Poglavlje 6
Derivacija

ak. god. 2021./2022.

6.1 Definicija i osnovna svojstva

Definicija

Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ funkcija i neka je $x_0 \in I$. Kažemo da je f **derivabilna** u točki x_0 ako postoji $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ i pišemo:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Zadatak (6.1.)

Neka je $f : \mathbb{R} \to \mathbb{R}$ zadana s :

a) $f(x) = c$ (konstantna funkcija)

b) $f(x) = x$

c) $f(x) = x^2$

d) $f(x) = \cos x$

Odredite $f'(x)$.

Rješenje:

$$\text{a) } f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c - c}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{0}{x - x_0}$$

$$= 0$$

b) $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1 = 1$$

c) $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} (x + x_0)$$

$$= 2x_0$$

$$\text{d)}\ f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{\cos x - \cos x_0}{x - x_0}$$

$$\left(\cos x - cosy = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}\right)$$

$$= \lim_{x\to x_0} \frac{-2\sin\frac{x+x_0}{2}\sin\frac{x-x_0}{2}}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{-\sin\frac{x+x_0}{2}\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}}$$

$$= \lim_{x\to x_0} \left(-\sin\frac{x+x_0}{2}\right) \lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} (*)$$

$$\lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}} = \left\{ \begin{matrix} \dfrac{x-x_0}{2} = t \\ x\to x_0 \Rightarrow t\to 0 \end{matrix} \right\}$$

$$= \lim_{t\to 0} \frac{\sin t}{t}$$

$$= 1$$

$$(*) = \lim_{x\to x_0} \left(-\sin\tfrac{x+x_0}{2}\right)\cdot 1$$

$$= -\sin\tfrac{x_0+x_0}{2}$$

$$= -\sin x_0$$

Nastavljajući ovaj postupak i za druge funkcije dobivamo **Tablicu derivacija**.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\mathrm{tg}x$	$\dfrac{1}{\cos^2 x}$
x	1	$\mathrm{ctg}x$	$-\dfrac{1}{\sin^2 x}$
x^n	nx^{n-1}	$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\mathrm{arctg}x$	$\dfrac{1}{1+x^2}$
$\ln x$	$\dfrac{1}{x}$	$\mathrm{arcctg}x$	$-\dfrac{1}{1+x^2}$
$\sin x$	$\cos x$	$\mathrm{sh}x$	$\mathrm{ch}x$
$\cos x$	$-\sin x$	$\mathrm{ch}x$	$\mathrm{sh}x$

Pravila deriviranja:

- $[c \cdot f(x)]' = c \cdot f'(x)$

- $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

- $[f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$

- $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

- $[(f \circ g)(x)]' = [f(g(x))]' = f'(g(x)) \cdot g'(x)$

Zadatak (6.2.)

Odredite $f'(x)$ za:

Rješenje:

a)
$$\begin{aligned} f(x) &= x^5 - 4x^3 + 2x - 3 \\ f'(x) &= 5x^4 - 4 \cdot 3x^2 + 2 \\ &= 5x^4 - 12x^2 + 2 \end{aligned}$$

b)
$$\begin{aligned} f(x) &= \frac{1}{4} - \frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4 \\ f'(x) &= -\frac{3}{3} \cdot x^2 + 2x - \frac{4}{4} \cdot x^3 \\ &= -x^2 + 2x - x^3 \end{aligned}$$

$$
\begin{aligned}
\text{c) } f(x) &= \frac{ax^6 + bx + c}{\sqrt{a^2 + b^2}} \\
&= \frac{1}{\sqrt{a^2 + b^2}} \left(ax^6 + bx + c\right) \\
f'(x) &= \frac{1}{\sqrt{a^2 + b^2}} \left(6ax^5 + b\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{d) } f(x) &= \frac{2x + 3}{x^2 - 5x + 5} \\
f'(x) &= \frac{(2x + 3)'(x^2 - 5x + 5) - (2x + 3)(x^2 - 5x + 5)'}{(x^2 - 5x + 5)^2} \\
&= \frac{2(x^2 - 5x + 5) - (2x + 3)(2x - 5)}{(x^2 - 5x + 5)^2} \\
&= \frac{2x^2 - 10x + 10 - 4x^2 + 4x + 15}{(x^2 - 5x + 5)^2} \\
&= \frac{-2x^2 - 6x + 25}{(x^2 - 5x + 5)^2}
\end{aligned}
$$

e) $$\begin{aligned} f(x) &= x^2 \cdot 2^{-x} \\ f'(x) &= 2x \cdot 2^{-x} + x^2 \cdot 2^{-x} \cdot \ln 2 \cdot (-1) \\ &= \frac{2x}{2^x} - \frac{x^2 \cdot \ln 2}{2^x} \\ &= \frac{2x - x^2 \ln 2}{2^x} \end{aligned}$$

2. način:

$$\begin{aligned} f(x) &= \frac{x^2}{2^x} \\ f'(x) &= \frac{2x \cdot 2^x - x^2 \cdot 2^x \ln 2}{2^{2x}} \\ &= \frac{2^x \left(2x - x^2 \ln 2\right)}{2^{2x}} \\ &= \frac{2x - x^2 \ln 2}{2^x} \end{aligned}$$

$$
\begin{aligned}
\text{f) } f(x) &= x^2 + \frac{2}{2x-1} - \frac{1}{x} \\
&= x^2 + 2(2x-1)^{-1} - x^{-1} \\
f'(x) &= 2x + 2 \cdot (-1) \cdot (2x-1)^{-2} \cdot 2 - (-1)x^{-2} \\
&= 2x - 4(2x-1)^{-2} + x^{-2} \\
&= 2x - \frac{4}{(2x-1)^2} + \frac{1}{x^2}
\end{aligned}
$$

$$
\begin{aligned}
\text{g) } f(x) &= 5\sin x + 3\cos x \\
f'(x) &= 5\cos x - 3\sin x
\end{aligned}
$$

h)

$$\begin{aligned}
f(x) &= \frac{\sin x + \cos x}{\sin x - \cos x} \\
f'(x) &= \frac{(\cos x - \sin x)(\sin x - \cos x) - (\cos x + \sin x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \\
&= \frac{-(\cos x - \sin x)^2 - (\cos x + \sin x)^2}{(\sin x - \cos x)^2} \\
&= \frac{-\cos^2 x + 2\cos x \sin x - \sin^2 x - \cos^2 x - 2\cos x \sin x - \sin^2 x}{(\sin x - \cos x)^2} \\
&= \frac{-2(\cos^2 x + \sin^2 x)}{(\sin x - \cos x)^2} \\
&= \frac{-2}{(\sin x - \cos x)^2}
\end{aligned}$$

i) $$\begin{aligned} f(x) &= x \cdot \text{arcsin} x \\ f'(x) &= \text{arcsin} x + \frac{x}{\sqrt{1-x^2}} \end{aligned}$$

j) $$\begin{aligned} f(x) &= (x-1) \cdot e^x \\ f'(x) &= 1 \cdot e^x + (x-1)e^x \\ &= (x-1+1)e^x \\ &= xe^x \end{aligned}$$

k) $$\begin{aligned} f(x) &= \frac{e^x}{x^2} \\ f'(x) &= \frac{e^x \cdot x^2 - e^x \cdot 2x}{x^4} \\ &= \frac{xe^x(x-2)}{x^4} \\ &= \frac{e^x(x-2)}{x^3} \end{aligned}$$

l) $$\begin{aligned} f(x) &= e^x \cos x \\ f'(x) &= e^x \cos x + e^x(-\sin x) \\ &= e^x(\cos x - \sin x) \end{aligned}$$

m) $$\begin{aligned} f(x) &= \frac{x^2}{\ln x} \\ f'(x) &= \frac{2x \ln x - x^2 \cdot \frac{1}{x}}{\ln^2 x} \\ &= \frac{x(2\ln x - 1)}{\ln^2 x} \end{aligned}$$

$$\text{n) } f(x) = \frac{1}{x} + 2\ln x - \frac{\ln x}{x}$$

$$f'(x) = -\frac{1}{x^2} + \frac{2}{x} - \frac{\frac{1}{x}\cdot x - \ln x}{x^2}$$

$$= \frac{\ln x - 2}{x^2} + \frac{2}{x}$$

$$\text{o) } f(x) = x \cdot \text{sh}x$$

$$f'(x) = \text{sh}x + x\text{ch}x$$

Zadatak (6.3.)

Odredite $f'(x)$ za:

$$\text{a) } f(x) = \operatorname{arctg}\left(1+\frac{1}{x}\right), \quad g(x) = \operatorname{arctg} x, \quad h(x) = 1+\frac{1}{x}, \quad f = g \circ h$$

$$\begin{aligned}
f'(x) &= \frac{1}{1+\left(1+\frac{1}{x}\right)^2} \cdot \left(1+\frac{1}{x}\right)' \\
&= \frac{1}{1+1+\frac{2}{x}+\frac{1}{x^2}} \cdot \frac{-1}{x^2} \\
&= \frac{1}{\frac{2x^2+2x+1}{x^2}} \cdot \frac{-1}{x^2} \\
&= \frac{x^2}{2x^2+2x+1} \cdot \frac{-1}{x^2} \\
&= \frac{-1}{2x^2+2x+1}
\end{aligned}$$

$$
\begin{aligned}
\text{b) } f(x) &= \sqrt{1-x^2}, \quad g(x)=\sqrt{x}, \;\; h(x)=1-x^2, \;\; f=g\circ h \\
f'(x) &= \frac{1}{2\sqrt{1-x^2}}\cdot(1-x^2)' \\
&= \frac{-2x}{2\sqrt{1-x^2}} \\
&= \frac{-x}{\sqrt{1-x^2}}
\end{aligned}
$$

$$
\text{c) } f(x) = 2x + 5\cos^3 x
$$

$$
g(x)=x^3, \;\; h(x)=\cos x, \;\; f(x)=2x+(g\circ h)(x)
$$

$$
\begin{aligned}
f'(x) &= 2+5\cdot 3\cos^2 x\cdot(-\sin x) \\
&= 2-15\cos^2 x\sin x
\end{aligned}
$$

$$\text{d) } f(x) = \frac{x-2}{\sqrt{x^2+1}}$$

$$f'(x) = \frac{\sqrt{x^2+1} - (x-2)\frac{1}{2\sqrt{x^2+1}} \cdot 2x}{x^2+1}$$

$$= \frac{\frac{x^2+1-x^2+2x}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{2x+1}{(x^2+1)(x^2+1)^{\frac{1}{2}}}$$

$$= \frac{2x+1}{\sqrt{(x^2+1)^3}}$$

$$\text{e) } f(x) = \sqrt{\arcsin x}$$

$$f'(x) = \frac{1}{2\sqrt{\arcsin x}} \cdot \frac{1}{\sqrt{1-x^2}}$$

f) $$f(x) = \sqrt{xe^x + x}$$

$$f'(x) = \frac{1}{2\sqrt{xe^x + x}}\left(e^x + xe^x + 1\right) = \frac{e^x + xe^x + 1}{2\sqrt{xe^x + x}}$$

g) $$f(x) = \arcsin\frac{1}{x^2}$$

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{1}{x^2}\right)^2}} \cdot \frac{-2}{x^3} = \frac{-2}{x^3\sqrt{1 - \frac{1}{x^4}}} = \frac{-2}{x\sqrt{x^4 - 1}}$$

$$\text{h) } f(x) = \ln(1-x^2)$$

$$f'(x) = \frac{1}{1-x^2}\cdot(-2x)$$

$$= \frac{2x}{x^2-1}$$

$$\text{i) } f(x) = \sqrt{\ln x + 1} + \ln\left(\sqrt{x}+1\right)$$

$$f'(x) = \frac{1}{2\sqrt{\ln x+1}}\cdot\frac{1}{x} + \frac{1}{\sqrt{x}+1}\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2x\sqrt{\ln x+1}} + \frac{1}{2\sqrt{x}(\sqrt{x}+1)}$$

$$= \frac{1}{2x\sqrt{\ln x+1}} + \frac{1}{2x+2\sqrt{x}}$$

Logaritamsko deriviranje koristimo kada se argument funkcije x nalazi i u bazi i u eksponentu.

$$\begin{aligned}
f(x) &= g(x)^{h(x)} && /\ln \\
\ln(f(x)) &= h(x)\ln(g(x)) && /' \\
\frac{1}{f(x)}f'(x) &= h'(x)\ln(g(x)) + h(x)\frac{1}{g(x)}g'(x) && /\cdot f(x) \\
f'(x) &= f(x)\left[h'(x)\ln(g(x)) + h(x)\frac{g'(x)}{g(x)}\right]
\end{aligned}$$

Zadatak (6.4.)

Odredite $f'(x)$ za

a) $f(x) = x^x$

b) $f(x) = (\sin x)^x$

c) $f(x) = \left(1 + \dfrac{1}{x}\right)^x$

Rješenje:

$$\begin{aligned} \text{a) } f(x) &= x^x && / \ln \\ \ln f(x) &= x \ln x && /' \\ \frac{1}{f(x)} f'(x) &= \ln x + x \cdot \frac{1}{x} && / \cdot f(x) \\ f'(x) &= x^x (\ln x + 1) \end{aligned}$$

$$
\begin{aligned}
\text{b) } f(x) &= (\sin x)^x && / \ln \\
\ln f(x) &= x \ln(\sin x) && /' \\
\frac{1}{f(x)} f'(x) &= \ln(\sin x) + x \frac{1}{\sin x} \cos x && / \cdot f(x) \\
f'(x) &= (\sin x)^x \left[\ln(\sin x) + x \cdot \mathrm{ctg} x\right]
\end{aligned}
$$

$$
\begin{aligned}
\text{c) } f(x) &= \left(1+\frac{1}{x}\right)^x && /\ln \\
\ln f(x) &= x\ln\left(1+\frac{1}{x}\right) && /' \\
\frac{1}{f(x)}f'(x) &= \ln\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2} && \\
\frac{1}{f(x)}f'(x) &= \ln\left(1+\frac{1}{x}\right) + \frac{-x}{x^2+x} && /\cdot f(x) \\
f'(x) &= \left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)+\frac{-1}{x+1}\right] &&
\end{aligned}
$$

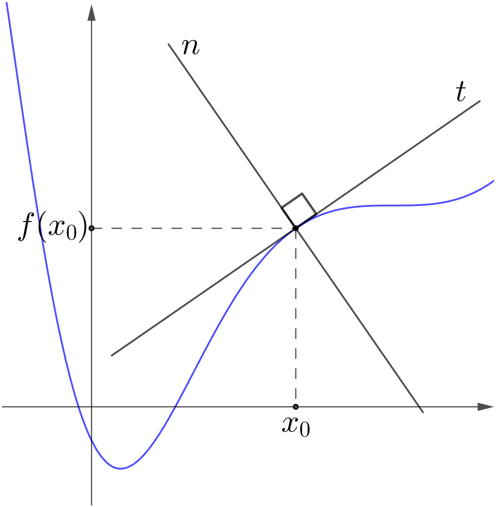
6.2 Tangenta i normala krivulje

Definicija

Tangenta krivulje $(x, f(x))$ u točki x_0 je pravac koji dodiruje krivulju u točno jednoj točki u nekoj okolini točke x_0.

Definicija

Normala krivulje $(x, f(x))$ u točki x_0 je pravac okomit na tangentu te krivulje u točki x_0.


n
t
f(x0)
x0

Derivacija funkcije f u x_0 jednaka je nagibu tangente na graf $f(x)$ u točki x_0, odnosno

$$f'(x_0) = k.$$

Uočimo i da je $y_0 = f(x_0)$ pa izlazi da je jednadžba tangente t

$$t ... y - y_0 = k(x - x_0).$$

Ako vrijedi da je $p_1 \perp p_2$, onda su njihovi koeficijenti smjera negativne recipročne vrijednosti, $k_1 = \dfrac{-1}{k_2}$, pa je jednadžba normale u točki x_0

$$n ... y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0).$$

Zadatak (6.5.)

Napišite jednadžbu tangente i normale na krivulju $y = \sqrt{x}$ u točki s apscisom $x = 4$.

Rješenje: $x = 4 \Longrightarrow y = \sqrt{4} = 2$
$T(x_0, y_0) = T(4, 2)$

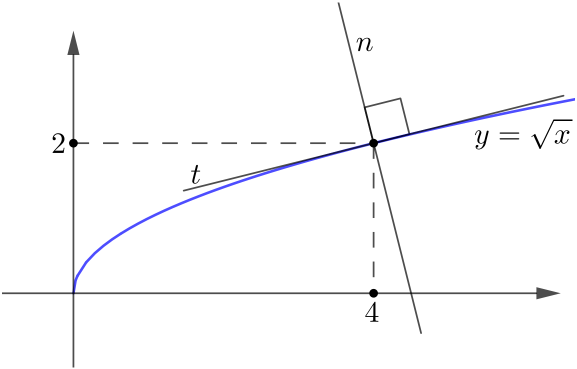
$$f'(x) = \frac{1}{2\sqrt{x}} \Longrightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$k_t = \frac{1}{4} \text{ i } k_n = -4$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$t \dots y = \frac{1}{4}x + 1$$

$$y - 2 = -4(x - 4)$$
$$n \dots y = -4x + 18$$


n
2
t
$y = \sqrt{x}$
4

Zadatak (6.6.)

Odredite jednadžbe svih tangenti na graf funkcije $f(x) = x^4 - x + 3$ koje prolaze ishodištem.

Rješenje:

$f'(x) = 4x^3 - 1$

Pronađimo koordinate dirališta $D(x_0, y_0)$.

$$t \ldots y - y_0 = f'(x)(x - x_0)$$

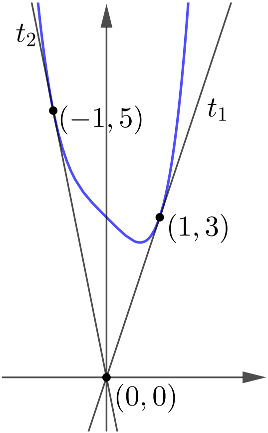
Kako je $O(0,0) \in t$ imamo:

$$\begin{aligned}
0 - (x_0^4 - x_0 + 3) &= (4x_0^3 - 1)(0 - x_0)\\
-x_0^4 + x_0 - 3 &= -4x_0^4 + x_0\\
3x_0^4 &= 3\\
x_0^4 &= 1\\
x_0 = 1 \Rightarrow y_0 = 3 \quad &\text{i} \quad x_0 = -1 \Rightarrow y_0 = 5
\end{aligned}$$

$$y - y_0 = f'(x)(x - x_0)$$

$y - 3 = 3(x - 1)$ $\quad$ $y - 5 = -5(x + 1)$

$t_1 \ldots y = 3x$ $\quad$ $t_2 \ldots y = -5x$



Zadatak (6.7.)

Odredite jednadžbe tangenti povučenih iz $T(9,-2)$ na graf funkcije $f(x)=\dfrac{1}{x-5}$.

Rješenje:

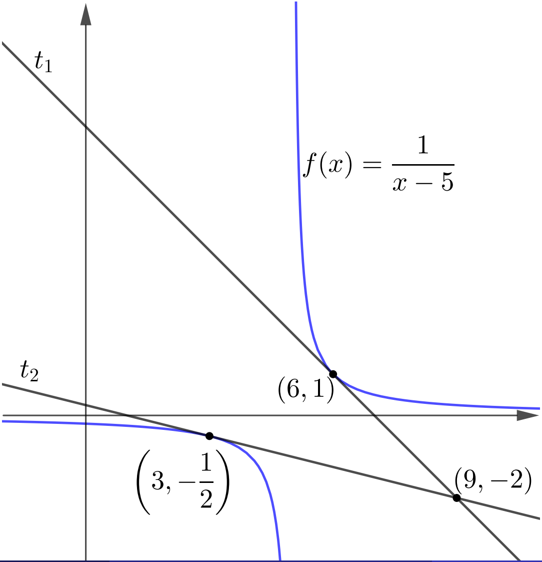
$$f'(x)=\frac{-1}{(x-5)^2}$$

Pronađimo koordinate dirališta $D(x_0,y_0)$.

$$\begin{aligned}
t\ldots y-y_0 &= f'(x)(x-x_0) \text{ i } T(9,-2)\in t\\
-2-\frac{1}{x_0-5} &= \frac{-1}{(x_0-5)^2}(9-x_0) \qquad /\cdot(x_0-5)^2\\
-2(x_0-5)^2-x_0+5 &= -9+x_0\\
-2x_0^2+20x_0-50-x_0+5 &= -9+x_0\\
x_0^2-9x_0+18 &= 0\\
x_0=6\Rightarrow y_0=1 \quad &\text{i} \quad x_0=3\Rightarrow y_0=-\frac{1}{2}
\end{aligned}$$

$$y - y_0 = f'(x)(x - x_0)$$

$$t_1 \ldots y = -x + 7 \qquad t_2 \ldots y = -\frac{1}{4}x + \frac{1}{4}$$



Zadatak (6.8.)

Odredite jednadžbe tangenti na graf funkcije $f(x) = \dfrac{x-1}{x+3}$ paralelnih s pravcem $p \ldots y = x - 2$.

Rješenje:

$$
\begin{aligned}
f'(x) &= \frac{(x-1)'(x+3) - (x-1)(x+3)'}{(x+3)^2} \\
&= \frac{x+3-x+1}{(x+3)^2} \\
&= \frac{4}{(x+3)^2}
\end{aligned}
$$

Pronađimo koordinate dirališta $D(x_0, y_0)$.

$$t \ldots y - y_0 = f'(x)(x - x_0)$$

Kako je $t || p$ imamo da je $k_t = 1 = f'(x_0)$

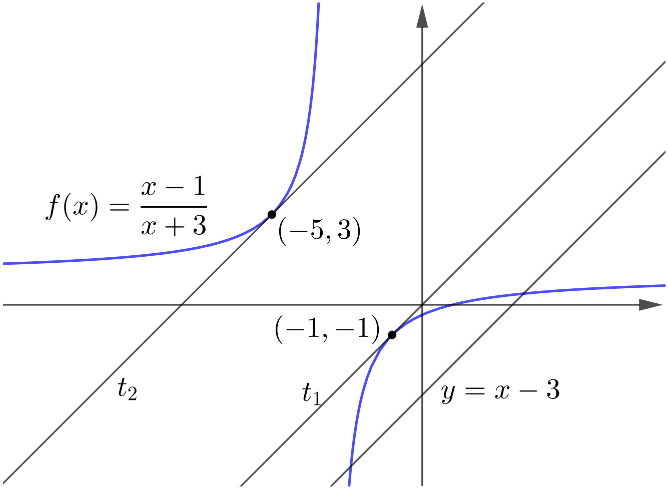
$$\begin{aligned}\frac{4}{(x_0+3)^2} &= 1 \qquad / \cdot (x_0+3)^2 \\ (x_0+3)^2 &= 4 \\ x_0+3 &= \pm 2\end{aligned}$$

$x_0 = -5 \Rightarrow y_0 = 3$ i $x_0 = -1 \Rightarrow y_0 = -1$

$$y - y_0 = f'(x)(x - x_0)$$

$y - 3 = x + 5 \qquad y + 1 = x + 1$

$t_1 \ldots y = x + 8 \qquad t_2 \ldots y = x$


$f(x) = \dfrac{x-1}{x+3}$
$(-5, 3)$
$(-1, -1)$
t_2
t_1
$y = x - 3$

Zadatak (6.9.)

Odredite jednadžbe tangenti na graf funkcije $f(x) = x^3 - 3x^2 + 3x - 4$ paralelnih s pravcem $p ... y = 12x - 12$.

Rješenje:

$f'(x) = 3x^2 - 6x + 3$

$$t ... y - y_0 = f'(x)(x - x_0)$$

Pronađimo koordinate dirališta $D(x_0, y_0)$.

Kako je $t || p$ imamo da je $k_t = 12 = f'(x_0)$

$$\begin{aligned} 3x_0^2 - 6x_0 + 3 &= 12 \quad / : 3 \\ x_0^2 - 2x_0 + 3 &= 0 \end{aligned}$$

$x_0 = 3 \Rightarrow y_0 = 5$ i $x_0 = -1 \Rightarrow y_0 = -11$

$$y - y_0 = f'(x)(x - x_0)$$

$$y - 5 = 12(x - 3) \qquad y + 11 = 12(x + 1)$$

$$t_1 \ldots y = 12x - 31 \qquad t_2 \ldots y = 12x + 1$$

Zadatak (6.10.)

Odredite a i b tako da pravac $y = -2x + 13$ bude tangenta parabole $f(x) = -x^2 + ax + b$ u točki a) $D(3, 7)$ i b) $D(1, 11)$ (sami).

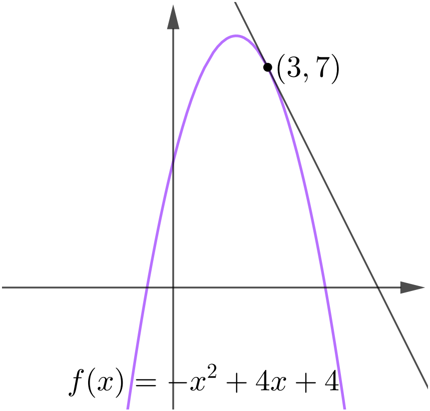
Rješenje: a)

$f'(x) = -2x + a$

$$\begin{aligned} f'(x_0) &= k \\ f'(3) &= -2 \\ -2 \cdot 3 + a &= -2 \\ a &= -2 + 6 \\ a &= 4 \end{aligned}$$

$$\begin{aligned} y_0 &= f(x_0) \\ 7 &= -3^2 + 4 \cdot 3 + b \\ 7 + 9 - 12 &= b \\ b &= 4 \end{aligned}$$

$$f(x) = -x^2 + 4x + 4$$



b)

$$\begin{aligned} f'(x_0) &= k \\ f'(1) &= -2 \\ -2 \cdot 1 + a &= -2 \\ a &= 0 \end{aligned}$$

$$\begin{aligned} y_0 &= f(x_0) \\ 11 &= -1^2 + 0 \cdot 1 + b \\ 11 + 1 &= b \\ b &= 12 \end{aligned}$$

$$f(x) = -x^2 + 12$$

Zadatak (6.11.)

Odredite jednadžbu normale na graf $f(x) = e^{2x} + e^{x} - 2$ u točki u kojoj graf siječe os x.

Rješenje: Odredimo koordinate $D(x_0, 0)$.

$$\begin{aligned} f(x_0) &= e^{2x_0} + e^{x_0} - 2 \qquad (t = e^{x_0}) \\ t^2 + t - 2 &= 0 \\ t^2 + 2t - t - 2 &= 0 \\ (t+2)(t-1) &= 0 \\ t = -2 \quad &\text{ili} \quad t = 1 \\ e^{x_0} = 1 \quad &\text{no} \quad e^{x_0} \neq -2, \text{ jer je } e^{x_0} > 0 \\ x_0 = 0 & \end{aligned}$$

$$D(0,0) \text{ i } f'(x) = 2e^{2x} + e^{x} \implies f'(0) = 3$$

$$y - 0 = -\frac{1}{3}(x - 0)$$

$$n...y = -\frac{1}{3}x$$

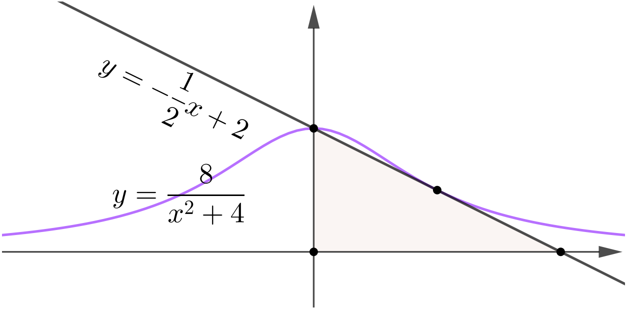
Zadatak (6.12.)

Zadana je funkcija $f(x) = \frac{8}{x^2+4}$ i $T(2,1)$ točka na grafu funkcije. Odredite površinu trokuta što ga zatvara tangenta na graf u toj točki s koordinatnim osima.

Rješenje:

Odredimo jednadžbu tangente u $T(2,1)$.

$$\begin{aligned} f'(x_0) &= \frac{-16x}{(x^2+4)^2} \\ f'(2) &= \frac{-16\cdot 2}{(2^2+4)^2} = -\frac{1}{2} \\ y-1 &= -\frac{1}{2}(x-2) \\ t \ldots y &= -\frac{1}{2}x+2 \end{aligned}$$


$y = -\frac{1}{2}x + 2$
$y = \frac{8}{x^2 + 4}$

Odredimo sjecišta t s koordinatnim osima:

$$0 = -\frac{1}{2}x + 2 \qquad y = -\frac{1}{2} \cdot 0 + 2$$

$$x = 4 \qquad y = 2$$

$$T_x(4, 0) \qquad T_y(0, 2)$$

$$P_\Delta = \frac{2 \cdot 4}{2} = 4$$

Zadatak (6.13.)

U kojoj točki krivulje $y = \ln(2x+1)$ treba postaviti tangentu tako da ona zatvara kut $\alpha = 30^\circ$ sa osi x?

Rješenje:

Tražimo diralište $D(x_0, y_0)$ pri čemu je $f'(x_0) = k = \text{tg}\alpha$.

$$\begin{aligned} \text{tg}30^\circ &= \frac{1}{\sqrt{3}} \\ k &= \frac{1}{\sqrt{3}} \\ f'(x) &= \frac{2}{2x+1} \\ \frac{1}{\sqrt{3}} &= \frac{2}{2x_0+1} \end{aligned}$$

$$\begin{aligned} 2x_0 + 1 &= 2\sqrt{3} \\ 2x_0 &= 2\sqrt{3} - 1 \\ x_0 &= \sqrt{3} - \frac{1}{2} \\ f(x_0) &= \ln\left(2\left(\sqrt{3} - \frac{1}{2}\right) + 1\right) \\ &= \ln 2\sqrt{3} \\ t \ldots y - \ln 2\sqrt{3} &= \frac{1}{\sqrt{3}}\left(x - \sqrt{3} + \frac{1}{2}\right) \end{aligned}$$

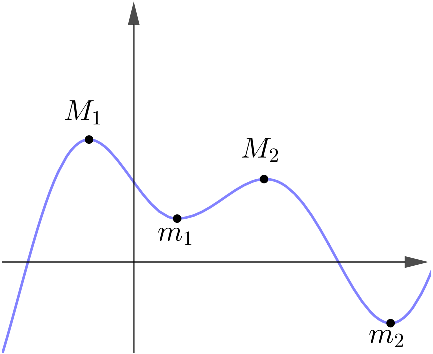
6.3 Lokalni ekstremi, konveksnost i konkavnost

Definicija

Neka je $f : I \to \mathbb{R}$ i $x_0 \in I$. Ako postoji $\varepsilon > 0$ takav da je $f(x_0) \leq f(x)$, $\forall x \in \langle x_0 - \varepsilon, x_0 + \varepsilon \rangle$, kažemo da f u x_0 ima **lokalni minimum**.

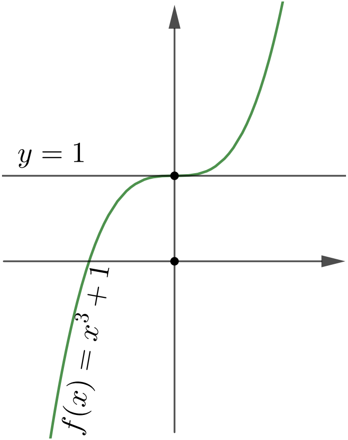
Definicija

Neka je $f : I \to \mathbb{R}$ i $x_0 \in I$. Ako postoji $\varepsilon > 0$ takav da je $f(x_0) \geq f(x)$, $\forall x \in \langle x_0 - \varepsilon, x_0 + \varepsilon \rangle$, kažemo da f u x_0 ima **lokalni maksimum**.



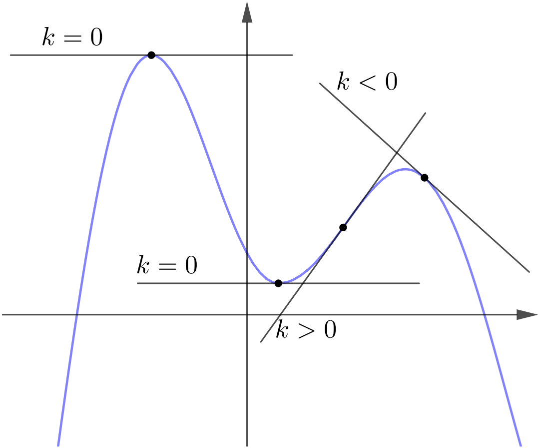
Nužan uvjet za lokalni ekstrem u x_0: $f'(x_0) = 0$. Ako f ima lokalni ekstrem u x_0 onda je $f'(x_0) = 0$ i za x_0 kažemo da je **stacionarna točka**.

No, to nije dovoljan uvjet: za npr. $f(x) = x^3 + 1$ u $x_0 = 0$ vrijedi $f'(x) = 3x^2$ i $f'(0) = 0$, no funkcija nema ekstrem u 0.



Ako funkcija f na intervalu I

- raste, onda je $f'(x) \geq 0, \forall x \in I$
- pada, onda je $f'(x) \leq 0, \forall x \in I$
- ima ekstrem, onda je $f'(x_0) = 0$ za neki $x_0 \in I$



Neka je x_0 stacionarna točka, tj. $f'(x_0) = 0$. Ako je

- $f''(x_0) > 0$, onda f u x_0 ima lokalni minimum
- $f''(x_0) < 0$, onda f u x_0 ima lokalni maksimum
- $f''(x_0) = 0$, onda računamo derivacije dok ne dobijemo k takav da je $f^k(x_0) \neq 0$. Ako je
  - k paran, i
    - $f^k(x_0) < 0$, onda f u x_0 ima lokalni maksimum
    - $f^k(x_0) > 0$, onda f u x_0 ima lokalni minimum
  - k neparan, onda f u x_0 nema ekstrem

Zadatak

Odredite ekstreme i intervale rasta i pada funkcije:

Rješenje: a)

$$\begin{aligned} f(x) &= x^2(x-12)^2 \\ &= x^2(x^2-24x+144) \\ &= x^4-24x^3+144x^2 \end{aligned}$$

$$f'(x) = 4x^3-72x^2+288x$$

Odredimo stacionarne točke:

$$\begin{aligned} 4x^3-72x^2+288x &= 0 \quad / : 4 \\ x^3-18x^2+72x &= 0 \\ x(x^2-18x+72) &= 0 \\ x(x^2-12x-6x+72) &= 0 \\ x(x-12)(x-6) &= 0 \end{aligned}$$

$x_1 = 0$, $x_2 = 12$, $x_3 = 6$

	$-\infty$		0		6		12		$+\infty$
f'		$-$		$+$		$-$		$+$	
f		$\searrow$	m	$\nearrow$	M	$\searrow$	m	$\nearrow$	

Funkcija f ima u 0 i 12 lokalni minimum, a u 6 lokalni maksimum.

Karakter ekstrema možemo odrediti i preko druge derivacije:

$$
\begin{aligned}
f''(x) &= 12x^2 - 144x + 288 \\
f''(0) &= 288 > 0, \text{ pa } f \text{ u } 0 \text{ ima lokalni minimum.} \\
f''(6) &= -144 < 0, \text{ pa } f \text{ u } 6 \text{ ima lokalni maksimum.} \\
f''(12) &= 288 > 0, \text{ pa } f \text{ u } 12 \text{ ima lokalni minimum.}
\end{aligned}
$$

b) (sami)

$$f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 1$$
$$f'(x) = x^3 - 3x^2 - 4x$$

Odredimo stacionarne točke:

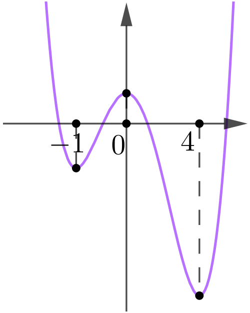
$$x^3 - 3x^2 - 4x = 0$$
$$x(x^2 - 3x - 4) = 0$$
$$x(x^2 - 4x + x - 4) = 0$$
$$x(x-4)(x+1) = 0$$

$x_1 = 0$, $x_2 = 4$, $x_3 = -1$

	$-\infty$		-1		0		4		$+\infty$
f'		$-$		$+$		$-$		$+$	
f		$\searrow$	m	$\nearrow$	M	$\searrow$	m	$\nearrow$	

Funkcija f ima u -1 i 4 lokalni minimum, a u 0 lokalni maksimum.

Intervali rasta su $\langle -1, 0\rangle$ i $\langle 4, +\infty\rangle$, a intervali pada su $\langle -\infty, -1\rangle$ i $\langle 0, 4\rangle$



c)

$$f(x) = \frac{x^2 - 2x + 2}{x - 1}$$

$$D_f = \mathbb{R} \setminus \{1\}$$

$$\begin{aligned} f'(x) &= \frac{(2x-2)(x-1) - (x^2 - 2x + 2)}{(x-1)^2} \\ &= \frac{2x^2 - 2x - 2x + 2 - x^2 + 2x - 2}{(x-1)^2} \\ &= \frac{x^2 - 2x}{(x-1)^2} \end{aligned}$$

Odredimo stacionarne točke:

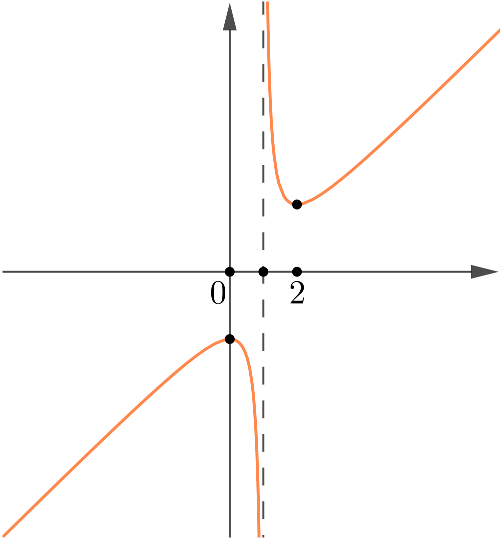
$$
\begin{aligned}
\frac{x^2-2x}{(x-1)^2} &= 0\\
x^2-2x &= 0\\
x(x-2) &= 0
\end{aligned}
$$

$x_1 = 0,\ x_2 = 2$

	$-\infty$		0		1		2		$+\infty$
f'	$\Vert$	$+$	$\vert$	$-$	$\vdots$	$-$	$\vert$	$+$	$\vert$
f	$\Vert$	$\nearrow$	M	$\searrow$	$\vdots$	$\searrow$	m	$\nearrow$	$\vert$

Funkcija f ima u 2 lokalni minimum, a u 0 lokalni maksimum.

Intervali rasta su $\langle -\infty, 0\rangle$ i $\langle 2, +\infty\rangle$, a intervali pada su $\langle 0, 1\rangle$ i $\langle 1, 2\rangle$.


0
2

d) (sami)

$$f(x) = \frac{x^2}{1+x^2} \qquad \frac{2x}{(1+x^2)^2} = 0$$

$$D_f = \mathbb{R} \qquad x_1 = 0$$

$$f'(x) = \frac{2x}{(1+x^2)^2}$$

	$-\infty$		0		$+\infty$
f'	‖	$-$		$+$	
f	‖	$\searrow$	m	$\nearrow$	

Interval rasta je $\langle 0, +\infty\rangle$, a interval pada je $\langle -\infty, 0\rangle$. U točki $x = 0$ funkcija ima lokalni minimum.

e) (sami)

$$f(x) = \frac{4}{\sqrt{x^2+8}}$$

$$D_f = \mathbb{R}$$

$$f'(x) = \frac{-4 \cdot \frac{1}{2\sqrt{x^2+8}} \cdot 2x}{x^2+8} = \frac{-4x}{(x^2+8)\sqrt{x^2+8}}$$

Odredimo stacionarne točke:

$$\frac{-4x}{(x^2+8)\sqrt{x^2+8}} = 0$$

$$-4x = 0$$

$$x_1 = 0$$

	$-\infty$		0		$+\infty$
f'	‖	$+$	|	$-$	|
f	‖	$\nearrow$	M	$\searrow$	|

Interval pada je $\langle 0, +\infty \rangle$, a interval rasta je $\langle -\infty, 0 \rangle$. U točki $x = 0$ funkcija ima lokalni maksimum.

f) (sami)

$$
\begin{aligned}
f(x) &= x - \ln(1+x) \\
D_f &= \langle -1, +\infty \rangle \\
f'(x) &= 1 - \frac{1}{1+x} \\
&= \frac{x}{1+x}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
\frac{x}{1+x} &= 0 \\
x_1 &= 0
\end{aligned}
$$

	-1		0		$+\infty$
f'	‖	$-$		$+$	
f	‖	↘	m	↗	

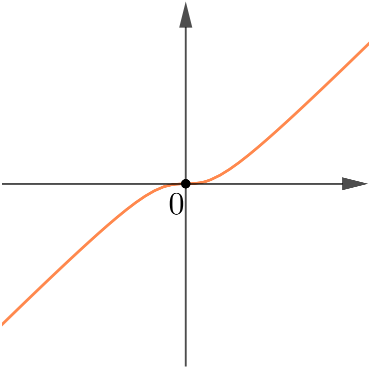
g)

$$\begin{aligned} f(x) &= x - \text{arctg} x \\ D_f &= \mathbb{R} \\ f'(x) &= 1 - \frac{1}{1+x^2} \\ &= \frac{x^2}{1+x^2} \end{aligned}$$

$$\begin{aligned} \frac{x^2}{1+x^2} &= 0 \\ x_1 &= 0 \end{aligned}$$

	$-\infty$		0		$+\infty$
f'	‖	$+$	|	$+$	|
f	‖	$\nearrow$	| m	$\nearrow$	|

Raste na cijeloj domeni. Nema lokalnih esktrema.


0

Zadatak

Ispitajte ekstreme funkcije $f(x) = x^4 \cdot e^{-x^2}$ pomoću f''.

Rješenje:

$$
\begin{aligned}
f(x) &= x^4 \cdot e^{-x^2}, \qquad D_f = \mathbb{R} \\
f'(x) &= 4x^3 \cdot e^{-x^2} + x^4 \cdot e^{-x^2} \cdot (-2x) \\
&= e^{-x^2}(4x^3 - 2x^5)
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
e^{-x^2}(4x^3 - 2x^5) &= 0 / : e^{-x^2} \neq 0 \\
x^3(4 - 2x^2) &= 0
\end{aligned}
$$

Kandidati: $x_1 = 0$, $x_2 = -\sqrt{2}$, $x_3 = \sqrt{2}$

$$
\begin{aligned}
f''(x) &= e^{-x^2}\cdot(-2x)\cdot(4x^3-2x^5)+e^{-x^2}\cdot(12x^2-10x^4)\\
&= e^{-x^2}\cdot(-8x^4+4x^6)+e^{-x^2}\cdot(12x^2-10x^4)\\
&= e^{-x^2}(4x^6-18x^4+12x^2)
\end{aligned}
$$

$$
\begin{aligned}
f''(\sqrt{2}) &= e^{-2}\cdot(-16)<0\\
f''(-\sqrt{2}) &= e^{-2}\cdot(-16)<0
\end{aligned}
$$

f u $\pm\sqrt{2}$ ima lok.makimum

$$
\begin{aligned}
f''(0) &= 0\\
f'''(x) &= e^{-x^2}(-8x^7+60x^5-96x^3+24x)\\
f'''(0) &= 0\\
f^{iv}(x) &= e^{-x^2}(16x^8-176x^6+492x^4-336x^2+24)\\
f^{iv}(0) &= 24>0
\end{aligned}
$$

f u 0 ima lok. minimum

Definicija

Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ i neka je $J \subseteq I$. Ako vrijedi: $\forall x_1, x_2 \in J$ je

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2},$$

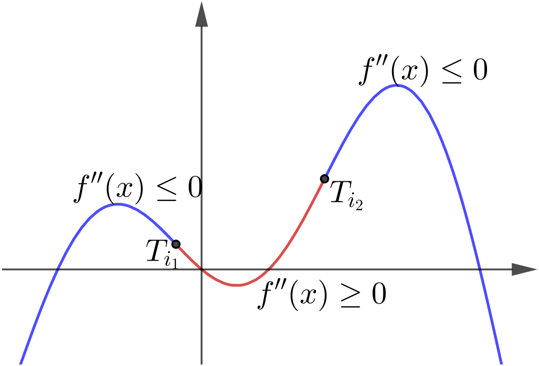
onda za f kažemo da je **konveksna** na J.

Definicija

Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$ i neka je $J \subseteq I$. Ako vrijedi: $\forall x_1, x_2 \in J$ je

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2},$$

onda za f kažemo da je **konkavna** na J.



Točka T u kojoj graf mjenja oblik iz konveksnog u konkavni i obratno naziva se **točka infleksije**. Ako je:

- $f''(x) \geq 0$, onda je f konveksna;
- $f''(x) \leq 0$, onda je f konkavna;
- $f''(x_0) = 0$, onda je x_0 kandidat za točku infleksije.

Nužan uvjet za točku infleksije u x_0 je $f''(x_0) = 0$, no to nije i dovoljan uvijet.

Zadatak

Odredite točke infleksije i intervale konveksnosti i konkavnosti za:

a) $f(x) = (x^2 - 3x + 2)e^x$

b) $f(x) = x \ln^2 x$ (sami)

c) $f(x) = \dfrac{1}{3x^2+1}$ (sami)

Rješenje: a)

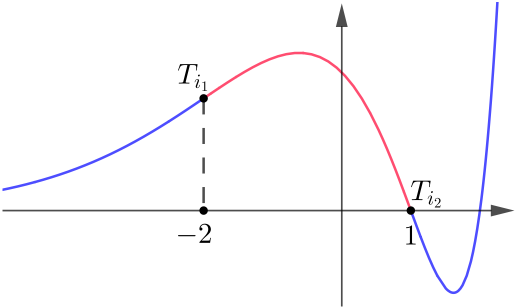
$$
\begin{aligned}
f(x) &= (x^2 - 3x + 2)e^x, \qquad D_f = \mathbb{R} \\
f'(x) &= (2x-3)e^x + (x^2 - 3x + 2)e^x \\
f''(x) &= 2e^x + (2x-3)e^x + (2x-3)e^x + (x^2 - 3x + 2)e^x \\
&= (2 + 2x - 3 + 2x - 3 + x^2 - 3x + 2)e^x \\
&= (x^2 + x - 2)e^x
\end{aligned}
$$

Odredimo točke infleksije:

$$\begin{aligned} (x^2+x-2)e^x &= 0 / : e^x \neq 0 \\ x^2+2x-x-2 &= 0 \\ x(x+2)-(x+2) &= 0 \\ (x-1)(x+2) &= 0 \end{aligned}$$

	$-\infty$	-2	1	$+\infty$
f''	$+$	$-$	$+$	
f	$\cup$	$\cap$	$\cup$	

Kandidati: $x_1 = 1$, $x_2 = -2$



Intervali konveksnosti su $\langle -\infty, -2\rangle$ i $\langle 1, +\infty\rangle$, a interval konkavnosti je $\langle -2, 1\rangle$. Točke infleksije su $T_{i_1}(-2, 12e^{-2})$ i $T_{i_2}(1, 0)$.

b)

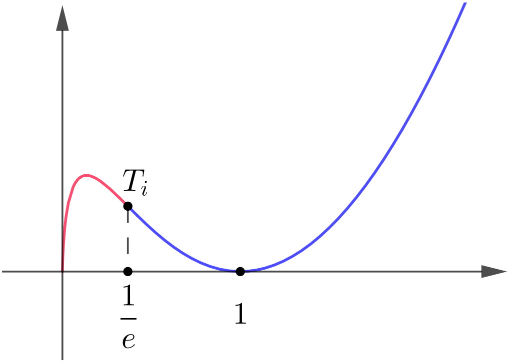
$$\begin{aligned} f(x) &= x\ln^2 x, \qquad D_f = \langle 0, +\infty\rangle \\ f'(x) &= \ln^2 x + 2x\ln x \cdot \frac{1}{x} \\ &= \ln^2 x + 2\ln x \\ f''(x) &= 2\ln x \cdot \frac{1}{x} + \frac{2}{x} \\ &= \frac{2}{x}(\ln x + 1) \end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned} \frac{2}{x}(\ln x + 1) &= 0 / \cdot \frac{x}{2} \neq 0 \\ \ln x &= -1 \end{aligned}$$

	0		$\frac{1}{e}$		$+\infty$
f''	‖	$-$	|	$+$	|
f	‖	$\cap$	|	$\cup$	|

Kandidati: $x_1 = \dfrac{1}{e}$



Interval konveksnosti je $\left\langle 1, \frac{1}{e} \right\rangle$, a interval konkavnosti je $\left\langle \frac{1}{e}, +\infty \right\rangle$. Točka infleksije je $T_i\left(\frac{1}{e}, \frac{1}{e}\right)$.

c)

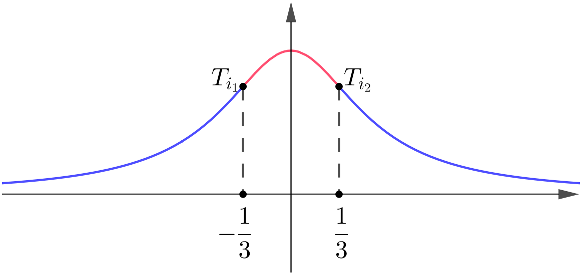
$$\begin{aligned}
f(x) &= \frac{1}{3x^2+1}, \qquad D_f = \mathbb{R} \\
f'(x) &= \frac{-6x}{(3x^2+1)^2} \\
f''(x) &= \frac{-6(3x^2+1)^2 + 6x \cdot 2(3x^2+1) \cdot 6x}{(3x^2+1)^4} \\
&= \frac{(3x^2+1)\left[-6(3x^2+1) + 72x^2\right]}{(3x^2+1)^4} \\
&= \frac{-18x^2 - 6 + 72x^2}{(3x^2+1)^3} \\
&= \frac{54x^2 - 6}{(3x^2+1)^3}
\end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned} \frac{54x^2-6}{(3x^2+1)^3} &= 0 \;\Big/ \cdot \frac{(3x^2+1)^3}{6} \neq 0 \\ 9x^2 &= 1 \\ x^2 &= \frac{1}{9} \end{aligned}$$

	$-\infty$	$-\frac{1}{3}$	$\frac{1}{3}$	$+\infty$
f''	$+$	$-$	$+$	
f	$\cup$	$\cap$	$\cup$	

Kandidati: $x_1 = \frac{1}{3}$, $x_2 = -\frac{1}{3}$



Intervali konveksnosti su $\left\langle -\infty, -\frac{1}{3} \right\rangle$ i $\left\langle -\frac{1}{3}, +\infty \right\rangle$, a interval konkavnosti je $\left\langle -\frac{1}{3}, \frac{1}{3} \right\rangle$. Točke infleksije su $T_{i_1}\left(-\frac{1}{3}, \frac{3}{4}\right)$ i $T_{i_2}\left(\frac{1}{3}, \frac{3}{4}\right)$.

6.4 Primjena ekstrema

Zadatak

Konopcem duljine 12m treba omeđiti dio ravnog terena oblika kružnog isječka najveće moguće površine. Kolika je ta površina?

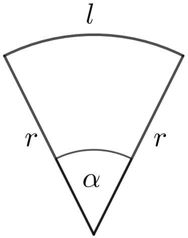
Rješenje:

$$l = \frac{\alpha}{2\pi} \cdot O_\circ = \frac{\alpha}{2\pi} \cdot 2r\pi = \alpha r$$

$$P_{Kl} = \frac{\alpha}{2\pi} \cdot P_\circ = \frac{\alpha}{2\pi} r^2\pi = \frac{\alpha r^2}{2} = \frac{rl}{2}$$

Prema uvjetu zadatka vrijedi: $l + 2r = 12$, odnosno $l = 12 - 2r$.

$$\begin{aligned} P_{Kl} &= \frac{rl}{2} \\ P_{Kl}(r) &= \frac{r(12-2r)}{2} \\ &= 6r - r^2 \end{aligned}$$



Maksimum ove funkcije je najveća površina koju tražimo.

$$
\begin{aligned}
P'_{Kl}(r) &= 6-2r\\
0 &= 6-2r\\
r &= 3
\end{aligned}
$$

$$
\begin{aligned}
P''_{Kl}(r) &= -2\\
P''_{Kl}(3) &= -2<0
\end{aligned}
$$

P u 3 zaista ima maksimum.

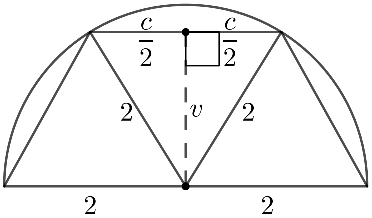
Tražena površina je:

$$
\begin{aligned}
P_{Kl}(3) &= \frac{3(12-2\cdot 3)}{2}\\
&= 9\text{cm}^2
\end{aligned}
$$

Zadatak

U polukrug polumjera $r = 2$cm upišite jednakokračni trapez maksimalne površine takav da je $a = 2r$. Kolika je površina tog trapeza?

Rješenje:



$$
\begin{aligned}
a &= 4 \\
v^2 &= 2^2 - \left(\frac{c}{2}\right)^2 \\
v &= \sqrt{4 - \frac{c^2}{4}}
\end{aligned}
$$

$$
\begin{aligned}
P &= \frac{(a+c)}{2}v \\
&= \frac{(4+c)}{2}\sqrt{4-\frac{c^2}{4}} \\
&= \frac{1}{4}(4+c)\sqrt{16-c^2} \\
P' &= \frac{1}{4}\left(\sqrt{16-c^2}+(4+c)\frac{1}{2\sqrt{16-c^2}}(-2c)\right) \\
&= \frac{1}{4}\left(\sqrt{16-c^2}-\frac{c(4+c)}{\sqrt{16-c^2}}\right) \\
&= \frac{1}{4}\left(\frac{16-c^2-4c-c^2}{\sqrt{16-c^2}}\right) \\
&= \frac{1}{4}\left(\frac{16-4c-2c^2}{\sqrt{16-c^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{1}{4}\left(\frac{16-4c-2c^2}{\sqrt{16-c^2}}\right) \qquad / \cdot 4\sqrt{16-c^2} \\
0 &= 16-4c-2c^2 \\
0 &= c^2+2c-8
\end{aligned}
$$

$c_1 = -4$ i $c_2 = 2$
$c_1 = -4$ odbacujemo jer $c > 0$

	-4		2		$+\infty$
P'		$+$		$-$	
P		$\nearrow$	M	$\searrow$	

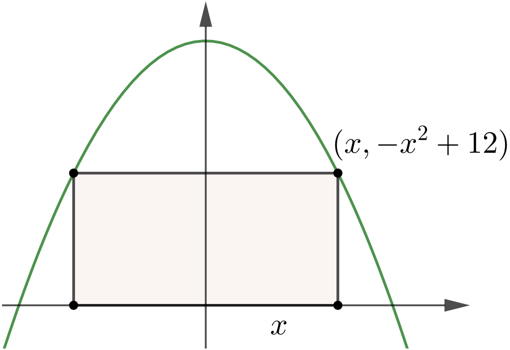
P u 2 zaista ima maksimum.

$$
\begin{aligned}
P(2) &= \frac{1}{4}(4+2)\sqrt{16-2^2} \\
&= 3\sqrt{3}
\end{aligned}
$$

Zadatak

U lik omeđen parabolom $y = -x^2 + 12$ i osi x upisujemo pravokutnik maksimalne površine. Kolika je površina tog pravokutnika?

Rješenje:



$$\begin{aligned} P &= ab \\ P(x) &= 2x(-x^2 + 12) \\ &= -2x^3 + 24x \end{aligned}$$

$$\begin{aligned} P'(x) &= -6x^2 + 24 \\ 0 &= -6x^2 + 24 \\ x^2 &= 4 \end{aligned}$$

$x_1 = -2$ i $x_2 = 2$

$x_1 = -2$ odbacujemo jer $x > 0$

$$\begin{aligned} P''(x) &= -12x \\ P''(2) &= -24 < 0 \end{aligned}$$

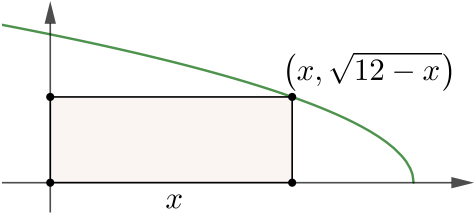
P u 2 zaista ima maksimum.

$$\begin{aligned} P(2) &= -2 \cdot 2^3 + 24 \cdot 2 \\ P(2) &= 32 \end{aligned}$$

Zadatak

U lik omeđen grafom funkcije $f(x) = \sqrt{12 - x}$ i koordinatnim osima upisujemo pravokutnik maksimalne površine. Kolika je površina tog pravokutnika?

Rješenje:



$$\begin{aligned} P &= ab \\ P(x) &= x\sqrt{12 - x} \end{aligned}$$

$$\begin{aligned} P'(x) &= \sqrt{12 - x} + x \cdot \frac{1}{2\sqrt{12 - x}} \cdot (-1) \\ &= \frac{2(12 - x) - x}{2\sqrt{12 - x}} \\ &= \frac{24 - 3x}{2\sqrt{12 - x}} \end{aligned}$$

$$
\begin{aligned}
0 &= \frac{24-3x}{2\sqrt{12-x}} \quad / \cdot 2\sqrt{12-x} \\
0 &= 24-3x \\
x &= 8
\end{aligned}
$$

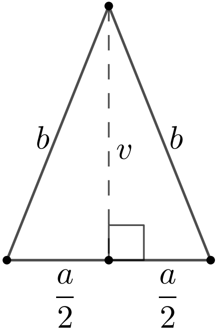
	0		8		12
P'		$+$		$-$	
P		$\nearrow$	M	$\searrow$	

$$
\begin{aligned}
P(8) &= 8\sqrt{12-8} \\
&= 16
\end{aligned}
$$

Zadatak

Žicu duljine 12 m savijamo u oblik jednakokračnog trokuta. Kolika je maksimalna površina tog trokuta?

Rješenje:



$$
\begin{aligned}
P &= \frac{av}{2} \\
v^2 &= b^2 - \frac{a^2}{4} \\
v &= \sqrt{\frac{4b^2 - a^2}{4}}
\end{aligned}
$$

$$
\begin{aligned}
o &= a + 2b \\
12 &= a + 2b \\
a &= 12 - 2b
\end{aligned}
$$

$$
\begin{aligned}
P(b) &= \frac{(12-2b)\left(\sqrt{\frac{4b^2-(12-2b)^2}{4}}\right)}{2} \\
&= \frac{2\,(6-b)\left(\sqrt{\frac{4b^2-144+48b-4b^2}{4}}\right)}{2} \\
&= (6-b)\sqrt{\frac{4(12b-36)}{4}} \\
&= (6-b)\sqrt{12b-36}
\end{aligned}
$$

$$\begin{aligned}
P'(b) &= -\sqrt{12b-36} + \frac{12(6-b)}{2\sqrt{12b-36}} \\
&= -\sqrt{12b-36} + \frac{6(6-b)}{\sqrt{12b-36}} \\
&= \frac{-(12b-36)+6(6-b)}{\sqrt{12b-36}} \\
&= \frac{-18b+72}{\sqrt{12b-36}}
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{-18b+72}{\sqrt{12b-36}} \quad / \cdot \sqrt{12b-36} \\
0 &= -18b+72 \\
b &= 4
\end{aligned}$$

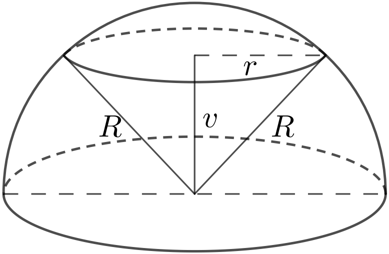
	3		4		12
P'	‖	$+$		$-$	
P	‖	↗	M	↘	

$$\begin{aligned}
P(4) &= (6-4)\sqrt{12\cdot 4-36} \\
&= 2\sqrt{12} \\
&= 4\sqrt{3}
\end{aligned}$$

Zadatak

U polukuglu radijusa R upisan je stožac s vrhom u centru baze polukugle. Odredite radijus baze stošca maksimalnog volumena.

Rješenje:



$$
\begin{aligned}
R^2 &= v^2 + r^2 \\
r^2 &= R^2 - v^2 \\
\\
V_S &= \frac{1}{3} B \cdot v \\
&= \frac{1}{3} r^2 \pi \cdot v \\
&= \frac{1}{3} \left(R^2 - v^2\right) \pi \cdot v \\
V(v) &= \frac{\pi}{3} \left(R^2 v - v^3\right)
\end{aligned}
$$

$$
\begin{aligned}
V'(v) &= \frac{\pi}{3}\left(R^2 - 3v^2\right) \\
0 &= \frac{\pi}{3}\left(R^2 - 3v^2\right) \\
v^2 &= \frac{R^2}{3} \\
v &= \frac{R\sqrt{3}}{3} \\
V''(v) &= \frac{\pi}{3}(-6v) \\
&= -2\pi v \\
V''\left(\frac{R\sqrt{3}}{3}\right) &= -2\pi\frac{R\sqrt{3}}{3} < 0
\end{aligned}
$$

V u $\dfrac{R\sqrt{3}}{3}$ ima maksimum.

Traženi radijus je:

$$
\begin{aligned}
r^2 &= R^2 - v^2 \\
r^2 &= R^2 - \left(\frac{R\sqrt{3}}{3}\right)^2 \\
&= R^2 - \frac{3R^2}{9} \\
&= \frac{2}{3}R^2 \\
r &= \frac{\sqrt{2}}{\sqrt{3}}R \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{\sqrt{6}}{3}R
\end{aligned}
$$

Zadatak

Na paraboli $y = 3x^2$ odredite onu točku koja je nabliža točki $A(0,3)$.

Rješenje: Neka je $T(x,y)$ točka na paraboli $y = 3x^2$.

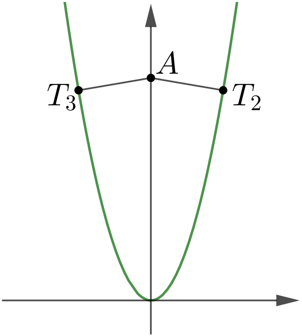
$$\begin{aligned}
\mathrm{d}(T,A) &= \sqrt{(x-0)^2 + (y-3)^2)} \\
&= \sqrt{x^2 + (3x^2-3)^2)} \\
&= \sqrt{x^2 + 9x^4 - 18x^2 + 9} \\
d(x) &= \sqrt{9x^4 - 17x^2 + 9} \\
d'(x) &= \frac{36x^3 - 34x}{2\sqrt{9x^4 - 17x^2 + 9}} \\
0 &= \frac{36x^3 - 34x}{2\sqrt{9x^4 - 17x^2 + 9}} \\
0 &= x(36x^2 - 34)
\end{aligned}$$

Kandidati: $x_1 = 0$, $x_2 = \frac{\sqrt{34}}{6}$, $x_3 = \frac{-\sqrt{34}}{6}$

$$T_1(0,0) \qquad d(T_1, A) = \sqrt{(0-0)^2 + (0-3)^2)} = 3$$

$$T_2\left(\frac{\sqrt{34}}{6}, \frac{17}{6}\right) \qquad d(T_2, A) = \sqrt{\left(\frac{\sqrt{34}}{6} - 0\right)^2 + \left(\frac{17}{6} - 1\right)^2} = \frac{\sqrt{35}}{6}$$

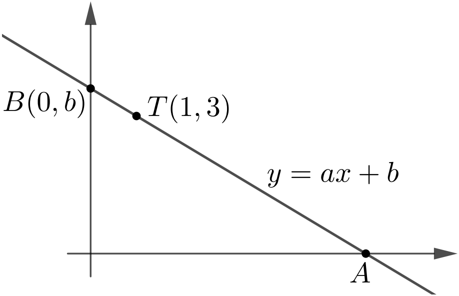
$$T_3\left(-\frac{\sqrt{34}}{6}, \frac{17}{6}\right) \qquad d(T_1, A) = \sqrt{\left(\frac{-\sqrt{34}}{6} - 0\right)^2 + \left(\frac{17}{6} - 1\right)^2} = \frac{\sqrt{35}}{6}$$



Zadatak

Točkom $T(1,3)$ polažemo padajući pravac koji s koordinatnim osima zatvara trokut najmanje površine. Koliki je koeficijent smjera tog pravca?

Rješenje:



Točka B je sjecište pravca p i osi $y \Longrightarrow B(0,b)$.

Točka A je sjecište pravca p i osi $x \Longrightarrow A\left(-\frac{b}{a}, 0\right)$.

Kako je $T \in p$ imamo $3 = a \cdot 1 + b$, odnosno $b = 3 - a$.

$$\begin{aligned} P_\Delta &= \frac{1}{2}\cdot\frac{-b}{a}\cdot b \\ &= \frac{1}{2}\cdot\frac{-b^2}{a} \\ P(a) &= \frac{-(3-a)^2}{2a} \\ &= \frac{-9+6a-a^2}{2a} \\ &= \frac{-9}{2a}+3-\frac{a}{2} \end{aligned}$$

$$\begin{aligned} P'(a) &= \frac{9}{2a^2}-\frac{1}{2} \\ 0 &= \frac{9}{2a^2}-\frac{1}{2} \\ \frac{1}{2} &= \frac{9}{2a^2} \\ a^2 &= 9 \\ a &= -3 \end{aligned}$$

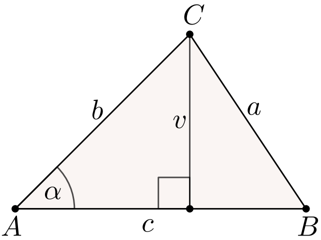
$$\begin{aligned} P''(a) &= -\frac{9}{a^3} \\ P''(-3) &= -\frac{9}{(-3)^3} \\ &= \frac{1}{3} > 0 \end{aligned}$$

Minimalna površina dobije se za pravac $y = -3x + 6$.

Zadatak

Neka je trokut ΔABC takav da je $\alpha = \frac{\pi}{6}$, te $b + c = 100$cm. Odredite duljinu stranice c tako da površina trokuta bude maksimalna.

Rješenje:



$$\begin{aligned} \sin\frac{\pi}{6} &= \frac{v}{b} \\ \frac{1}{2} &= \frac{v}{b} \\ v &= \frac{1}{2}b \end{aligned}$$

$$\begin{aligned} P_\Delta &= \frac{1}{2}\cdot c\cdot v \\ &= \frac{1}{2}\cdot c\cdot\frac{1}{2}b \\ &= \frac{1}{4}\cdot c\cdot(100-c) \\ P(c) &= \frac{1}{4}\cdot\left(100c-c^2\right) \end{aligned}$$

$$
\begin{aligned}
P'(c) &= \frac{1}{4}\cdot(100-2c) \\
0 &= \frac{1}{4}\cdot(100-2c) \\
2c &= 100 \\
c &= 50
\end{aligned}
$$

$$
\begin{aligned}
P''(c) &= \frac{1}{4}\cdot(-2) \\
&= -\frac{1}{2} \\
P''(50) &= -\frac{1}{2} < 0
\end{aligned}
$$

P u 50 ima maksumum.

$$
\begin{aligned}
P(50) &= \frac{1}{4}\cdot\left(100\cdot 50 - 50^2\right) \\
&= \frac{1}{4}\cdot(5000-2500) \\
&= \frac{2500}{4} \\
&= 625\text{cm}^2
\end{aligned}
$$

6.5 L'Hospitalovo pravilo

Neodređeni oblici:

- $\frac{\infty}{\infty}$, $\frac{0}{0}$
- $0 \cdot \infty$
- $\infty - \infty$
- 0^0, 1^∞, ∞^0

Ako je $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ oblika $\dfrac{0}{0}$ ili $\dfrac{\infty}{\infty}$, onda je

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Ako je $\lim\limits_{x \to x_0} f(x) \cdot g(x)$ oblika $0 \cdot \infty$, onda je $\lim\limits_{x \to x_0} \dfrac{f(x)}{\frac{1}{g(x)}}$ oblika $\dfrac{0}{0}$, a $\lim\limits_{x \to x_0} \dfrac{g(x)}{\frac{1}{f(x)}}$ oblika $\dfrac{\infty}{\infty}$.

Zadatak

Izračunajte sljedeće limese:

a) $\lim\limits_{x\to 0} \dfrac{(1+x)^n - 1}{x}$

b) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$ (sami)

c) $\lim\limits_{x\to 0} \dfrac{\text{ch}x - 1}{1 - \cos x}$ (sami)

d) $\lim\limits_{x\to +\infty} \dfrac{x^3}{e^{2x}}$

e) $\lim\limits_{x\to +\infty} \dfrac{\ln x}{x^n}$ (sami)

f) $\lim\limits_{x\to 4} \dfrac{x^2 - 16}{x^2 + x - 20}$ (sami)

g) $\lim\limits_{x\to 1} \dfrac{\ln x}{x - 1}$ (sami)

h) $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\sin x}$ (sami)

i) $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x}$

k) $\lim\limits_{x\to 0} \dfrac{\text{tg}x - \sin x}{\sin^3 x}$

l) $\lim\limits_{x\to 0} \dfrac{\ln 2 - \ln x}{\ln 2 + \ln x}$ (sami)

m) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

n) $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{x}{\ln x}\right)$ (sami)

o) $\lim\limits_{x\to 0^+} (\text{sh}x \cdot \ln x)$

p) $\lim\limits_{x\to 0} \left(x \cdot \ln^2 x\right)$ (sami)

r) $\lim\limits_{x\to +\infty} \left(x \cdot \ln\left(e + \dfrac{1}{x}\right) - x\right)$ (sami)

s) $\lim\limits_{x\to +\infty} \left(x \cdot e^{-\frac{1}{x^2}} - x\right)$

t) $\lim\limits_{x\to +\infty} \left(x^2 \cdot 2^{-x}\right)$ (sami)

Rješenje:

a)
$$\lim_{x\to 0}\frac{(1+x)^n-1}{x} = \left(\frac{0}{0}\right) =^{\text{L'H}} \lim_{x\to 0}\frac{n(1+x)^{n-1}}{1} = n\cdot\lim_{x\to 0}\frac{(1+x)^{n-1}}{1} = n$$

b)
$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \left(\frac{0}{0}\right) =^{\text{L'H}} \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \left(\frac{0}{0}\right) =^{\text{L'H}} \lim_{x\to 0}\frac{\sin x}{6x} = \frac{1}{6}$$

$$
\begin{aligned}
\text{c) } \lim_{x\to 0} \frac{\mathrm{ch}x - 1}{1 - \cos x} \quad &= \quad \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \quad \lim_{x\to 0} \frac{\mathrm{sh}x}{\sin x} = \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \quad \lim_{x\to 0} \frac{\mathrm{ch}x}{\cos x} \\
&= \quad \frac{1}{1} \\
&= \quad 1
\end{aligned}
$$

$$
\begin{aligned}
\text{d) } \lim_{x\to+\infty} \frac{x^3}{e^{2x}} &= \left(\frac{+\infty}{+\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to+\infty} \frac{3x^2}{2e^{2x}} = \left(\frac{+\infty}{+\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to+\infty} \frac{6x}{4e^{2x}} = \left(\frac{+\infty}{+\infty}\right) \\
&=^{\text{L'H}} \lim_{x\to+\infty} \frac{6}{8e^{2x}} \\
&= \frac{3}{4} \lim_{x\to+\infty} \frac{1}{e^{2x}} \\
&= \frac{3}{4} \cdot 0 = 0
\end{aligned}
$$

e) $$\lim_{x\to+\infty} \frac{\ln x}{x^n} = \left(\frac{+\infty}{+\infty}\right) \stackrel{\text{L'H}}{=} \lim_{x\to+\infty} \frac{\frac{1}{x}}{nx^{n-1}} = \frac{1}{n}\lim_{x\to+\infty}\frac{1}{x^n} = \frac{1}{n}\cdot 0 = 0$$

f) $$\lim_{x\to 4} \frac{x^2-16}{x^2+x-20} = \left(\frac{0}{0}\right) \stackrel{\text{L'H}}{=} \lim_{x\to 4}\frac{2x}{2x+1} = \frac{8}{9}$$

g) $\lim_{x\to 1} \frac{\ln x}{x-1} = \left(\frac{0}{0}\right)$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1}$$

$$= 1$$

h) $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} = \left(\frac{0}{0}\right)$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$$

$$= \frac{2}{1}$$

$$= 2$$

i) $$\lim_{x\to 0} \frac{a^x - b^x}{x} = \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{a^x \ln a - b^x \ln b}{1}$$

$$= \ln \frac{a}{b}$$

j) $$\lim_{x\to t} \frac{\sin x - \sin t}{x - t} = \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to t} \frac{\cos x}{1}$$

$$= \cos t$$

$$\text{k) } \lim_{x\to 0} \frac{\mathrm{tg}x - \sin x}{\sin^3 x} = \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{cos^2 x} - \cos x}{3\sin^2 x \cos x}$$

$$= \lim_{x\to 0} \frac{1-\cos^3 x}{3\sin^2 x \cos^3 x} = \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{3\cos^2 x \sin x}{6\sin x \cos^4 x - 9\sin^3 x \cos^2 x}$$

$$= \lim_{x\to 0} \frac{3\cos^2 x \sin x}{3\cos^2 x \sin x(2\cos^2 x - 3\sin^2 x)}$$

$$= \lim_{x\to 0} \frac{1}{2\cos^2 x - 3\sin^2 x}$$

$$= \frac{1}{2}$$

l) $\lim_{x\to 0} \frac{\ln 2 - \ln x}{\ln 2 + \ln x} = \left(\frac{\infty}{\infty}\right)$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{-\frac{1}{x}}{\frac{1}{x}}$$

$$= -1$$

m) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = (+\infty - (+\infty))$

$$= \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} = \left(\frac{0}{0}\right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x}$$

$$= \frac{0}{2}$$

$$= 0$$

$$\text{n) } \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right) = (\infty-\infty)$$

$$= \lim_{x\to 1}\frac{1-x}{\ln x} = \left(\frac{0}{0}\right)$$

$$=^{\text{L'H}} \lim_{x\to 1}\frac{-1}{\frac{1}{x}}$$

$$= \lim_{x\to 1}(-x)$$

$$= -1$$

$$
\begin{aligned}
\text{o) } \lim_{x\to 0^+} (\mathrm{sh}x \cdot \ln x) &= (0\cdot\infty) \\
&= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\mathrm{sh}x}} = \left(\frac{\infty}{\infty}\right) \\
&\stackrel{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\mathrm{ch}x}{\mathrm{sh}^2 x}} \\
&= \lim_{x\to 0^+} \frac{\mathrm{sh}^2 x}{-x\mathrm{ch}x} \\
&= -\lim_{x\to 0^+} \frac{\mathrm{sh}x}{\mathrm{ch}x} \cdot \underbrace{\lim_{x\to 0^+} \frac{\mathrm{sh}x}{x}}_{1} \\
&= -\lim_{x\to 0^+} \mathrm{th}x \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{p) } \lim_{x\to 0}\left(x\ln^2 x\right) &= (0\cdot\infty)\\
&= \lim_{x\to 0}\frac{\ln^2 x}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to 0}\frac{\frac{2\ln x}{x}}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0}\frac{2\ln x}{-\frac{1}{x}} = \left(\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to 0}\frac{\frac{2}{x}}{\frac{1}{x^2}}\\
&= \lim_{x\to 0}(2x)\\
&= 0
\end{aligned}
$$

r) $$\lim_{x\to+\infty}\left[x\ln\left(e+\frac{1}{x}\right)-x\right] = \lim_{x\to+\infty}\left[x\left(\ln\left(e+\frac{1}{x}\right)-1\right)\right]$$

$$= (\infty\cdot 0)$$

$$= \lim_{x\to+\infty}\frac{\ln\left(e+\frac{1}{x}\right)-1}{\frac{1}{x}} = \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to+\infty}\frac{\frac{1}{e+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to+\infty}\frac{1}{e+\frac{1}{x}}$$

$$= \frac{1}{e}$$

$$
\begin{aligned}
\text{s) } \lim_{x\to+\infty}\left(xe^{-\frac{1}{x^2}}-x\right) &= \lim_{x\to+\infty}\left[x\left(e^{-\frac{1}{x^2}}-1\right)\right] \\
&= (\infty\cdot 0) \\
&= \lim_{x\to+\infty}\frac{e^{-\frac{1}{x^2}}-1}{\frac{1}{x}} = \left(\frac{0}{0}\right) \\
&\overset{\text{L'H}}{=} \lim_{x\to+\infty}\frac{e^{-\frac{1}{x^2}}\cdot\dfrac{2}{x^3}}{-\dfrac{1}{x^2}} \\
&= \lim_{x\to+\infty}\frac{-2e^{-\frac{1}{x^2}}}{x} \\
&= \frac{-2}{\infty} \\
&= 0
\end{aligned}
$$

t) $\lim\limits_{x\to+\infty}\left(x^2 2^{-x}\right) = \lim\limits_{x\to+\infty}\frac{x^2}{2^x} = \left(\frac{\infty}{\infty}\right)$

$=^{\text{L'H}} \lim\limits_{x\to+\infty}\frac{2x}{2^x\ln 2} = \left(\frac{\infty}{\infty}\right)$

$=^{\text{L'H}} \frac{2}{\ln^2 2}\lim\limits_{x\to+\infty}\frac{1}{2^x}$

$= 0$

Limese oblika $0^0, 1^\infty$ i ∞^0 svodimo na L'Hospitalovo pravilo logaritmiranjem:

$$\begin{aligned}\lim_{x\to x_0} f(x)^{g(x)} &= A \quad / \ln \\ \lim_{x\to x_0} \ln\left(f(x)^{g(x)}\right) &= \ln A \\ \lim_{x\to x_0} g(x)\ln f(x) &= \ln A\end{aligned}$$

Zadatak

Izračunajte limes $\lim\limits_{x\to 0} (\cos 2x)^{\frac{3}{x^2}}$.

$$\begin{aligned}
\lim_{x\to 0} (\cos 2x)^{\frac{3}{x^2}} &= (1^\infty) = A \quad / \ln \\
\ln A &= \lim_{x\to 0} \left(\ln (\cos 2x)^{\frac{3}{x^2}}\right) \\
&= \lim_{x\to 0} \frac{3}{x^2} \ln(\cos 2x) = (\infty \cdot 0) \\
&= \lim_{x\to 0} \frac{3 \ln \cos 2x}{x^2} = \left(\frac{0}{0}\right) \\
&\stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{3 \frac{1}{\cos 2x} (-2 \sin 2x)}{2x} \\
&= -6 \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{1}{\cos 2x} \\
&= -6 \cdot 1 \cdot 1
\end{aligned}$$

$$
\begin{aligned}
\ln A &= -6 \\
A &= e^{-6}
\end{aligned}
$$

6.6 Asimptote

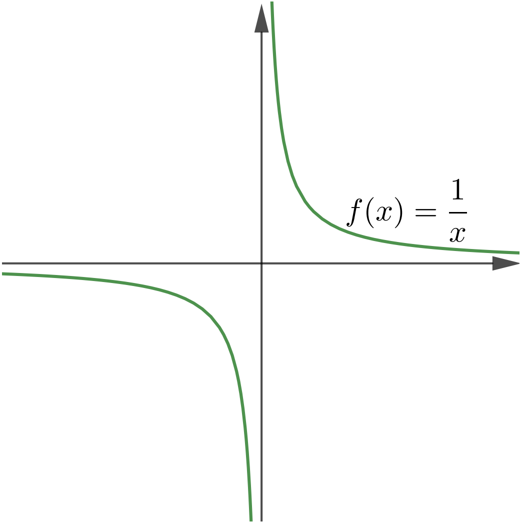
Vertikalne asimptote

Pravac $x = c$ je vertikalna asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{ili} \quad \lim_{x \to c^-} f(x) = \pm\infty$$

Napomena: Vertikalnu asimptotu tražimo u prekidima i na rubovima domene.

Primjer: $f(x) = \frac{1}{x}$, $D_f = \mathbb{R} \setminus \{0\}$



$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

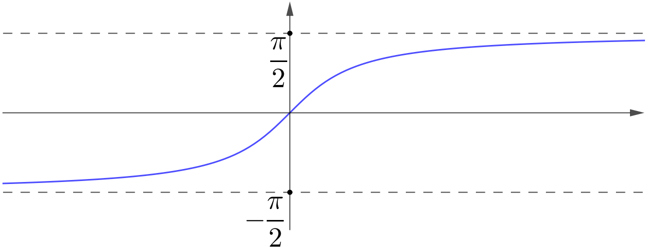
Pravac $x = 0$ je vertikalna asimptota

Horizontalne asimptote

Pravac $y = c$ je lijeva (desna) horizontalna asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$\lim_{x \to -\infty} f(x) = c \ \left(\lim_{x \to +\infty} f(x) = c \right).$$

Primjer: $f(x) = \text{arctg} x$, $D_f = \mathbb{R}$



$$\lim_{x \to +\infty} \text{arctg} x = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \text{arctg} x = -\frac{\pi}{2}$$

$x = \frac{\pi}{2}$ je desna horizontalna asimptota, a $x = -\frac{\pi}{2}$ je lijeva horizontalna asimptota.

Kose asimptote

Pravac $y = kx + l$ je lijeva kosa asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$k = \lim_{x \to -\infty} \frac{f(x)}{x} \text{ i } l = \lim_{x \to -\infty} [f(x) - kx]$$

Pravac $y = kx + l$ je desna kosa asimptota funkcije $f : I \subseteq \mathbb{R} \to \mathbb{R}$ ako je

$$k = \lim_{x \to +\infty} \frac{f(x)}{x} \text{ i } l = \lim_{x \to +\infty} [f(x) - kx]$$

Napomena: Ako f ima lijevu(desnu) horizontalnu asimptotu, onda nema lijevu(desnu) kosu asimptotu i obratno.

Zadatak

Odredite sve asimptote funkcije:

a) $f(x) = \dfrac{x^3+1}{3x^2+1}$

b) $f(x) = \dfrac{x^3-2x-1}{2x^3+2}$ (sami)

c) $f(x) = \mathrm{arctg} e^x$

d) $f(x) = \dfrac{1}{x^2}$ (sami)

e) $f(x) = -\ln x + 2$

f) $f(x) = \dfrac{2x^3+5x^2+3x+2}{x^2+1}$ (sami)

g) $f(x) = \sqrt[3]{x^2(6-x)}$ (sami)

h) $f(x) = \dfrac{x^2-6x+3}{x-3}$ (sami)

i) $f(x) = x + 2\mathrm{arcctg} x$

j) $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$ (sami)

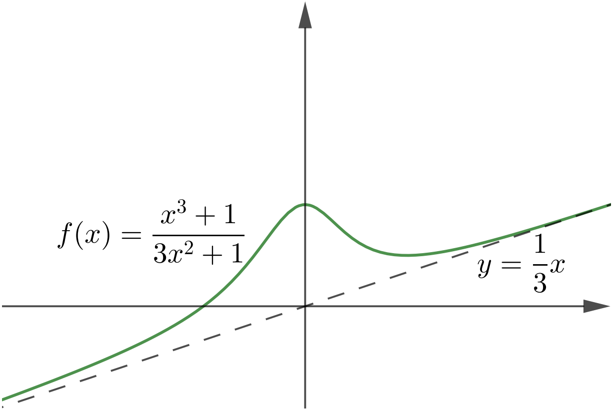
Rješenje: a) $f(x) = \dfrac{x^3+1}{3x^2+1}$

$D_f = \mathbb{R}$, pa f nema vertikalnih asimptota.

$$\begin{aligned} k &= \lim_{x\to\pm\infty} \frac{f(x)}{x} \\ &= \lim_{x\to\pm\infty} \frac{x^3+1}{x(3x^2+1)} \\ &= \lim_{x\to\pm\infty} \frac{x^3+1}{3x^3+x} : \frac{x^3}{x^3} \\ &= \lim_{x\to\pm\infty} \frac{1+\frac{1}{x^3}}{3+\frac{1}{x^2}} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} l &= \lim_{x\to\pm\infty} [f(x) - kx] \\ &= \lim_{x\to\pm\infty} \left[\frac{x^3+1}{3x^3+x} - \frac{1}{3}x\right] \\ &= \lim_{x\to\pm\infty} \frac{3x^3+3-3x^3-x}{3(3x^3+x)} \\ &= \lim_{x\to\pm\infty} \frac{3-x}{9x^3+3x} \\ &= 0 \end{aligned}$$

Pravac $y = \frac{1}{3}x$ je kosa asimptota.


$f(x) = \dfrac{x^3+1}{3x^2+1}$
$y = \dfrac{1}{3}x$

b) $f(x) = \dfrac{x^3 - 2x - 1}{2x^3 + 2}$

$D_f = \mathbb{R} \setminus \{1\}$

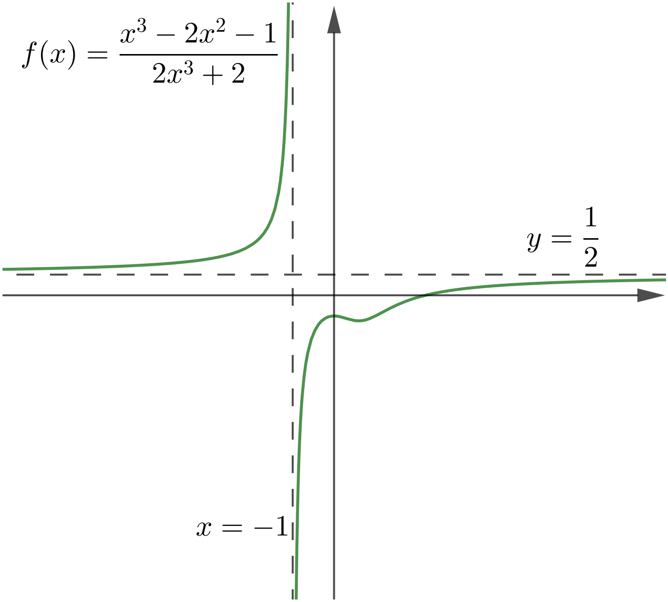
$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} \frac{x^3 - 2x - 1}{2x^3 + 2} \\ &= \lim_{x\to -1^-} \frac{-4}{0^-} \\ &= +\infty \end{aligned}$$

$$\begin{aligned} \lim_{x\to -1^+} f(x) &= \lim_{x\to 1^+} \frac{x^3 - 2x - 1}{2x^3 + 2} \\ &= \lim_{x\to -1^+} \frac{-4}{0^+} \\ &= -\infty \end{aligned}$$

$x = -1$ je vertikalna asimptota

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^3-2x-1}{2x^3+2} = \frac{1}{2}$$

$y = \frac{1}{2}$ je horizontalna asimptota


$f(x) = \dfrac{x^3 - 2x^2 - 1}{2x^3 + 2}$
$y = \dfrac{1}{2}$
$x = -1$

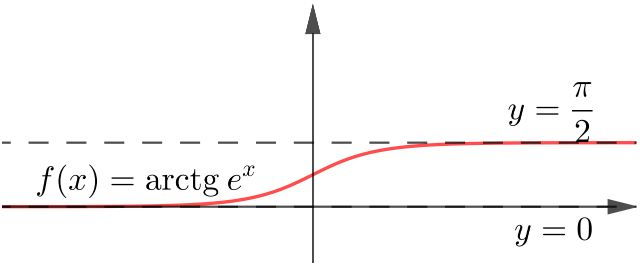
c) $f(x) = \text{arctg} e^x$, $D_f = \mathbb{R}$

Nema vertikalnih asimptota.

$$\begin{aligned}
\lim_{x \to +\infty} f(x) &= \lim_{x \to +\infty} \underbrace{\text{arctg} \overbrace{e^x}^{+\infty}}_{\frac{\pi}{2}} \\
&= \frac{\pi}{2} \\
\lim_{x \to -\infty} f(x) &= \lim_{x \to -\infty} \underbrace{\text{arctg} \overbrace{e^x}^{0}}_{0} \\
&= 0
\end{aligned}$$

$y = \dfrac{\pi}{2}$ je desna horizontalna, a $y = 0$ lijeva horizontalna asimptota.

Kako f ima lijevu i desnu horizontalnu asimptotu, nema kosih.

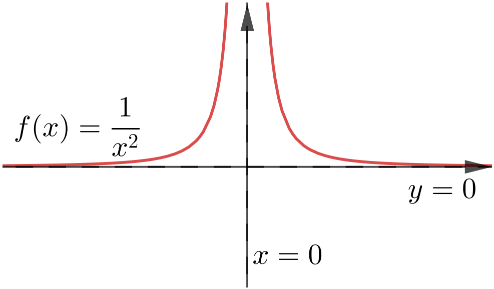

$y = \frac{\pi}{2}$
$f(x) = \operatorname{arctg} e^x$
$y = 0$

d) $f(x) = \dfrac{1}{x^2}$, $D_f = \mathbb{R} \setminus \{0\}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x^2} = +\infty$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x^2} = +\infty$$

$x = 0$ je vertikalna asimptota



$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{1}{x^2} = 0$$

$y = 0$ je obostrana horizontalna asimptota

Kako f ima lijevu i desnu horizontalnu asimptotu, nema kosih.

e) $f(x) = -\ln x + 2,\ D_f = \langle 0, +\infty\rangle$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} (-\ln x + 2) \\ &= -(-\infty) + 2 \\ &= +\infty\end{aligned}$$

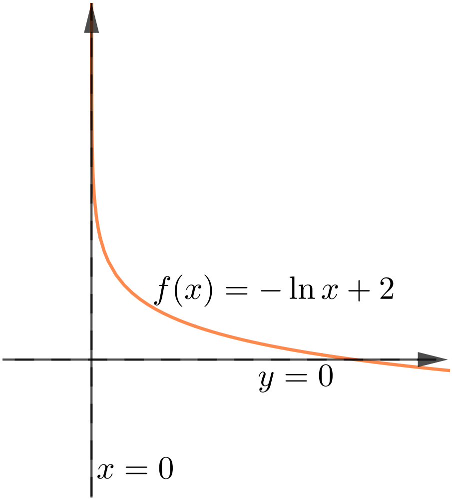
$x = 0$ je vertikalna asimptota

$$\begin{aligned}\lim_{x\to +\infty} f(x) &= \lim_{x\to +\infty} (-\ln x + 2) \\ &= -\infty\end{aligned}$$

f nema horizontalnih asimptota.

$$\begin{aligned}\lim_{x\to +\infty} \frac{f(x)}{x} &= \lim_{x\to +\infty} \frac{-\ln x + 2}{x} = \left(\frac{\infty}{\infty}\right) \\ &=^{\text{L'H}} \frac{-\frac{1}{x}}{1} \\ &= 0\end{aligned}$$

Kako 0 ne može biti koeficijent smjera kose asimptote, zaključujemo da f nema kosih asimptota.


$f(x) = -\ln x + 2$
$y = 0$
$x = 0$

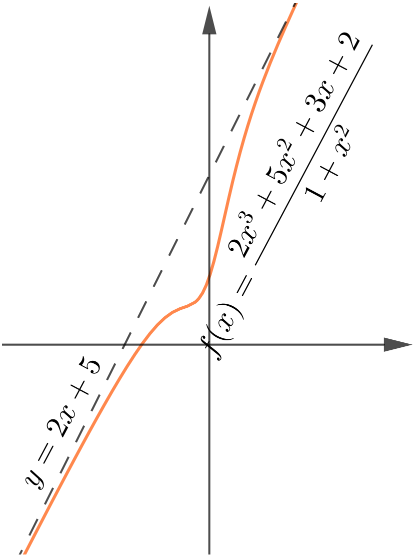
f) $f(x) = \dfrac{2x^3 + 5x^2 + 3x + 2}{x^2 + 1}$, $D_f = \mathbb{R}$

Nema vertikalnih asimptota.

$$\begin{aligned}
k &= \lim_{x\to\pm\infty} \frac{f(x)}{x} \\
&= \lim_{x\to\pm\infty} \frac{\frac{2x^3+5x^2+3x+2}{x^2+1}}{x} \\
&= \lim_{x\to\pm\infty} \frac{2x^3 + 5x^2 + 3x + 2}{x^3 + x} : \frac{x^3}{x^3} \\
&= \lim_{x\to\pm\infty} \frac{2 + \frac{5}{x} + \frac{3}{x^2} + \frac{2}{x^3}}{1 + \frac{1}{x^2}} \\
&= \frac{2}{1} = 2
\end{aligned}$$

$$
\begin{aligned}
l &= \lim_{x\to\pm\infty}[f(x)-kx] \\
&= \lim_{x\to\pm\infty}\left[\frac{2x^3+5x^2+3x+2}{x^2+1}-2x\right] \\
&= \lim_{x\to\pm\infty}\frac{2x^3+5x^2+3x+2-2x^3-2x}{x^2+1} \\
&= \lim_{x\to\pm\infty}\frac{5x^2+x+2}{x^2+1}:\frac{x^2}{x^2} \\
&= \lim_{x\to\pm\infty}\frac{5+\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}} \\
&= 5
\end{aligned}
$$

Pravac $y = 2x + 5$ je kosa asimptota funkcije f, pa zaključujemo da f nema horizontalnih asimptota asimptota.

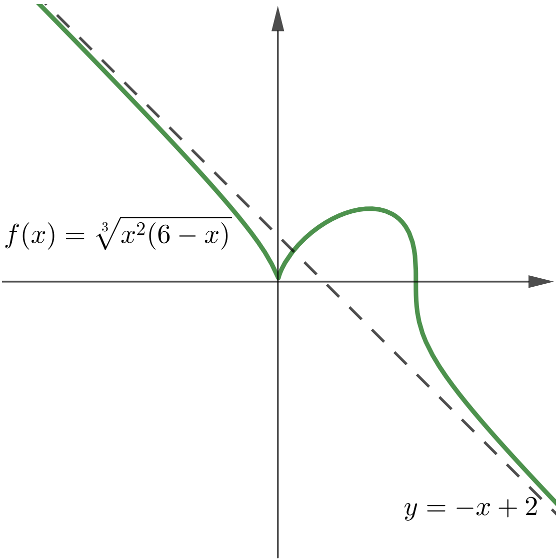

$f(x) = \frac{2x^3 + 5x^2 + 3x + 2}{1 + x^2}$
$y = 2x + 5$

g) $f(x) = \sqrt[3]{x^2(6-x)}$, $D_f = \mathbb{R}$, pa f nema vertikalnih asimptota.

$$
\begin{aligned}
k &= \lim_{x\to\pm\infty} \frac{f(x)}{x} \\
&= \lim_{x\to\pm\infty} \frac{\sqrt[3]{6x^2 - x^3}}{x} \\
&= \lim_{x\to\pm\infty} \sqrt[3]{\frac{6}{x} - 1} \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
l &= \lim_{x\to\pm\infty}[f(x)-kx] \\
&= \lim_{x\to\pm\infty}\left[\sqrt[3]{6x^2-x^3}+x\right] = (-\infty+\infty) \\
&= \lim_{x\to\pm\infty}\left[\sqrt[3]{6x^2-x^3}+x\right]\cdot\frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\pm\infty}\frac{\sqrt[3]{\frac{6}{x}-1}+1}{\frac{1}{x}} = \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to\pm\infty}\frac{\frac{1}{3\sqrt[3]{\left(\frac{6}{x}-1\right)^2}}\cdot\left(-\frac{6}{x^2}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\pm\infty}\frac{2}{\sqrt[3]{\left(\frac{6}{x}-1\right)^2}} \\
&= 2
\end{aligned}
$$

Pravac $y=-x+2$ je kosa asimptota funkcije f, pa zaključujemo da f nema horizontalnih asimptota asimptota.


$f(x) = \sqrt[3]{x^2(6-x)}$
$y = -x + 2$

h) $f(x) = \dfrac{x^2 - 6x + 3}{x - 3}$, $D_f = \mathbb{R} \setminus \{3\}$.

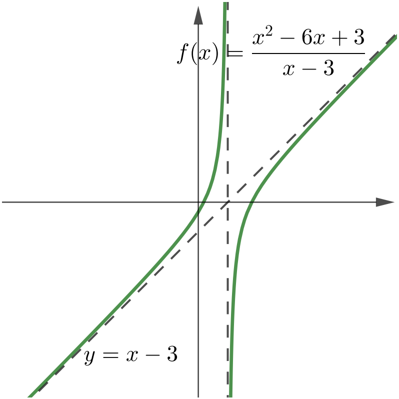
$$\lim_{x \to 3^-} \frac{x^2 - 6x + 3}{x - 3} = \frac{-6}{0^-} = +\infty \qquad \lim_{x \to 3^+} \frac{x^2 - 6x + 3}{x - 3} = \frac{-6}{0^+} = -\infty$$

Pravac $x = 3$ je vertikalna asimptota.

$$\begin{aligned} k &= \lim_{x \to \pm\infty} \frac{f(x)}{x} \\ &= \lim_{x \to \pm\infty} \frac{\frac{x^2-6x+3}{x-3}}{x} \\ &= \lim_{x \to \pm\infty} \frac{x^2 - 6x + 3}{x^2 - 3x} : \frac{x^2}{x^2} \\ &= \lim_{x \to \pm\infty} \frac{1 - \frac{6}{x} + \frac{3}{x^2}}{1 - \frac{3}{x}} \\ &= 1 \end{aligned}$$

$$\begin{aligned} l &= \lim_{x \to \pm\infty} [f(x) - kx] \\ &= \lim_{x \to \pm\infty} \left[\frac{x^2 - 6x + 3}{x - 3} - x\right] \\ &= \lim_{x \to \pm\infty} \frac{x^2 - 6x + 3 - x^2 + 3x}{x - 3} \\ &= \lim_{x \to \pm\infty} \frac{-3x + 3}{x - 3} \\ &= -3 \end{aligned}$$

Pravac $y = x - 3$ je kosa asimptota funkcije f, pa zaključujemo da f nema horizontalnih asimptota asimptota.



i) $f(x) = x + 2\text{arcctg}x$, $D_f = \mathbb{R}$, pa f nema vertikalnih asimptota.

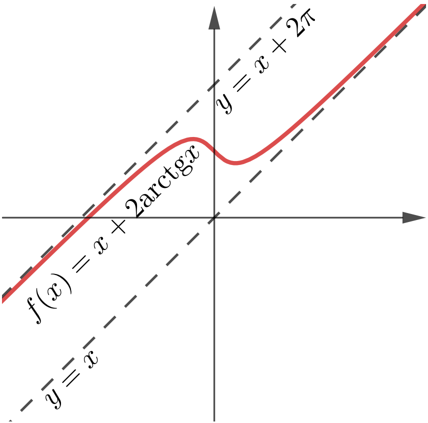
$$\begin{aligned} k_1 &= \lim_{x\to+\infty} \frac{f(x)}{x} \\ &= \lim_{x\to+\infty} \frac{x + 2\text{arcctg}x}{x} \\ &= 1 + \lim_{x\to+\infty} \frac{2\overbrace{\text{arcctg}x}^{\pi}}{x} \\ &= 1 \end{aligned}$$

$$\begin{aligned} k_2 &= \lim_{x\to-\infty} \frac{f(x)}{x} \\ &= \lim_{x\to-\infty} \frac{x + 2\text{arcctg}x}{x} \\ &= 1 + \lim_{x\to-\infty} \frac{2\overbrace{\text{arcctg}x}^{0}}{x} \\ &= 1 \end{aligned}$$

$$\begin{aligned} l_1 &= \lim_{x\to+\infty} [f(x) - kx] \\ &= \lim_{x\to+\infty} [x + 2\text{arcctg}x - x] \\ &= \lim_{x\to+\infty} 2\text{arcctg}x \\ &= 0 \end{aligned}$$

$$\begin{aligned} l_2 &= \lim_{x\to-\infty} [f(x) - kx] \\ &= \lim_{x\to-\infty} [x + 2\text{arcctg}x - x] \\ &= \lim_{x\to-\infty} 2\text{arcctg}x \\ &= 2\pi \end{aligned}$$

Pravac $y = x + 2\pi$ je lijeva kosa, a $y = x$ je desna kosa asimptota



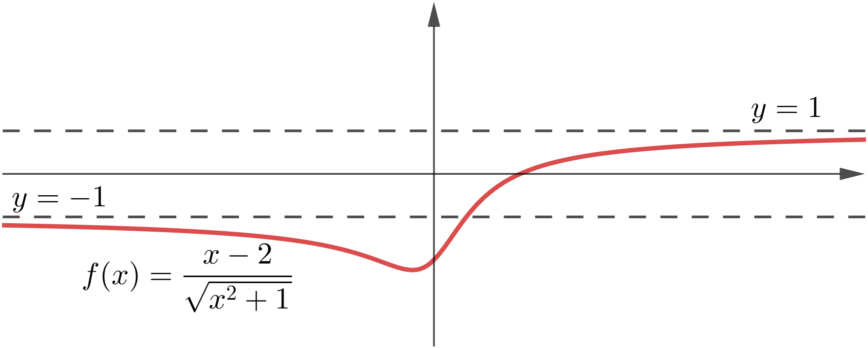
$y = x + 2\pi$
$f(x) = x + 2\mathrm{arctg}x$
$y = x$

j) $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$, $D_f = \mathbb{R}$, pa f nema vertikalnih asimptota.

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x-2}{\sqrt{x^2+1}} : \frac{x}{x} \\
&= \lim_{x\to+\infty} \frac{1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= (x \leftrightarrow -x) \\
&= \lim_{x\to+\infty} \frac{-x-2}{\sqrt{x^2+1}} : \frac{x}{x} \\
&= \lim_{x\to+\infty} \frac{-1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} \\
&= -1
\end{aligned}$$

Pravac $y = 1$ je desna horizontalna, a pravac $y = -1$ je lijeva horizontalna asimptota.


$y = 1$
$y = -1$
$f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$

6.7 Tok funkcije

Zadatak

Odredite prirodnu domenu funkcije, nultočke, točke ekstrema, intervale rasta i pada, asimptote, te skicirajte graf funkcije:

a) $f(x) = \dfrac{x^3 - 4}{(x-1)^3}$

b) $f(x) = \dfrac{x^2 - 1}{e^{x^2}}$ (sami)

c) $f(x) = \dfrac{\ln^2 x}{x}$

d) $f(x) = 2\arccos\dfrac{1}{x-1}$

e) $f(x) = \dfrac{x-2}{\sqrt{x^2+2}}$ (sami)

Rješenje: a) $f(x) = \dfrac{x^3 - 4}{(x-1)^3}$

Domena: $D_f = \mathbb{R} \setminus \{1\}$

Nultočke:

$$\begin{aligned} f(x) &= 0 \\ \frac{x^3 - 4}{(x-1)^3} &= 0 \\ x^3 - 4 &= 0 \\ x^3 &= 4 \\ x_0 &= \sqrt[3]{4} \end{aligned}$$

Ekstremi:

$$\begin{aligned} f(x) &= \frac{x^3 - 4}{(x-1)^3} \\ f'(x) &= \frac{3x^2(x-1)^3 - (x^3-4)3(x-1)^2}{(x-1)^6} \\ &= \frac{(x-1)^2\left(3x^2(x-1) - 3(x^3-4)\right)}{(x-1)^6} \\ &= \frac{3x^3 - 3x^2 - 3x^3 + 12}{(x-1)^4} \\ &= \frac{-3x^2 + 12}{(x-1)^4} \end{aligned}$$

Odredimo stacionarne točke:

$$
\begin{aligned}
-3x^2 + 12 &= 0 \\
3x^2 &= 12 \\
x^2 &= 4
\end{aligned}
$$

$x_1 = -2$, $x_2 = 2$

	$-\infty$		-2		1		2		$+\infty$
f'	‖	$-$		$+$	¦	$+$		$-$	
f	‖	↘	m	↗	¦	↗	M	↘	

Funkcija f u $x = -2$ ima lokalni minimum i $f(-2) = \frac{4}{9}$, a u $x = 2$ lokalni maksimum i $f(2) = 4$.

Asimptote:

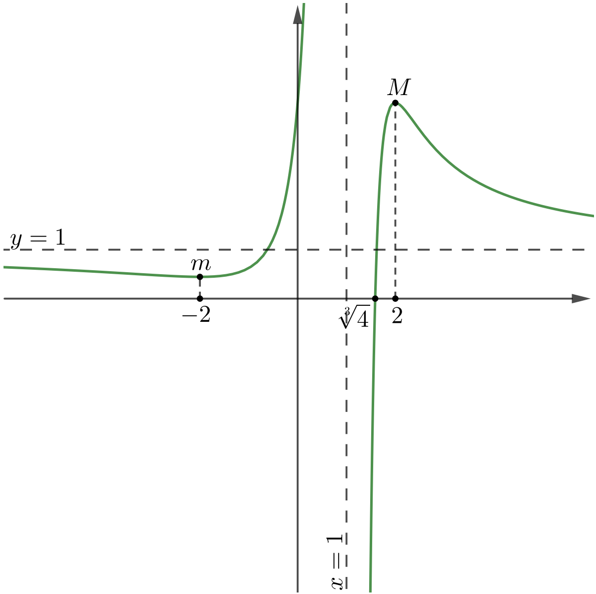
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^3-4}{(x-1)^3} = \frac{-3}{0^-} = +\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3-4}{(x-1)^3} = \frac{-3}{0^+} = -\infty$$

Pravac $x = 1$ je vertikalna asimptota

$$
\begin{aligned}
\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{x^3-4}{(x-1)^3} : \frac{x^3}{x^3} \\
&= \lim_{x\to\pm\infty} \frac{1-\frac{4}{x^3}}{\left(1-\frac{1}{x^3}\right)^3} \\
&= 1
\end{aligned}
$$

Pravac $y = 1$ je lijeva i desna horizontalna asimptota, pa f nema kosih asimptota.



M
$y = 1$
m
-2
$\sqrt[3]{4}$
2
$x = 1$

b) $f(x) = \dfrac{x^2 - 1}{e^{x^2}}$

Domena: $D_f = \mathbb{R}$

Nultočke:

$$
\begin{aligned}
f(x) &= 0 \\
\frac{x^2 - 1}{e^{x^2}} &= 0 \\
x^2 - 1 &= 0 \\
x^2 &= 1 \\
x_0 &= \pm 1
\end{aligned}
$$

Ekstremi:

$$
\begin{aligned}
f'(x) &= \frac{2x\cdot e^{x^2}-(x^2-1)e^{x^2}\cdot 2x}{e^{2x^2}} \\
&= \frac{e^{x^2}\left(2x-2x^3+2x\right)}{e^{2x^2}} \\
&= \frac{4x-2x^3}{e^{x^2}}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
4x-2x^3 &= 0 \\
2x(2-x^2) &= 0
\end{aligned}
$$

$x_1=0,\ x_2=\sqrt{2},\ x_3=-\sqrt{2}$

	$-\infty$		$-\sqrt{2}$		0		$\sqrt{2}$		$+\infty$
f'		$+$		$-$		$+$		$-$	
f		$\nearrow$	M	$\searrow$	m	$\nearrow$	M	$\searrow$	

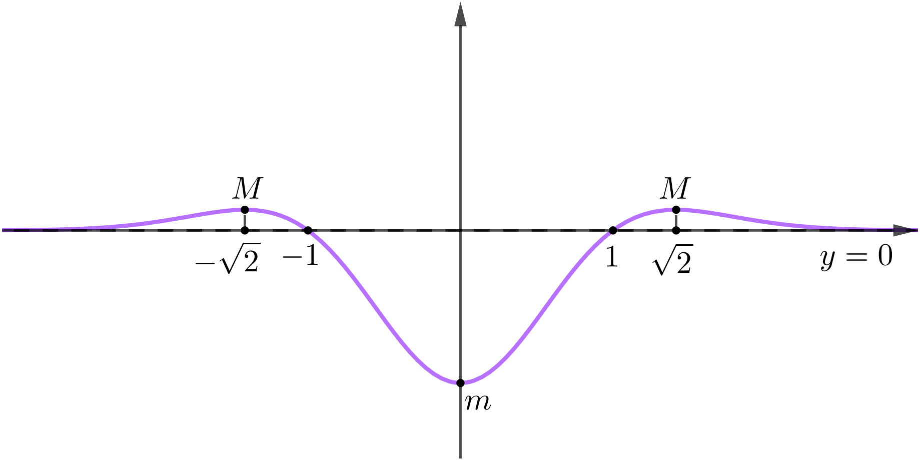
Funkcija f u $x=0$ ima lokalni minimum, a u $x=-\sqrt{2}$ i $x=\sqrt{2}$ lokalni maksimum. $f(\sqrt{2})=f(-\sqrt{2})=\dfrac{1}{e^2}$ i $f(0)=-1$.

Asimptote:

Nema vertikalnih asimptota.

$$\begin{aligned}
\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{x^2-1}{e^{x^2}} = \left(\frac{\infty}{\infty}\right) \\
&\stackrel{\text{L'H}}{=} \lim_{x\to\pm\infty} \frac{2x}{2x\cdot e^{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{1}{e^{x^2}} \\
&= 0
\end{aligned}$$

Pravac $y = 0$ je lijeva i desna horizontalna asimptota, pa f nema kosih asimptota.



M
M
$-\sqrt{2}$
-1
1
$\sqrt{2}$
$y = 0$
m

c) $f(x) = \dfrac{\ln^2 x}{x}$

Domena: $D_f = \langle 0, +\infty \rangle$

Nultočke:

$$\begin{aligned} f(x) &= 0 \\ \ln^2 x &= 0 \\ \ln x &= 0 \\ x &= 1 \end{aligned}$$

Ekstremi:

$$\begin{aligned} f(x) &= \frac{\ln^2 x}{x} \\ f'(x) &= \frac{2\ln x \cdot \frac{1}{x} \cdot x - \ln^2 x}{x^2} \\ &= \frac{\ln x(2 - \ln x)}{x^2} \end{aligned}$$

Odredimo stacionarne točke:

$$
\begin{aligned}
\ln x(2-\ln x) &= 0\\
x_1 &= e^0\\
x_1 &= 1\\
x_2 &= e^2
\end{aligned}
$$

	0		1		e^2		$+\infty$
f'	‖	$-$		$+$		$-$	
f	‖	$\searrow$	m	$\nearrow$	M	$\searrow$	

Funkcija f u $x = 1$ ima lokalni minimum i $f(1) = 0$, a u $x = e^2$ lokalni maksimum i $f(e^2) = \dfrac{4}{e^2}$.

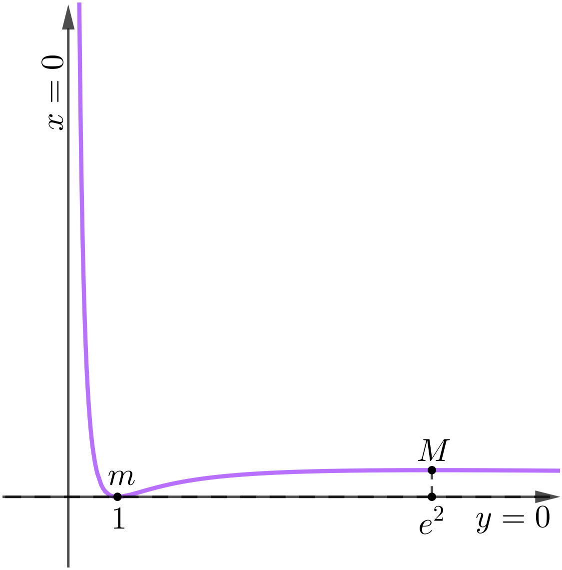
Asimptote:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln^2 x}{x}$$
$$= +\infty$$

Pravac $x = 0$ je vertikalna asimptota

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{\ln^2 x}{x} = \left(\frac{\infty}{\infty}\right)$$
$$\stackrel{\text{L'H}}{=} \lim_{x\to +\infty} \frac{2\ln x \cdot \frac{1}{x}}{1}$$
$$= \lim_{x\to +\infty} \frac{2\ln x}{x} = \left(\frac{\infty}{\infty}\right)$$
$$\stackrel{\text{L'H}}{=} \lim_{x\to +\infty} \frac{2}{x}$$
$$= 0$$

Pravac $y = 0$ je desna horizontalna asimptota, pa f nema kosih asimptota.



$x = 0$
m
M
1
e^2
$y = 0$

d) $f(x) = 2\arccos\dfrac{1}{x-1}$

Domena:

$$D_{\text{arccos}} = [-1, 1] \Longrightarrow -1 \leq \frac{1}{x-1} \leq 1$$

$$-1 \quad \leq \quad \frac{1}{x-1} \qquad\qquad \frac{1}{x-1} \quad \leq \quad 1$$

$$0 \quad \leq \quad \frac{x}{x-1} \qquad\qquad \frac{2-x}{x-1} \quad \leq \quad 0$$

$$x \in \langle -\infty, 0] \bigcup \langle 1, +\infty\rangle \qquad x \in \langle -\infty, 1\rangle \bigcup [2, +\infty\rangle$$

$$D_f = \langle -\infty, 0] \bigcup [2, +\infty\rangle$$

Nultočke:

$$\begin{aligned} f(x) &= 0 \\ 2\arccos\frac{1}{x-1} &= 0 \\ \frac{1}{x-1} &= 1 \\ x_0 &= 2 \end{aligned}$$

Ekstremi:

$$\begin{aligned} f(x) &= 2\arccos\frac{1}{x-1} \\ f'(x) &= -2\frac{1}{\sqrt{1-\left(\frac{1}{x-1}\right)^2}} \cdot \left(-\frac{1}{(x-1)^2}\right) \\ &= \frac{2}{(x-1)^2\sqrt{1-\left(\frac{1}{x-1}\right)^2}} \\ &> 0 \end{aligned}$$

f nema ekstrema i raste na cijeloj domeni.

Asimptote:

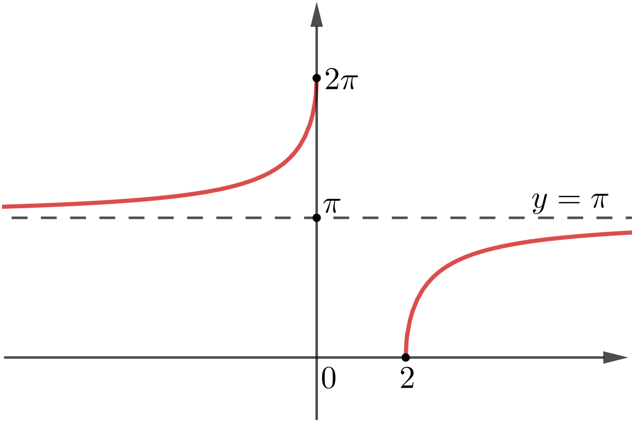
Nema vertikalnih asimptota.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} 2\arccos\frac{1}{x-1} \\ &= 2\arccos 0 \\ &= 2\cdot\frac{\pi}{2} \\ &= \pi\end{aligned}$$

Pravac $y=\pi$ je lijeva i desna horizontalna asimptota, pa f nema kosih asimptota.

$f(0) = 2\arccos(-1) = 2\pi$

$f(2) = 2\arccos 1 = 0$



e) $f(x) = \dfrac{x-2}{\sqrt{x^2+2}}$

Domena:

$D_f = \mathbb{R}$

Nultočke:

$$\begin{aligned} f(x) &= 0 \\ \frac{x-2}{\sqrt{x^2+2}} &= 0 \\ x-2 &= 0 \\ x_0 &= 2 \end{aligned}$$

Ekstremi:

$$
\begin{aligned}
f(x) &= \frac{x-2}{\sqrt{x^2+2}} \\
f'(x) &= \frac{\sqrt{x^2+2}-(x-2)\cdot\frac{1}{2\sqrt{x^2+2}}\cdot 2x}{x^2+2} \\
&= \frac{\frac{x^2+2-x^2+2x}{2\sqrt{x^2+2}}}{x^2+2} \\
&= \frac{2+2x}{(x^2+2)\sqrt{x^2+2}}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
\frac{2+2x}{(x^2+2)\sqrt{x^2+2}} &= 0 \\
2+2x &= 0 \\
x_1 &= -1
\end{aligned}
$$

	$-\infty$		-1		$+\infty$
f'	‖	$-$	|	$+$	|
f	‖	$\searrow$	m	$\nearrow$	|

Funkcija f u $x=-1$ ima lokalni minimum i $f(-1)=-\sqrt{3}$.

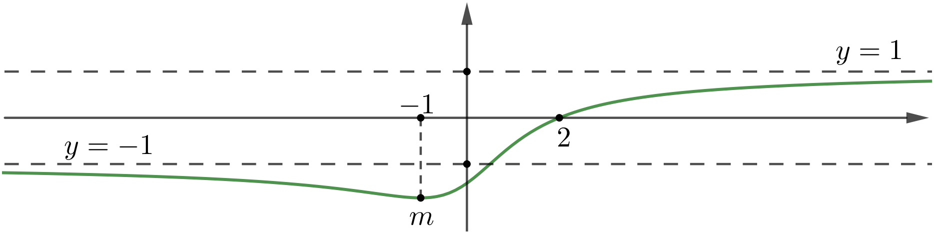
Asimptote:

Nema vertikalnih asimptota.

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x-2}{\sqrt{x^2+1}} : \frac{x}{x} \\
&= \lim_{x\to+\infty} \frac{1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= (x \leftrightarrow (-x)) \\
&= \lim_{x\to+\infty} \frac{-x-2}{\sqrt{(-x)^2+1}} : \frac{x}{x} \\
&= \lim_{x\to+\infty} \frac{-1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} \\
&= -1
\end{aligned}$$

Pravac $y = -1$ je lijeva, a pravac $y = 1$ desna horizontalna asimptota, pa f nema kosih asimptota.



$y = 1$
-1
2
$y = -1$
m

Zadatak

Odredite prirodnu domenu funkcije, nultočke, točke ekstrema, intervale rasta i pada, točke infleksije, asimptote, te skicirajte graf funkcije:

a) $f(x) = x \cdot e^{-\frac{1}{x^2}}$

b) $f(x) = x - 2\text{arctg}x$

c) $f(x) = x^2 \cdot 2^{-x}$ (sami)

d) $f(x) = \arcsin(e^x - 1)$ (sami)

Rješenje: a) $f(x) = x \cdot e^{-\frac{1}{x^2}}$

Domena: $D_f = \mathbb{R} \setminus \{0\}$

Nultočke: Jedina moguća nultočka bi bila $x_0 = 0$, no nije jer 0 nije u domeni.

Ekstremi:

$$\begin{aligned} f(x) &= x \cdot e^{-\frac{1}{x^2}} \\ f'(x) &= e^{-\frac{1}{x^2}} + x \cdot e^{-\frac{1}{x^2}} \cdot \left(-\frac{-2}{x^3}\right) \\ &= \underbrace{e^{-\frac{1}{x^2}}}_{>0} \underbrace{\left(1 + \frac{2}{x^2}\right)}_{>0} \\ &> 0 \end{aligned}$$

Funkcija nema ekstreme i raste na cijeloj domeni.

Točke infleksije:

$$\begin{aligned} f''(x) &= e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} \cdot \left(1 + \frac{2}{x^2}\right) + e^{-\frac{1}{x^2}} \cdot \left(-\frac{4}{x^3}\right) \\ &= e^{-\frac{1}{x^2}} \left(\frac{2}{x^3} + \frac{4}{x^5} - \frac{4}{x^3}\right) \\ &= e^{-\frac{1}{x^2}} \left(\frac{-2x^2 + 4}{x^5}\right) \end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned} -2x^4 + 4 &= 0 \\ x^2 &= 2 \end{aligned}$$

Kandidati: $x_1 = \sqrt{2}$, $x_2 = -\sqrt{2}$

	$-\infty$	$-\sqrt{2}$	0	$\sqrt{2}$	$+\infty$
f''	$+$	$-$	$+$	$-$	
f	$\cup$	$\cap$	$\cup$	$\cap$	

Intervali konveksnosti su $\langle -\infty, -\sqrt{2} \rangle$ i $\langle 0, \sqrt{2} \rangle$, a intervali konkavnosti su $\langle -\sqrt{2}, 0 \rangle$ i $\langle \sqrt{2}, +\infty \rangle$.

Asimptote:

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} x\cdot e^{-\frac{1}{x^2}} \\
&= 0 \\
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x\cdot e^{-\frac{1}{x^2}} \\
&= 0
\end{aligned}$$

Nema vertikalnih asimptota.

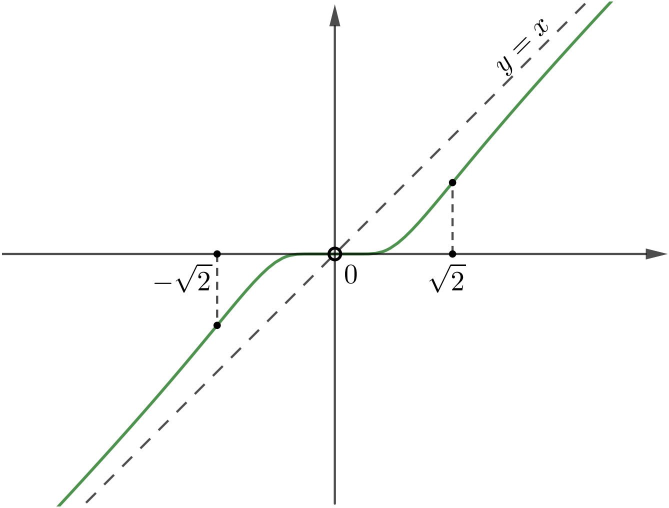
$$\begin{aligned}
\lim_{x\to \pm\infty} f(x) &= \lim_{x\to \pm\infty} x\cdot e^{-\frac{1}{x^2}} \\
&= \pm\infty \cdot 1 \\
&= \pm\infty
\end{aligned}$$

Nema horizontalnih asimptota.

$$
\begin{aligned}
k &= \lim_{x\to\pm\infty} \frac{f(x)}{x} \\
&= \lim_{x\to\pm\infty} e^{-\frac{1}{x^2}} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
l &= \lim_{x\to\pm\infty} [f(x) - kx] \\
&= \lim_{x\to\pm\infty} \left[x \cdot e^{-\frac{1}{x^2}} - x\right] = (\infty - \infty) \\
&= \lim_{x\to\pm\infty} x \left(e^{-\frac{1}{x^2}} - 1\right) = (\infty \cdot 0) \\
&= \lim_{x\to\pm\infty} \frac{e^{-\frac{1}{x^2}}}{\frac{1}{x}} = \left(\frac{0}{0}\right) \\
&=^{\text{L'H}} \lim_{x\to\pm\infty} \frac{\frac{2}{x^3} \cdot e^{-\frac{1}{x^2}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\pm\infty} \frac{-2}{xe^{\frac{1}{x^2}}} \\
&= 0
\end{aligned}
$$

Pravac $y = x$ je kosa asimptota.



$y = x$
$-\sqrt{2}$
0
$\sqrt{2}$

b) $f(x) = x - 2\text{arctg}x$

Domena: $D_f = \mathbb{R}$

Nultočke: $x_0 = 0$

Ekstremi:

$$
\begin{aligned}
f(x) &= x - 2\text{arctg}x \\
f'(x) &= 1 - \frac{2}{x^2+1} \\
&= \frac{1 + x^2 - 2}{x^2+1} \\
&= \frac{x^2-1}{x^2+1}
\end{aligned}
$$

Odredimo stacionarne točke:

$$
\begin{aligned}
x^2 - 1 &= 0 \\
x^2 &= 1
\end{aligned}
$$

$x_1 = -1$, $x_2 = 1$

	$-\infty$		-1		1		$+\infty$
f'	‖	$+$		$-$		$+$	
f	‖	↗	M	↘	m	↗	

Funkcija f u $x = 1$ ima lokalni minimum i $f(1) = 1 - \frac{\pi}{2}$, te u $x = -1$ ima lokalni maksimum i $f(-1) = \frac{\pi}{2} - 1$.

Točke infleksije:

$$\begin{aligned} f''(x) &= \frac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} \\ &= \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

Odredimo točke infleksije:

$$\begin{aligned} 4x &= 0 \\ x &= 0 \end{aligned}$$

	$-\infty$	0	$+\infty$
f''	$-$	$+$	
f	$\cap$	$\cup$	

Interval konkavnosti je $\langle -\infty, 0\rangle$, a interval konveksnosti je $\langle 0, +\infty\rangle$.

Asimptote:

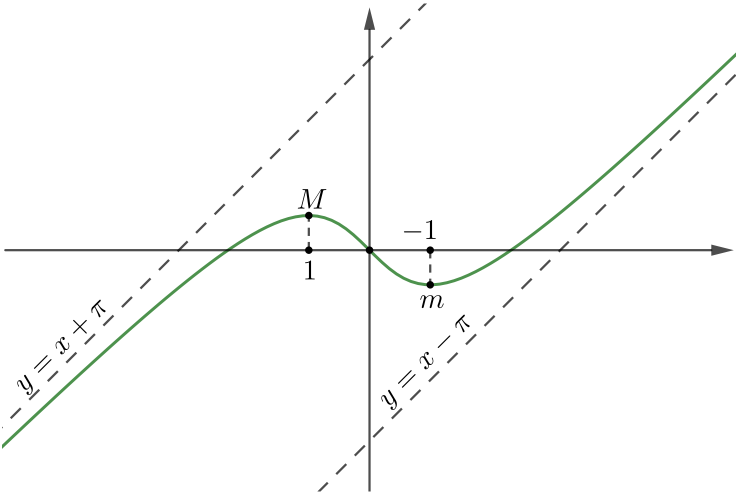
Nema vertikalnih asimptota.

$$\begin{aligned} k &= \lim_{x\to\pm\infty} \frac{x - 2\text{arctg}x}{x} \\ &= \lim_{x\to\pm\infty} \left(1 - \frac{2\text{arctg}x}{x}\right) \\ &= 1 \end{aligned}$$

$$\begin{aligned} l_1 &= \lim_{x\to+\infty} [f(x) - kx] \\ &= \lim_{x\to+\infty} [x - 2\text{arctg}x - x] \\ &= \lim_{x\to+\infty} (-2\text{arctg}x) \\ &= -2\cdot\frac{\pi}{2} \\ &= -\pi \end{aligned}$$

$$\begin{aligned} l_2 &= \lim_{x\to-\infty} [f(x) - kx] \\ &= \lim_{x\to-\infty} [x - 2\text{arctg}x - x] \\ &= \lim_{x\to-\infty} (-2\text{arctg}x) \\ &= -2\cdot\frac{-\pi}{2} \\ &= \pi \end{aligned}$$

Pravac $y = x + \pi$ je lijeva kosa, a pravac $y = x - \pi$ je desna kosa asimptota.



M
-1
1
m
$y = x + \pi$
$y = x - \pi$

c) $f(x) = x^2 \cdot 2^{-x}$

Domena: $D_f = \mathbb{R}$

Nultočke: $x_0 = 0$

Ekstremi:

$$\begin{aligned} f(x) &= x^2 \cdot 2^{-x} \\ f'(x) &= 2x \cdot 2^{-x} + x^2 \cdot 2^{-x} \ln 2 \cdot (-1) \\ &= x \cdot 2^{-x} (2 - x \ln 2) \end{aligned}$$

Odredimo stacionarne točke:

$x_1 = 0$, $x_2 = \dfrac{2}{\ln 2}$

	$-\infty$		x_1		x_2		$+\infty$
f'		$-$		$+$		$-$	
f		↘	m	↗	M	↘	

Funkcija f u $x = 0$ ima lokalni minimum i $f(0) = 0$, te u $x = \dfrac{\ln 2}{2}$ ima lokalni maksimum i $f\left(\dfrac{2}{\ln 2}\right) = \dfrac{\ln^2 2}{4} \cdot 2^{-\frac{\ln 2}{2}}$.

Točke infleksije:

$$\begin{aligned} f''(x) &= 2^{-x}(2-2x\ln 2)+(2x-x^2\ln 2)(-2^{-x}\ln 2)\\ &= 2^{-x}(2-2x\ln 2-2x\ln 2+x^2\ln^2 2)\\ &= 2^{-x}(2-4x\ln 2+x^2\ln^2 2)\end{aligned}$$

$$\begin{aligned} 0 &= 2^{-x}(2-4x\ln 2+x^2\ln^2 2)\\ x_{1,2} &= \frac{4\ln 2\pm\sqrt{16\ln^2 2-8\ln^2 2}}{2\ln^2 2}\\ &= \frac{4\ln 2\pm 2\sqrt{2}\ln 2}{2\ln^2 2}\end{aligned}$$

	$-\infty$	x_1	x_2	$+\infty$
f''	$+$	$-$	$+$	
f	$\cup$	$\cap$	$\cup$	

Asimptote:

Nema vertikalnih asimptota.

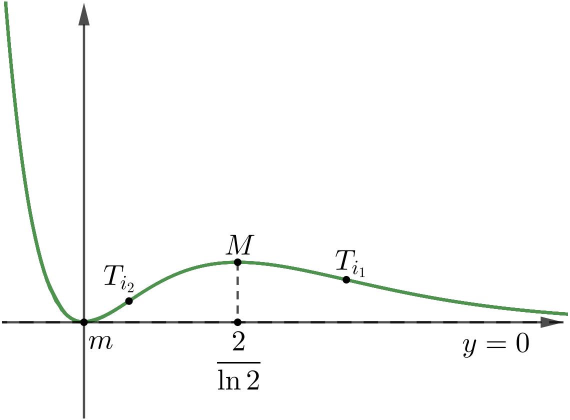
$$\begin{aligned}
\lim_{x\to+\infty} x^2\cdot 2^{-x} &= \lim_{x\to+\infty}\frac{x^2}{2^x}\left(=\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{2x}{2^x\ln 2}\left(=\frac{\infty}{\infty}\right)\\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{2}{2^x\ln^2 2}\\
&= 0\\
\lim_{x\to-\infty} x^2\cdot 2^{-x} &= \infty\cdot 2^{+\infty}\\
&= +\infty
\end{aligned}$$

Pravac $y=0$ je desna horizontalna asimpotota.

Provjeravamo ima li f lijevu kosu asimptotu:

$$\begin{aligned} k &= \lim_{x\to-\infty} \frac{f(x)}{x} \\ &= \lim_{x\to-\infty} \frac{x^2 \cdot 2^{-x}}{x} \\ &= \lim_{x\to-\infty} x \cdot 2^{-x} \\ &= \infty \cdot 2^{+\infty} = -\infty \end{aligned}$$

Nema kosu asimptotu.



M
T_{i_1}
T_{i_2}
m
$\frac{2}{\ln 2}$
$y = 0$

d) $f(x) = \arcsin(e^x - 1)$

Domena:

$D_{\text{arcsin}} = [-1, 1] \Longrightarrow -1 \leq e^x - 1 \leq 1$

$$\begin{aligned} -1 &\leq e^x - 1 \\ 0 &\leq e^x \\ x &\in \mathbb{R} \end{aligned} \qquad \begin{aligned} e^x - 1 &\leq 1 \\ e^x &\leq 2 \\ x &\in \langle -\infty, \ln 2 \rangle \end{aligned}$$

$$D_f = \langle -\infty, \ln 2 \rangle$$

Nultočke:

$$\begin{aligned} e^x - 1 &= 0 \\ e^x &= 1 \\ x &= 0 \end{aligned}$$

Ekstremi:

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1 - (e^x - 1)^2}} \cdot e^x \\ &= \frac{e^x}{\sqrt{1 - (e^x - 1)^2}} > 0 \end{aligned}$$

Funkcija f nema ektrema i stalno raste.

Točke infleksije:

$$
\begin{aligned}
f'(x) &= \frac{e^x}{\sqrt{2e^x - e^{2x}}} \\
f''(x) &= \frac{e^x \cdot \sqrt{2e^x - e^{2x}} - e^x \cdot \frac{1}{2\sqrt{2e^x - e^{2x}}} \cdot (2e^x - 2e^{2x})}{2e^x - e^{2x}} \\
&= \frac{e^x \cdot \sqrt{2e^x - e^{2x}} - \frac{e^x(e^x - e^{2x})}{\sqrt{2e^x - e^{2x}}}}{2e^x - e^{2x}} \\
&= \frac{\frac{e^x(2e^x - e^{2x}) - e^x(e^x - e^{2x})}{\sqrt{2e^x - e^{2x}}}}{2e^x - e^{2x}} \\
&= \frac{e^x(2e^x - e^{2x}) - e^x(e^x - e^{2x})}{(2e^x - e^{2x})^{\frac{3}{2}}} \\
&= \frac{e^{2x}}{(2e^x - e^{2x})^{\frac{3}{2}}} > 0
\end{aligned}
$$

Funkcija f nema točku infleksije i konveksna je na cijeloj domeni.

Asimptote: Nema vertikalnih asimptota.

$$\begin{aligned}\lim_{x\to-\infty} \arcsin(e^x - 1) &= \arcsin(0-1)\\ &= -\frac{\pi}{2}\end{aligned}$$

Pravac $y = -\frac{\pi}{2}$ je lijeva horizontalna asimpotota. Nema kosu asimptotu.