)}}{2}$$

$$t_{1,2} = \frac{2y \pm 2\sqrt{y^2 + 1}}{2}$$

$e^x = y + \sqrt{y^2 + 1}$ ili $e^x = y - \sqrt{y^2 + 1}$

$e^x = y - \sqrt{y^2 + 1}$ odbacujemo jer je $e^x > 0$

$e^x = y + \sqrt{y^2 + 1} \quad / \ln$

Arsh: $\mathbb{R} \to \mathbb{R}$, $\quad \text{Arsh}\, y = \ln\left(y + \sqrt{y^2 + 1}\right)$

Kosinus hiperbolni

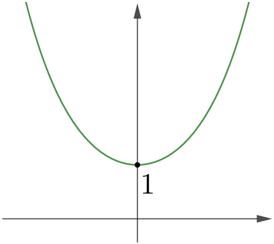
Funkcija ch: $\mathbb{R} \to [1, \infty\rangle$ definirana sa

$$\text{ch}x = \frac{e^x + e^{-x}}{2}$$

naziva se kosinus hiperbolni.

Funkcija $f(x) =$ch$x$ je **parna**:

$\text{ch}(-x) = \dfrac{e^{-x} + e^x}{2} = \dfrac{e^x + e^{-x}}{2} = \text{ch}x$



Inverz funkcije:

$$
\begin{aligned}
f(x) &= \frac{e^x + e^{-x}}{2} \\
y &= \frac{e^x + e^{-x}}{2} \quad / \cdot 2 \\
2y &= e^x + e^{-x} \quad / \cdot e^x \\
0 &= e^{2x} - 2ye^x + 1 \\
t &= e^x \\
0 &= t^2 - 2yt + 1
\end{aligned}
$$

$$t_{1,2} = \frac{2y \pm \sqrt{(2y)^2 - 4 \cdot 1 \cdot 1}}{2}$$

$$t_{1,2} = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$

$$t_{1,2} = \frac{2y \pm \sqrt{4\left(y^2 - 1\right)}}{2}$$

$$t_{1,2} = \frac{2y \pm 2\sqrt{y^2 - 1}}{2}$$

$e^x = y + \sqrt{y^2 - 1}$ ili $e^x = y - \sqrt{y^2 - 1}$

$e^x = y - \sqrt{y^2 - 1}$ odbacujemo jer je $e^x > 0$

$e^x = y + \sqrt{y^2 - 1} \quad / \ln$

Arch: $[1, \infty\rangle \to \mathbb{R}$

$\text{Arch}\, y = \ln\left(y + \sqrt{y^2 - 1}\right)$

Vrijedi: $\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$

$$\begin{aligned}
\mathrm{ch}^2 x - \mathrm{sh}^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\
&= 1
\end{aligned}$$

Tangens hiperbolni

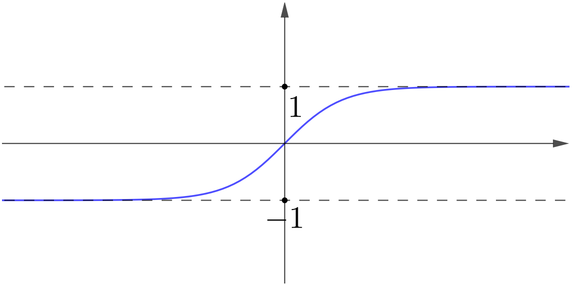
Funkcija th: $\mathbb{R} \to \langle -1, 1 \rangle$ definirana sa

$$\text{th}x = \frac{\text{sh}x}{\text{ch}x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

naziva se tangens hiperbolni.

Funkcija $f(x) =$th$x$ je **neparna**:

$$\text{th}(-x) = \frac{\text{sh}(-x)}{\text{ch}(-x)} = \frac{-\text{sh}x}{\text{ch}x} = -\text{th}x$$



Inverz funkcije:

$$\begin{aligned}
f(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad / \cdot (e^x + e^{-x}) \\
ye^x + ye^{-x} &= e^x - e^{-x} \quad / \cdot e^x \\
ye^{2x} + y &= e^{2x} - 1, \\
1 + y &= e^{2x}(1 - y) \\
e^{2x} &= \frac{1 + y}{1 - y} \quad / \ln \\
2x &= \ln\left(\frac{1 + y}{1 - y}\right) \\
\operatorname{Arth}(y) &= \frac{1}{2} \ln\left(\frac{1 + y}{1 - y}\right)
\end{aligned}$$

Kotangens hiperbolni

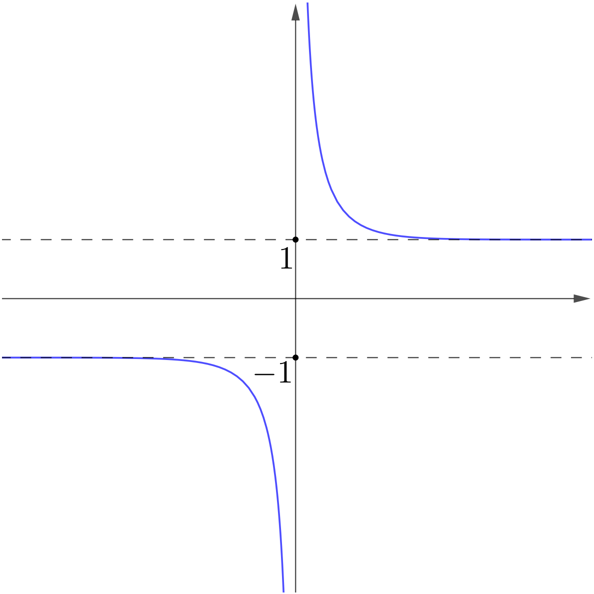
Funkcija cth: $\mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus [-1, 1]$ definirana sa

$$\text{cth}x = \frac{\text{ch}x}{\text{sh}x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

naziva se kotangens hiperbolni.

Funkcija $f(x) =$cth$x$ je **neparna**:

$$\text{cth}(-x) = \frac{\text{ch}(-x)}{\text{sh}(-x)} = \frac{\text{ch}x}{-\text{sh}x} = -\text{cth}x$$


1
−1

Inverz funkcije:

$$\begin{aligned}
f(x) &= \frac{e^x + e^{-x}}{e^x - e^{-x}} \\
y &= \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad / \cdot (e^x - e^{-x}) \\
ye^x - ye^{-x} &= e^x + e^{-x} \quad / \cdot e^x \\
ye^{2x} - y &= e^{2x} + 1, \\
e^{2x}(y-1) &= y + 1 \\
e^{2x} &= \frac{y+1}{y-1} \quad / \ln \\
2x &= \ln\left(\frac{y+1}{y-1}\right) \\
\operatorname{Arcth}(y) &= \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right)
\end{aligned}$$

Zadatak (5.5.)

Odredite prirodnu domenu funkcije:

a) $f(x) = e^{\frac{1}{x^2-4}} + \frac{1}{x}$, b) $f(x) = \operatorname{arctg}\frac{1}{x-3}$,

c) $f(x) = \frac{\sqrt{x-2}}{\ln(x+1)}$, d) $f(x) = \sqrt{1-10^x}$,

e) $f(x) = \arcsin\sqrt{\ln x}$, f) $f(x) = \ln(2\sin x - 1)$,

g) $f(x) = \ln\sqrt{1-x} + \sqrt{\ln(1-x)}$, h) $f(x) = \frac{1}{4^x - 3\cdot 2^x + 2}$,

i) $f(x) = \ln\frac{x+3}{2-x}$, j) $f(x) = e^{\frac{1}{x}} + \frac{\sqrt{x^2-1}}{x+2}$,

k) $f(x) = \arccos\frac{2x}{1+x}$.

Rješenje: a) $f(x) = e^{\frac{1}{x^2-4}} + \frac{1}{x}$

$x \neq 0 \qquad x^2 - 4 \neq 0$

$x^2 \neq 4$

$x \neq 2$ i $x \neq -2$

$D_f = \mathbb{R} \setminus \{-2, 0, 2\}$

b) $f(x) = \text{arctg}\frac{1}{x-3}$

$D_{\text{arctg}} = \mathbb{R} \qquad x - 3 \neq 0$

$x \neq 3$

$D_f = \mathbb{R} \setminus \{3\}$

c) $f(x) = \dfrac{\sqrt{x-2}}{\ln(x+1)}$

$x - 2 \geq 0 \qquad x + 1 > 0 \qquad \ln(x+1) \neq 0$

$x \geq 2 \qquad x > -1 \qquad x + 1 \neq e^0$

$x + 1 \neq 1$

$x \neq 0$

$$D_f = [2, \infty\rangle$$

d) $f(x) = \sqrt{1 - 10^x}$

$$1 - 10^x \geq 0$$

1. način:

$1 \geq 10^x \quad / \log$

$\log 1 \geq x$

$0 \geq x$

2. način

$10^0 \geq 10^x$

$0 \geq x$

$$D_f = \langle -\infty, 0]$$

e) $f(x) = \arcsin\sqrt{\ln x}$

$x > 0 \qquad \ln x \geq 0 \qquad -1 \leq \sqrt{\ln x} \leq 1$

$x \geq e^0 \qquad \sqrt{\ln x} \leq 1 \quad /^2$

$x \geq 1 \qquad \ln x \leq 1$

$x \leq e^1 = e$

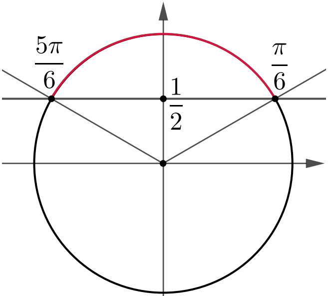
$$D_f = [1, e]$$

f) $f(x) = \ln(2\sin x - 1)$

$2\sin x - 1 > 0$
$2\sin x > 1$

$\sin x > \frac{1}{2}$

$$D_f = \bigcup_{k\in\mathbb{Z}} \left\langle \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \right\rangle$$



g) $f(x) = \ln\sqrt{1-x} + \sqrt{\ln(1-x)}$

$$1 - x > 0 \qquad \ln(1-x) \geq 0$$

$$x < 1 \qquad 1 - x \geq 1$$

$$x \leq 0$$

$$D_f = \langle -\infty, 0]$$

h) $f(x) = \dfrac{1}{4^x - 3 \cdot 2^x + 2}$

$$\begin{aligned} 4^x - 3 \cdot 2^x + 2 &\neq 0 \\ t &= 2^x \\ t^2 - 3t + 2 &= 0 \end{aligned} \qquad \begin{aligned} 2^x &\neq 1 \\ 2^x &\neq 2^0 \\ x &\neq 0 \end{aligned} \qquad \begin{aligned} 2^x &\neq 2 \\ 2^x &\neq 2^1 \\ x &\neq 1 \end{aligned}$$

$$t_{1,2} = \frac{3 \pm \sqrt{1}}{2}$$

$$t_1 = 1,\ t_2 = 2$$

$$D_f = \mathbb{R} \setminus \{0, 1\}$$

i) $f(x) = \ln \dfrac{x+3}{2-x}$

$\dfrac{x+3}{2-x} > 0$

	$-\infty$	-3	2	$+\infty$
$x+3$	$-$	$+$	$+$	
$2-x$	$+$	$+$	$-$	
$\frac{x+3}{2-x}$	$-$	$+$	$-$	

$D_f = \langle -3, 2\rangle$

j) $f(x) = e^{\frac{1}{x}} + \dfrac{\sqrt{x^2-1}}{x+2}$

$x \neq 0$ $\qquad$ $x^2 - 1 \geq 0$ $\qquad$ $x + 2 \neq 0$

$x^2 \geq 1$ $\qquad$ $x \neq -2$

$x \leq -1$ ili $x \geq 1$

$D_f = \langle -\infty, -1] \bigcup [1, \infty\rangle \setminus \{-2\}$

k) $f(x) = \arccos\frac{2x}{1+x}$

$$-1 \leq \frac{2x}{1+x} \leq 1$$

$$-1 \leq \frac{2x}{1+x}$$

$$0 \leq \frac{2x}{1+x} + 1$$

$$0 \leq \frac{2x+1+x}{1+x}$$

$$0 \leq \frac{3x+1}{1+x}$$

	$-\infty$ … -1	-1 … $-\frac{1}{3}$	$-\frac{1}{3}$ … $+\infty$
$3x+1$	$-$	$-$	$+$
$1+x$	$-$	$+$	$+$
$\frac{3x+1}{1+x}$	$+$	$-$	$+$

$$x \in \langle -\infty, -1 \rangle \bigcup \left[-\frac{1}{3}, \infty \right\rangle$$

$$\begin{aligned}
\frac{2x}{1+x} &\leq 1\\
\frac{2x}{1+x} - 1 &\leq 0\\
\frac{2x-1-x}{1+x} &\leq 0\\
\frac{x-1}{1+x} &\leq 0
\end{aligned}$$

	$-\infty$ to -1	-1 to 1	1 to $+\infty$
$x-1$	$-$	$-$	$+$
$1+x$	$-$	$+$	$+$
$\frac{x-1}{1+x}$	$+$	$-$	$+$

$$x \in \langle -1, 1]$$

$$D_f = \left(\langle -\infty, -1 \rangle \bigcup \left[-\frac{1}{3}, \infty \right\rangle \right) \bigcap \langle -1, 1] = \left[-\frac{1}{3}, 1 \right]$$

Zadatak (5.6.)

Odredite inverz i sliku funkcije

a) $f(x) = \ln(2\sin x - 1)$

b) $f(x) = \operatorname{arctg}\sqrt{e^x + 1}$

c) $f(x) = 3 \cdot 2^{1-x} + 1$

d) $f(x) = 2^{x^3}$

e) $f(x) = (x-1)^3$

Rješenje: a) $f(x) = \ln(2\sin x - 1)$

$$\begin{aligned} f(x) &= \ln(2\sin x - 1) \\ y &= \ln(2\sin x - 1) \quad /e^{\cdot} \\ e^y &= 2\sin x - 1 \\ \sin x &= \frac{e^y + 1}{2} \quad /\text{arcsin} \\ x &= \arcsin\frac{e^y + 1}{2} \\ f^{-1}(y) &= \arcsin\frac{e^y + 1}{2} \end{aligned}$$

$$\text{Im} f = D_{f^{-1}}$$

$$\begin{aligned} -1 \leq \quad \frac{e^x + 1}{2} &\leq 1 \\ \frac{e^x + 1}{2} &\leq 1 \\ e^x + 1 &\leq 2 \\ e^x &\leq 1 \\ x &\leq 0 \end{aligned}$$

$$\text{Im} f = D_{f^{-1}} = \langle -\infty, 0]$$

b) $f(x) = \operatorname{arctg}\sqrt{e^x + 1}$

$$\begin{aligned}
f(x) &= \operatorname{arctg}\sqrt{e^x+1} \\
y &= \operatorname{arctg}\sqrt{e^x+1} \quad /\operatorname{tg} \\
\operatorname{tg} y &= \sqrt{e^x+1} \quad /^2 \\
\operatorname{tg}^2 y &= e^x + 1 \\
e^x &= \operatorname{tg}^2 y - 1 \quad /\ln \\
x &= \ln\left(\operatorname{tg}^2 y - 1\right) \\
f^{-1}(x) &= \ln\left(\operatorname{tg}^2 x - 1\right)
\end{aligned}$$

$$\text{Im}f = D_{f^{-1}} \subseteq \left[0, \frac{\pi}{2}\right\rangle$$

$$\text{tg}^2 x - 1 > 0$$

$$\text{tg}^2 x > 1$$

$$\text{tg}x \leq -1 \text{ ili } \text{tg}x > 1$$

$$x \in \left\langle -\frac{\pi}{2}, -\frac{\pi}{4} \right\rangle \bigcup \left\langle \frac{\pi}{4}, \frac{\pi}{2} \right\rangle$$

$$\text{Im}f = \left(\left\langle -\frac{\pi}{2}, -\frac{\pi}{4} \right\rangle \bigcup \left\langle \frac{\pi}{4}, \frac{\pi}{2} \right\rangle \right) \bigcap \left[0, \frac{\pi}{2}\right\rangle$$

$$\text{Im}f = \left\langle \frac{\pi}{4}, \frac{\pi}{2} \right\rangle$$

c) $f(x) = 3 \cdot 2^{1-x} + 1$

$$\begin{aligned}
f(x) &= 3 \cdot 2^{1-x} + 1 \\
y &= 3 \cdot 2^{1-x} + 1 \\
3 \cdot 2^{1-x} &= y - 1 \\
2^{1-x} &= \frac{y-1}{3} \quad / \log_2 \\
1 - x &= \log_2 \frac{y-1}{3} \\
x &= 1 - \log_2 \frac{y-1}{3} \\
f^{-1}(x) &= 1 - \log_2 \frac{x-1}{3}
\end{aligned}$$

$\mathrm{Im} f = D_{f^{-1}}$

$$\begin{aligned}
\frac{x-1}{3} &> 0 \\
x - 1 &> 0 \\
x &> 1
\end{aligned}$$

$\mathrm{Im} f = D_{f^{-1}} = \langle 1, \infty \rangle$

d) $f(x) = 2^{x^3}$

$$\begin{aligned} f(x) &= 2^{x^3} \\ y &= 2^{x^3} \quad / \log_2 \\ \log_2 y &= x^3 \quad / \sqrt[3]{\ } \\ x &= \sqrt[3]{\log_2 y} \\ f^{-1}(x) &= \sqrt[3]{\log_2 x} \end{aligned}$$

$$\mathrm{Im} f = D_{f^{-1}}$$

$$x > 0$$

$$\mathrm{Im} f = D_{f^{-1}} = \langle 0, \infty \rangle$$

e) $f(x) = (x-1)^3$

$$\begin{aligned} f(x) &= (x-1)^3 \\ y &= (x-1)^3 \quad /\sqrt[3]{\ } \\ x-1 &= \sqrt[3]{y} \\ x &= \sqrt[3]{y}+1 \\ f^{-1}(x) &= \sqrt[3]{x}+1 \end{aligned}$$

$$\mathrm{Im} f = D_{f^{-1}} = \mathbb{R}$$

Zadatak (5.7.)

Ispitajte (ne)parnost sljedećih funkcija:

a) $f(x) = \sqrt{x^2 - 4}$

b) $f(x) = 3^{\frac{1}{x}}$

c) $f(x) = \frac{1}{x}$

d) $f(x) = \frac{1}{x^2}$

e) $f(x) = \frac{1}{x - 1}$

Rješenje:

$$\text{a) } f(-x) = \sqrt{(-x)^2 - 4} = \sqrt{x^2 - 4} = f(x)$$

f je parna.

$$\text{b) } f(-x) = 3^{\frac{1}{-x}} = 3^{-\frac{1}{x}}$$

f nije ni parna ni neparna.

c)
$$
\begin{aligned}
f(-x) &= \frac{1}{-x} \\
&= -\frac{1}{x} \\
&= -f(x)
\end{aligned}
$$

f je neparna.

d) $$\begin{aligned} f(-x) &= \frac{1}{(-x)^2} \\ &= \frac{1}{x^2} \\ &= f(x) \end{aligned}$$

f je parna.

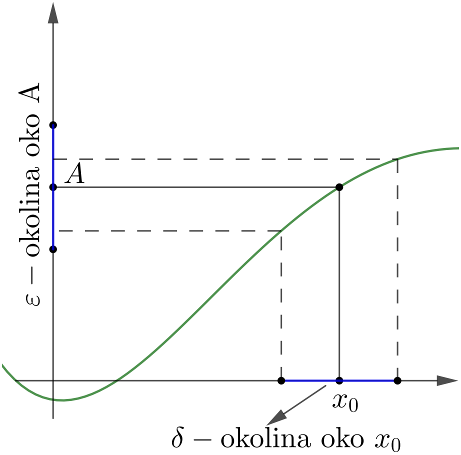
$$
\text{e) } f(-x) = \frac{1}{x-1} = \frac{1}{-x-1}
$$

f nije ni parna ni neparna.

5.3 Limes funkcije

Definicija

Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ funkcija i neka je $x_0 \in I$. Ako postoji $A \in \mathbb{R}$ takav da vrijedi: $\forall \varepsilon > 0, \exists \delta > 0$ tako da $|x - x_0| < \delta \Longrightarrow |f(x) - A| < \varepsilon$, onda kažemo da f ima limes u točki x_0 i pišemo $\lim\limits_{x \to x_0} f(x) = A$.



Definicija

Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ funkcija i neka je $x_0 \in I$. Ako postoji $A \in \mathbb{R}$ takav da vrijedi: $\forall \varepsilon > 0, \exists \delta > 0$ tako da $x \in \langle x_0 - \delta, x_0 \rangle \implies |f(x) - A| < \varepsilon$, onda kažemo da f ima limes u točki x_0 **slijeva** i pišemo $\lim\limits_{x \to x_0^-} f(x) = A$.

Definicija

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Funkcija f ima limes u točki x_0 ako postoje limesi slijeva i zdesna u toj točki i jednaki su.

Zadatak (5.8.)

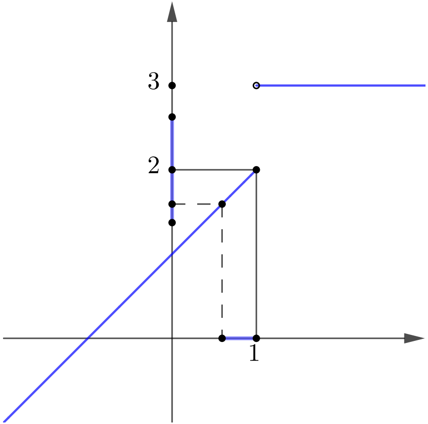
Neka je $f : \mathbb{R} \to \mathbb{R}$ zadana s $f(x) = \begin{cases} x+1, & x \leq 1 \\ 3, & x > 1 \end{cases}$. Postoji li

a) $\lim\limits_{x \to 1^-} f(x)$

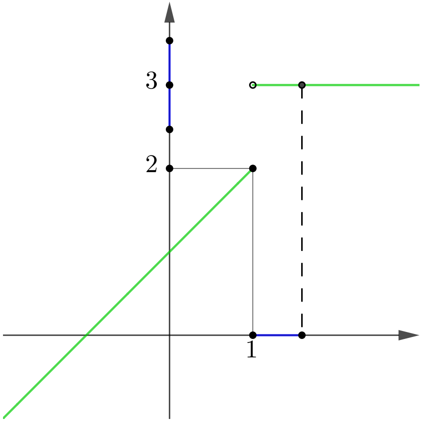
b) $\lim\limits_{x \to 1^+} f(x)$

c) $\lim\limits_{x \to 1} f(x)$

a) Vrijedi da je $\lim\limits_{x\to 1^-} f(x) = 2$. Naime, za bilo koji $\varepsilon > 0$ odaberimo $\delta = \varepsilon$. Tada lako vidimo da čim je $x \in \langle 1-\delta, 1\rangle$ imamo da je $f(x) = x + 1 \in \langle 2-\varepsilon, 2+\varepsilon\rangle$. Dakle, svaka točka iz lijeve δ-okoline od 1 preslika se u ε-okolinu oko 2.



b) Vrijedi da je $\lim_{x \to 1^+} f(x) = 3$. Slično kao pod a), za bilo koji $\varepsilon > 0$ odaberimo $\delta = \varepsilon$, pa čim je $x \in \langle 1, 1 + \delta \rangle$, očito je $f(x) = 3 \in \langle 3 - \varepsilon, 3 + \varepsilon \rangle$, odnosno svaka točka iz desne δ-okoline od 1 preslika se u ε-okolinu oko 3.



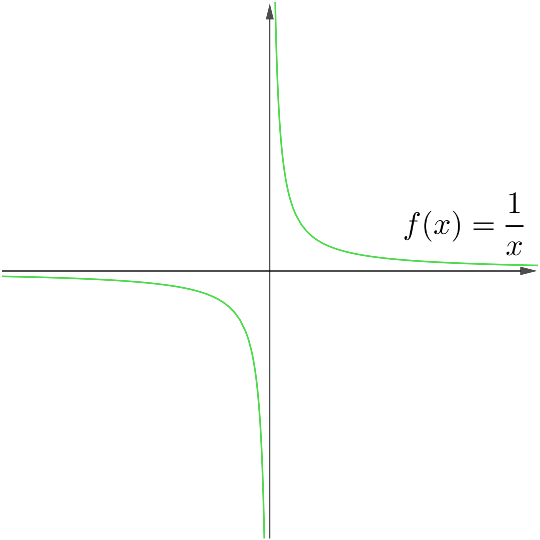
c) Kako je $\lim_{x \to 1^-} f(x) = 2$ i $\lim_{x \to 1^+} f(x) = 3$, pa imamo da je

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x),$$

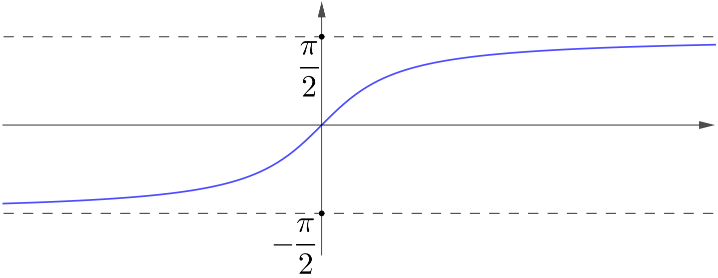
tj. $\lim_{x \to 1} f(x)$ ne postoji.

Primjer:

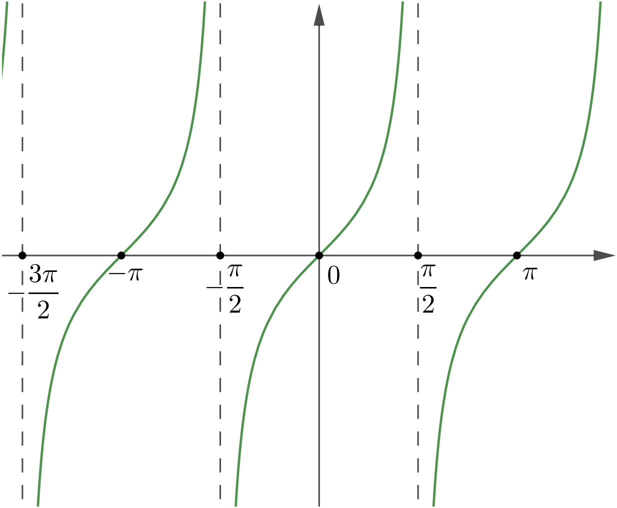
- $\lim\limits_{x \to +\infty} \dfrac{1}{x} = 0$
- $\lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$
- $\lim\limits_{x \to 0^+} \dfrac{1}{x} = +\infty$
- $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$



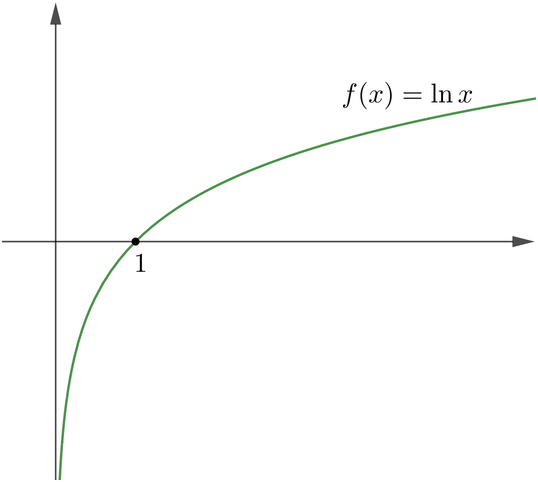
- $\lim\limits_{x \to +\infty} \operatorname{arctg} x = \frac{\pi}{2}$
- $\lim\limits_{x \to -\infty} \operatorname{arctg} x = -\frac{\pi}{2}$



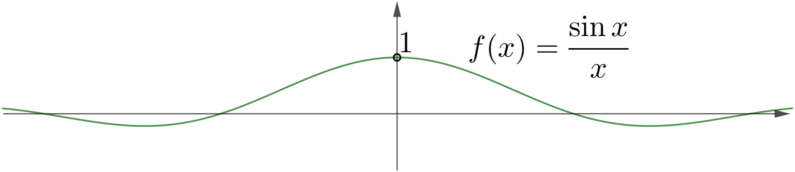
- $\lim\limits_{x\to\frac{\pi}{2}^-} \mathrm{tg}x = +\infty$
- $\lim\limits_{x\to\frac{\pi}{2}^+} \mathrm{tg}x = -\infty$



- $\lim_{x \to +\infty} \ln x = +\infty$
- $\lim_{x \to 0^+} \ln x = -\infty$



- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to \pm\infty} \frac{\sin x}{x} = 0$



- $\lim_{x\to +\infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim_{x\to +\infty} \left(1 + \frac{k}{x}\right)^x = e^k$

Zadatak (5.9.)

Odredite sljedeće limese:

a)
$$\lim_{x\to+\infty} \frac{(x+1)^2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2+2x+1}{x^2+1} : \frac{x^2}{x^2}$$
$$= \lim_{x\to+\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}}$$
$$= \frac{1}{1} = 1$$

b)
$$\lim_{x\to+\infty} \frac{100x}{x^2-1} : \frac{x^2}{x^2} = \lim_{x\to+\infty} \frac{\frac{100}{x}}{1-\frac{1}{x^2}}$$
$$= \frac{0}{1} = 0$$

c) $$\lim_{x\to+\infty} \frac{x^2-5x+1}{3x+7} : \frac{x^2}{x^2} = \lim_{x\to+\infty} \frac{1-\frac{5}{x}+\frac{1}{x^2}}{\frac{3}{x}+\frac{7}{x^2}}$$
$$= \frac{1}{0^+} = +\infty$$

d) $$\lim_{x\to+\infty} \frac{x^2-1}{1-x} : \frac{x^2}{x^2} = \lim_{x\to+\infty} \frac{1-\frac{1}{x^2}}{\frac{1}{x^2}-\frac{1}{x}}$$
$$= \frac{1}{0^-}$$
$$= -\infty$$

e) $$\lim_{x\to+\infty} \frac{2x^2-3x-4}{\sqrt{x^4+1}} : \frac{x^2}{x^2} = \lim_{x\to+\infty} \frac{2-\frac{3}{x}-\frac{4}{x^2}}{\sqrt{1+\frac{1}{x^4}}} = \frac{2}{1} = 2$$

f) $$\lim_{x\to-1} \frac{x^3+1}{x^2+1} = \lim_{x\to-1} \frac{(-1)^3+1}{(-1)^2+1} = \frac{0}{2} = 0$$

g)
$$
\begin{aligned}
\lim_{x\to 5}\frac{x^2-7x+10}{x^2-25} &= \left(\lim_{x\to 5}\frac{25-35+10}{25-25}=\frac{0}{0}\right)\\
&= \lim_{x\to 5}\frac{x^2-5x-2x+10}{(x-5)(x+5)}\\
&= \lim_{x\to 5}\frac{x(x-5)-2(x-5)}{(x-5)(x+5)}\\
&= \lim_{x\to 5}\frac{(x-5)(x-2)}{(x-5)(x+5)}\\
&= \lim_{x\to 5}\frac{x-2}{x+5}\\
&= \lim_{x\to 5}\frac{5-2}{5+5}\\
&= \frac{3}{10}
\end{aligned}
$$

$$
\begin{aligned}
\text{h) } \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} &= \left(\frac{1-1}{1-1}=\frac{0}{0}\right) \\
&= \left\{\begin{matrix} x=t^2 \\ x\to 1 \Rightarrow t\to 1 \end{matrix}\right\} \\
&= \lim_{t\to 1}\frac{t-1}{t^2-1} \\
&= \lim_{t\to 1}\frac{t-1}{(t-1)(t+1)} \\
&= \lim_{t\to 1}\frac{1}{t+1} \\
&= \frac{1}{2}
\end{aligned}
$$

i) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} = \left(\frac{1-1}{1-1}=\frac{0}{0}\right)$$

$$= \begin{Bmatrix} 1+x = t^6 \\ x\to 0 \Rightarrow t\to 1 \end{Bmatrix}$$

$$= \lim_{t\to 1}\frac{\sqrt{t^6}-1}{\sqrt[3]{t^6}-1}$$

$$= \lim_{t\to 1}\frac{t^3-1}{t^2-1}$$

$$= \lim_{t\to 1}\frac{(t-1)(t^2+t+1)}{(t-1)(t+1)}$$

$$= \lim_{t\to 1}\frac{t^2+t+1}{t+1}$$

$$= \frac{3}{2}$$

$$\text{j) } \lim_{x\to 7} \frac{2-\sqrt{x-3}}{x^2-49} = \left(\frac{2-2}{49-49} = \frac{0}{0}\right)$$

$$= \lim_{x\to 7} \frac{2-\sqrt{x-3}}{x^2-49} \cdot \frac{2+\sqrt{x-3}}{2+\sqrt{x-3}}$$

$$= \lim_{x\to 7} \frac{2^2 - \left(\sqrt{x-3}\right)^2}{(x^2-49)\left(2+\sqrt{x-3}\right)}$$

$$= \lim_{x\to 7} \frac{7-x}{(x-7)(x+7)\left(2+\sqrt{x-3}\right)}$$

$$= \frac{-1}{(7+7)\left(2+\sqrt{7-3}\right)}$$

$$= -\frac{1}{56}$$

k) $$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \left(\frac{1-1}{0} = \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \lim_{x\to 0} \frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \lim_{x\to 0} \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= 1$$

l) $\lim_{x\to 2} \frac{\sin x}{x} = \frac{\sin 2}{2}$

m) $\lim_{x\to +\infty} \frac{\sin x}{x}$

$$-1 \leq \sin x \leq 1 \quad / : x$$

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \quad / \lim_{x\to +\infty}$$

$$\lim_{x\to +\infty} -\frac{1}{x} \leq \lim_{x\to +\infty} \frac{\sin x}{x} \leq \lim_{x\to +\infty} \frac{1}{x}$$

$$0 \leq \lim_{x\to +\infty} \frac{\sin x}{x} \leq 0$$

$$\lim_{x\to +\infty} \frac{\sin x}{x} = 0$$

n) $\lim\limits_{x\to 0} \frac{\sin 5x}{\sin 2x} = \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 2x}{2x}} \cdot \frac{5x}{2x}$$

$$= \frac{5}{2} \lim_{x\to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 2x}{2x}}$$

$$= \frac{5}{2} \cdot \frac{1}{1}$$

$$= \frac{5}{2}$$

Napomena: Općenito, za $A \in \mathbb{R}$ vrijedi:

$$\lim_{x\to 0} \frac{\sin Ax}{Ax} = \left\{ \begin{matrix} Ax = t \\ x \to 0 \Rightarrow t \to 0 \end{matrix} \right\} = \lim_{t\to 0} \frac{\sin t}{t} = 1$$

$$\text{o) } \lim_{x\to 0}\left(\frac{x^2-2x+3}{x^2-3x+2}\right)^{\frac{\sin x}{x}} = \left(\lim_{x\to 0}\frac{x^2-2x+3}{x^2-3x+2}\right)^{\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \left(\frac{3}{2}\right)^1$$

$$= \frac{3}{2}$$

p) $$\begin{aligned}\lim_{x\to+\infty}\left(\frac{x-1}{x+1}\right)^{x} &= \lim_{x\to+\infty}\left(\frac{x+1-2}{x+1}\right)^{x}\\
&= \lim_{x\to+\infty}\left(\frac{x+1}{x+1}+\frac{-2}{x+1}\right)^{x}\\
&= \lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{x}\\
&= \lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{x+1-1}\\
&= \lim_{x\to+\infty}\left[\left(1+\frac{-2}{x+1}\right)^{x+1}\left(1+\frac{-2}{x+1}\right)^{-1}\right]\\
&= \underbrace{\lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{x+1}}_{e^{-2}}\underbrace{\lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{-1}}_{1}\\
&= \frac{1}{e^{2}}\end{aligned}$$

r) $$\lim_{x\to 0}\left(\frac{2+x}{3-x}\right)^{x} = \left(\frac{2+0}{3-0}\right)^{0} = 1$$

s) $$\lim_{x\to 1}\left(\frac{x-1}{x^2-1}\right)^{x+1} = \left(\frac{1-1}{1-1}\right)^{2} = \lim_{x\to 1}\left(\frac{x-1}{(x-1)(x+1)}\right)^{x+1} = \lim_{x\to 1}\left(\frac{1}{x+1}\right)^{x+1} = \frac{1}{4}$$

t) $$\lim_{x\to +\infty}\sin\frac{1}{\sqrt{x}} = \sin 0 = 0$$

u) $$\lim_{x\to+\infty} \frac{x+1}{\sqrt{x^2+x+1}} : \frac{x}{x} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \frac{1+0}{\sqrt{1+0+0}} = 1$$

v) $$\lim_{x\to+\infty} \left(\frac{x^2+3x+1}{2x^2-1}\right)^x = \left(\lim_{x\to+\infty} \frac{x^2+3x+1}{2x^2-1} : \frac{x^2}{x^2}\right)^{\lim\limits_{x\to+\infty} x} = \left(\lim_{x\to+\infty} \frac{1+\frac{3}{x}+\frac{1}{x^2}}{2-\frac{1}{x^2}}\right)^{\lim\limits_{x\to+\infty} x} = \left(\frac{1}{2}\right)^{+\infty} = 0$$

z) $$\lim_{x\to+\infty}\left(\frac{2x^2+1}{x^2-3}\right)^x = \left(\lim_{x\to+\infty}\frac{2x^2+1}{x^2-3}:\frac{x^2}{x^2}\right)^{\lim\limits_{x\to+\infty}x}$$

$$= \left(\lim_{x\to+\infty}\frac{2+\frac{1}{x^2}}{1-\frac{3}{x^2}}\right)^{\lim\limits_{x\to+\infty}x}$$

$$= \left(\frac{2}{1}\right)^{+\infty}$$

$$= +\infty$$