Poglavlje 7
Neodređeni integral

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Definicija

Neka je $f : I \subseteq \mathbb{R} \to \mathbb{R}$. Za $F : I \to \mathbb{R}$ kažemo da je **primitivna funkcija** funkcije f ako je

$$F'(x) = f(x), \forall x \in I.$$

Uočimo da, ako je F primitivna funkcija funkcije f, onda je skup svih primitivnih funkcija od f dan sa $F + c$, gdje je $c \in \mathbb{R}$ konstanta.

Definicija

Skup svih primitivnih funkcija od f nazivamo **neodređeni integral** i označavamo sa $\int f(x)\, dx$.

Tablica važnih integrala:

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$x^m, m \in \mathbb{R} \setminus \{-1\}$	$\frac{x^{m+1}}{m+1} + c$	$\sin x$	$-\cos x + c$
$\frac{1}{x}$	$\ln\|x\| + c$	$\cos x$	$\sin x + c$
$\frac{1}{1+x^2}$	$\operatorname{arctg} x + c$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + c$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + c$
a^x	$\frac{a^x}{\ln a} + c$	$\operatorname{sh} x$	$\operatorname{ch} x + c$
e^x	$e^x + c$	$\operatorname{ch} x$	$\operatorname{sh} x + c$

Pravila integriranja:

- $\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx$
- $\int (c \cdot f(x))\,dx = c \cdot \int f(x)dx$

Zadatak (7.1)

Odredite sljedeće neodređene integrale:

a) $\int (x^2 - x)dx$

Zadatak (7.1)

Odredite sljedeće neodređene integrale:

a) $\int (x^2 - x)dx = \frac{x^3}{3} - \frac{x^2}{2} + c$

Zadatak (7.1)

Odredite sljedeće neodređene integrale:

a) $\int (x^2 - x)dx = \frac{x^3}{3} - \frac{x^2}{2} + c$

b) $\int (x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{7}})dx$

Zadatak (7.1)

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b) $\int (x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{7}})dx = \frac{4}{5}x^{\frac{5}{4}} - \frac{24}{17}x^{\frac{17}{12}} + \frac{7}{6}x^{\frac{6}{7}} + c$

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c) $\int (\text{tg}^2 x)dx$

Zadatak (7.1)

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c) $\int (\mathrm{tg}^2 x)dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \int \left(\frac{1}{\cos^2 x} - 1\right) dx = \mathrm{tg}\, x - x + c$

Zadatak (7.1)

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d) $\int 3^x e^x dx$

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$\mathrm{tg}\, x - x + c$

d) $\int 3^x e^x dx = \int (3e)^x dx = \frac{(3e)^x}{\ln(3e)} + c = \frac{3^x e^x}{\ln 3 + 1} + c$

7.1 Metoda supstitucije

$$
\begin{aligned}
\int f(x)dx &= \left\{ \begin{matrix} t = g(x) & \Rightarrow x = g^{-1}(t) \\ dt = g'(x)dx & \end{matrix} \right\} \\
&= \int f\left(g^{-1}(t)\right) \cdot \frac{1}{g'(g^{-1}(t))} dt
\end{aligned}
$$

Zadatak (7.2)

Odredite sljedeće neodređene integrale:

a) $\displaystyle\int \frac{dx}{\sqrt{5x-2}}$

Zadatak (7.2)

Odredite sljedeće neodređene integrale:

a) $$\int \frac{dx}{\sqrt{5x-2}} = \left\{ \begin{array}{ll} t = 5x - 2 & \\ dt = 5dx & \Rightarrow \frac{dt}{5} = dx \end{array} \right\}$$

$$= \int \frac{1}{\sqrt{t}} \cdot \frac{dt}{5}$$

$$= \frac{1}{5} \int t^{-\frac{1}{2}} dt$$

$$= \frac{1}{5} \cdot 2t^{\frac{1}{2}} + c$$

$$= \frac{2}{5}\sqrt{5x-2} + c$$

b) $\int x \cdot 7^{x^2} dx$

b) $$\int x \cdot 7^{x^2} dx = \begin{Bmatrix} t = x^2 \\ dt = 2xdx \quad \Rightarrow \dfrac{dt}{2} = xdx \end{Bmatrix}$$

$$= \frac{1}{2} \int 7^t dt$$

$$= \frac{7^t}{2 \ln 7} + c$$

$$= \frac{7^{x^2}}{2 \ln 7} + c$$

c) $\int \cos(kx)dx$

c) $$\int \cos(kx)dx = \left\{ \begin{array}{l} t = kx \\ dt = kdx \quad \Rightarrow \dfrac{dt}{k} = dx \end{array} \right\}$$

$$= \frac{1}{k}\int \cos t dt$$

$$= \frac{1}{k}\sin t + c$$

$$= \frac{1}{k}\sin(kx) + c$$

d) $\displaystyle\int \frac{3x^2 dx}{1+x^6}$

$$\text{d) } \int \frac{3x^2dx}{1+x^6} = \begin{Bmatrix} t = x^3 \\ dt = 3x^2dx \end{Bmatrix}$$

$$= \int \frac{dt}{1+t^2}$$

$$= \operatorname{arctg} t + c$$

$$= \operatorname{arctg} x^3 + c$$

e) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

$$
\begin{aligned}
\text{e) } \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx &= \left\{ \begin{matrix} t = \sqrt{x} \\ dt = \dfrac{dx}{2\sqrt{x}} \Rightarrow 2dt = \dfrac{dx}{\sqrt{x}} \end{matrix} \right\} \\
&= 2\int \cos t dt \\
&= 2\sin t + c \\
&= 2\sin\sqrt{x} + c
\end{aligned}
$$

f) $\int \operatorname{tg} x \, dx$

$$
\begin{aligned}
\text{f) } \int \operatorname{tg} x\, dx &= \int \frac{\sin x}{\cos x} dx \\
&= \left\{ \begin{aligned} t &= \cos x \\ dt &= -\sin x dx \end{aligned} \right\} \\
&= -\int \frac{dt}{t} \\
&= -\ln|t| + c \\
&= -\ln|\cos x| + c
\end{aligned}
$$

g) $\displaystyle\int \frac{dx}{\cos x \cdot \sin x}$

g) $$\int \frac{dx}{\cos x \cdot \sin x} = \int \frac{\sin x dx}{\cos x \cdot \sin^2 x}$$

$$= \int \frac{1}{\operatorname{ctg} x} \cdot \frac{1}{\sin^2 x} dx$$

$$= \left\{ \begin{aligned} t &= \operatorname{ctg} x \\ dt &= -\frac{1}{\sin^2 x} dx \end{aligned} \right\}$$

$$= -\int \frac{dt}{t}$$

$$= -\ln|t| + c$$

$$= -\ln|\operatorname{ctg} x| + c$$

h) $\int x(2x+10)^{10}dx$

$$
\begin{aligned}
\text{h) } \int x(2x+10)^{10}dx &= \left\{\begin{array}{ll} t = 2x+10 & \Rightarrow x = \dfrac{t-10}{2} \\ dt = 2dx & \Rightarrow \dfrac{dt}{2} = dx \end{array}\right\} \\
&= \frac{1}{2}\int \frac{t-10}{2}\cdot t^{10}dt \\
&= \frac{1}{4}\int (t^{11}-10t^{10})dt \\
&= \frac{1}{4}\cdot\frac{t^{12}}{12} - \frac{10}{4}\cdot\frac{t^{11}}{11} + c \\
&= \frac{(2x+10)^{12}}{48} - \frac{5(2x+10)^{11}}{22} + c
\end{aligned}
$$

i) $\displaystyle\int \frac{dx}{\sqrt{e^x - 1}}$

$$
\begin{aligned}
\text{i) } \int \frac{dx}{\sqrt{e^x-1}} &= \left\{ \begin{array}{c} t^2 = e^x - 1 \Rightarrow e^x = t^2+1 \\ 2tdt = e^x dx \Rightarrow \dfrac{2tdt}{t^2+1} = dx \end{array} \right\} \\
&= \int \frac{2tdt}{t(t^2+1)} \\
&= 2\int \frac{dt}{t^2+1} \\
&= 2\text{arctg}\, t + c \\
&= 2\text{arctg}\sqrt{e^x-1} + c
\end{aligned}
$$

j) $\displaystyle\int \frac{dx}{x^2 + a^2}$

$$
\begin{aligned}
\text{j) } \int \frac{dx}{x^2+a^2} &= \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} \\
&= \left\{ \begin{array}{l} t = \dfrac{x}{a} \\ dt = \dfrac{dx}{a} \end{array} \right\} \\
&= \frac{1}{a^2}\int \frac{a\,dt}{t^2+1} \\
&= \frac{1}{a}\int \frac{dt}{t^2+1} \\
&= \frac{1}{a}\text{arctg}\, t + c \\
&= \frac{1}{a}\text{arctg}\frac{x}{a} + c
\end{aligned}
$$

k) $\int \frac{dx}{\sqrt{a^2 - x^2}}$

$$
\begin{aligned}
\text{k)} \int \frac{dx}{\sqrt{a^2-x^2}} &= \frac{1}{a}\int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \\
&= \left\{ \begin{array}{l} t = \dfrac{x}{a} \\ dt = \dfrac{dx}{a} \end{array} \right\} \\
&= \frac{1}{a}\int \frac{a\,dt}{\sqrt{1-t^2}} \\
&= \arcsin t + c \\
&= \arcsin\frac{x}{a} + c
\end{aligned}
$$

Zadatak (7.3)

Odredite funkciju F takvu da je $F'(x) = \dfrac{x}{\sqrt{(x^2+1)^3}}$ te $F(0) = 1$.

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Odredite funkciju F takvu da je $F'(x) = \dfrac{x}{\sqrt{(x^2+1)^3}}$ te $F(0) = 1$.

Rješenje:

$$\begin{aligned} F(x) &= \int \frac{x}{\sqrt{(x^2+1)^3}} dx \\ &= \left\{ \begin{aligned} &t = x^2 + 1 \\ &dt = 2x\,dx \Rightarrow \frac{dt}{2} = x\,dx \end{aligned} \right\} \\ &= \frac{1}{2} \int \frac{dt}{t^{\frac{3}{2}}} \\ &= \frac{1}{2} \cdot (-2) \cdot t^{-\frac{1}{2}} + c \\ &= -\frac{1}{\sqrt{x^2+1}} + c \end{aligned}$$

$$
\begin{aligned}
F(0) &= -\frac{1}{\sqrt{0^2+1}} + c \\
1 &= -1 + c \\
c &= 2
\end{aligned}
$$

$$F(x) = -\frac{1}{\sqrt{x^2+1}} + 2$$

Zadatak (7.4)

Odredite sljedeće integrale:

a) $\int \frac{x^2+1}{\sqrt{x}} dx$

Zadatak (7.4)

Odredite sljedeće integrale:

$$\text{a)} \int \frac{x^2+1}{\sqrt{x}}dx = \int \frac{x^2}{\sqrt{x}}dx + \int \frac{1}{\sqrt{x}}dx$$

$$= \int x^{\frac{3}{2}}dx + \int x^{-\frac{1}{2}}dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + c$$

b) $\int x(3x+1)^5 dx$

$$
\begin{aligned}
\text{b) } \int x(3x+1)^5 dx &= \left\{ \begin{array}{lll} t = 3x+1 & \Rightarrow & x = \dfrac{t-1}{3} \\ dt = 3dx & \Rightarrow & dx = \dfrac{dt}{3} \end{array} \right\} \\
&= \int \frac{t-1}{3} \cdot t^5 \cdot \frac{dt}{3} \\
&= \frac{1}{9} \int \left((t-1)t^5 \right) \, dt \\
&= \frac{1}{9} \int \left(t^6 - t^5 \right) \, dt \\
&= \frac{1}{9} \left(\frac{1}{7} t^7 - \frac{1}{6} t^6 \right) + c \\
&= \frac{1}{9} \left(\frac{(3x+1)^7}{7} - \frac{(3x+1)^6}{6} \right) + c
\end{aligned}
$$

c) $\int x\sqrt{2x+1}\,dx$

$$
\begin{aligned}
\text{c) } \int x\sqrt{2x+1}\,dx &= \left\{ \begin{array}{lll} t = 2x+1 & \Rightarrow & x = \dfrac{t-1}{2} \\ & & \\ dt = 2dx & \Rightarrow & dx = \dfrac{dt}{2} \end{array} \right\} \\
&= \int \frac{t-1}{2} \cdot \sqrt{t} \cdot \frac{dt}{2} \\
&= \frac{1}{4} \int \left(t^{\frac{3}{2}} - t^{\frac{1}{2}} \right) \, dt \\
&= \frac{1}{4} \left(\frac{2}{5} t^{\frac{5}{2}} - \frac{2}{3} t^{\frac{3}{2}} \right) + c \\
&= \frac{(2x+1)^{\frac{5}{2}}}{10} - \frac{(2x+1)^{\frac{3}{2}}}{6} + c
\end{aligned}
$$

d) $\int x^3\sqrt{x^2+1}\,dx$

$$
\begin{aligned}
\text{d) } \int x^3\sqrt{x^2+1}\,dx &= \left\{\begin{matrix} t = x^2+1 & \Rightarrow & x^2 = t-1 \\ dt = 2x\,dx & \Rightarrow & \dfrac{dt}{2} = x\,dx \end{matrix}\right\} \\
&= \frac{1}{2}\int \sqrt{t}\cdot(t-1)\,dt \\
&= \frac{1}{2}\int (t^{\frac{3}{2}} - t^{\frac{1}{2}})dt \\
&= \frac{1}{2}\left(\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right) + c \\
&= \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{(x^2+1)^{\frac{3}{2}}}{3} + c
\end{aligned}
$$

e) $\int \frac{\sin x}{\cos^3 x}\, dx$

$$\text{e)} \int \frac{\sin x}{\cos^3 x}\, dx = \left\{ \begin{aligned} t &= \cos x \\ dt &= -\sin x\, dx \end{aligned} \right\}$$

$$= -\int \frac{dt}{t^3}$$

$$= \frac{1}{2} t^{-2} + c$$

$$= \frac{1}{2\cos^2 x} + c$$

7.2 Metoda parcijalne integracije

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)dx$$

ili

$$\int u\,dv = uv - \int v\,du$$

Zadatak (7.5)

Odredite sljedeće integrale:

a) $\int xe^x \, dx$

Zadatak (7.5)

Odredite sljedeće integrale:

a) $$\int xe^x\,dx = \begin{Bmatrix} u = x & du = dx \\ dv = e^x\,dx & v = e^x \end{Bmatrix}$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

Zadatak (7.5)

Odredite sljedeće integrale:

a)
$$\int xe^x\,dx = \left\{\begin{matrix} u = x & du = dx \\ dv = e^x\,dx & v = e^x \end{matrix}\right\}$$
$$= xe^x - \int e^x\,dx$$
$$= xe^x - e^x + c$$

b)
$$\int x \sin x\,dx$$

Zadatak (7.5)

Odredite sljedeće integrale:

a) $$\int xe^x\,dx = \begin{Bmatrix} u = x & du = dx \\ dv = e^x\,dx & v = e^x \end{Bmatrix}$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

b) $$\int x\sin x\,dx = \begin{Bmatrix} u = x & du = dx \\ dv = \sin x\,dx & v = -\cos x \end{Bmatrix}$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c$$

c) $\int x \ln x \, dx$

c) $$\int x \ln x \, dx = \left\{ \begin{matrix} u = \ln x & du = \dfrac{dx}{x} \\ dv = x \, dx & v = \dfrac{x^2}{2} \end{matrix} \right\}$$

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x}$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

d) $\int \ln x \, dx$

d) $$\int \ln x \, dx = \left\{ \begin{matrix} u = \ln x & du = \frac{dx}{x} \\ & \\ dv = dx & v = x \end{matrix} \right\}$$

$$= x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - x + c$$

e) $\int \operatorname{arctg} x \, dx$

$$
\begin{aligned}
\text{e) } \int \operatorname{arctg} x\, dx &= \left\{ \begin{matrix} u = \operatorname{arctg} x & du = \dfrac{dx}{1+x^2} \\ \\ dv = dx & v = x \end{matrix} \right\} \\
&= x \operatorname{arctg} x - \int \frac{x}{1+x^2} dx \\
&= \left\{ \begin{matrix} t = 1 + x^2 \\ dt = 2x\, dx \end{matrix} \right\} \\
&= x \operatorname{arctg} x - \frac{1}{2} \int \frac{dt}{t} \\
&= x \operatorname{arctg} x - \frac{1}{2} \ln |t| + c \\
&= x \operatorname{arctg} x - \frac{1}{2} \ln(1 + x^2) + c
\end{aligned}
$$

f) $\int e^x \sin x \, dx$

$$
\begin{aligned}
\text{f) } \int e^x \sin x \, dx &= \left\{ \begin{matrix} u = \sin x & du = \cos x \, dx \\ dv = e^x \, dx & v = e^x \end{matrix} \right\} \\
&= e^x \sin x - \int e^x \cos x \, dx \\
&= \left\{ \begin{matrix} u = \cos x & du = -\sin x \, dx \\ dv = e^x \, dx & v = e^x \end{matrix} \right\} \\
&= e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \, dx \right) \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\
\int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\
2 \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x \\
\int e^x \sin x \, dx &= \frac{1}{2} e^x (\sin x - \cos x) + c
\end{aligned}
$$

g) $\int \cos(\ln x)\, dx$

g) $$\begin{aligned}
\int \cos(\ln x)\,dx &= \left\{\begin{matrix} u = \cos(\ln x) & du = \dfrac{-\sin(\ln x)}{x}\,dx \\ dv = dx & v = x \end{matrix}\right\} \\
&= x\,\cos(\ln x) - \int x \cdot \frac{-\sin(\ln x)}{x}\,dx \\
&= x\,\cos(\ln x) + \int \sin(\ln x)\,dx \\
&= \left\{\begin{matrix} u = \sin(\ln x) & du = \dfrac{\cos(\ln x)}{x}\,dx \\ dv = dx & v = x \end{matrix}\right\} \\
&= x\,\cos(\ln x) + \left(x\,\sin(\ln x) - \int x \cdot \frac{\cos(\ln x)}{x}\,dx\right) \\
&= x\,\cos(\ln x) + x\,\sin(\ln x) - \int \cos(\ln x)\,dx
\end{aligned}$$

$$\begin{aligned}
\int \cos(\ln x)\,dx &= x\,\cos(\ln x) + x\,\sin(\ln x) - \int \cos(\ln x)\,dx \\
2\int \cos(\ln x)\,dx &= x\,\cos(\ln x) + x\,\sin(\ln x) \\
\int \cos(\ln x)\,dx &= \frac{1}{2}x(\cos(\ln x) + \sin(\ln x)) + c
\end{aligned}$$

h) $\int \sqrt{x} \ln x \, dx$

$$\text{h) } \int \sqrt{x}\ln x\,dx = \begin{Bmatrix} u = \ln x & du = \dfrac{dx}{x} \\ dv = \sqrt{x}\,dx & v = \dfrac{2}{3}x^{\frac{3}{2}} \end{Bmatrix}$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{3}{2}} \cdot \frac{1}{x}\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + c$$

i) $\int \operatorname{arctg}\sqrt{x}\, dx$

$$
\begin{aligned}
\text{i) } \int \operatorname{arctg}\sqrt{x}\,dx &= \left\{\begin{matrix} t^2 = x \\ 2t\,dt = dx \end{matrix}\right\} \\
&= 2\int \operatorname{arctg} t \cdot t\,dt \\
&= \left\{\begin{matrix} u = \operatorname{arctg} t & du = \dfrac{1}{1+t^2}dt \\ dv = t\,dt & v = \dfrac{1}{2}t^2 \end{matrix}\right\} \\
&= t^2 \cdot \operatorname{arctg} t - 2\int \frac{t^2}{2} \cdot \frac{1}{1+t^2}dt \\
&= t^2 \cdot \operatorname{arctg} t - \int \frac{1+t^2-1}{1+t^2}dt
\end{aligned}
$$

$$
\begin{aligned}
&= t^2 \cdot \operatorname{arctg} t - \left(\int \frac{1+t^2}{1+t^2} dt - \int \frac{1}{1+t^2} dt \right) \\
&= t^2 \cdot \operatorname{arctg} t - (t - \operatorname{arctg} t + c) \\
&= t^2 \cdot \operatorname{arctg} t - t + \operatorname{arctg} t + c \\
&= x \operatorname{arctg} \sqrt{x} - \sqrt{x} + \operatorname{arctg} \sqrt{x} + c
\end{aligned}
$$

j) $\int \ln(1 + x^2)\, dx$

$$
\begin{aligned}
\text{j) } \int \ln(1+x^2)\,dx &= \left\{ \begin{matrix} u = \ln(1+x^2) & du = \dfrac{2x}{1+x^2}dx \\ & \\ dv = dx & v = x \end{matrix} \right\} \\
&= x\ln(1+x^2) - 2\int \frac{x^2}{1+x^2}dx \\
&= x\ln(1+x^2) - 2\left(\int \frac{1+x^2}{1+x^2}dx - \int \frac{1}{1+x^2}dx\right) \\
&= x\ln(1+x^2) - 2x + 2\operatorname{arctg} x + c
\end{aligned}
$$

k)

$$\int (x^2 + 3x) \sin x \, dx$$

k)

$$
\begin{aligned}
\int (x^2+3x)\sin x\,dx &= \left\{\begin{matrix} u = x^2+3x & du = (2x+3)\,dx \\ dv = \sin x\,dx & v = -\cos x \end{matrix}\right\} \\
&= -(x^2+3x)\cos x + \int (2x+3)\cos x\,dx \\
&= \left\{\begin{matrix} u = 2x+3 & du = 2\,dx \\ dv = \cos x\,dx & v = \sin x \end{matrix}\right\} \\
&= -(x^2+3x)\cos x + (2x+3)\sin x - 2\int \sin x\,dx \\
&= -(x^2+3x)\cos x + (2x+3)\sin x + 2\cos x + c
\end{aligned}
$$

l) $\int e^{\sqrt{x}}\, dx$

$$
\begin{aligned}
\text{l)} \int e^{\sqrt{x}}\,dx &= \begin{Bmatrix} t^2 = x \\ 2t\,dt = dx \end{Bmatrix} \\
&= 2\int t \cdot e^t\,dt \\
&= \begin{Bmatrix} u = t & du = dt \\ dv = e^t\,dt & v = e^t \end{Bmatrix} \\
&= 2t \cdot e^t - 2\int e^t\,dt \\
&= 2t \cdot e^t - 2e^t + c \\
&= 2\sqrt{x} \cdot e^{\sqrt{x}} - 2e^{\sqrt{x}} + c
\end{aligned}
$$

7.3 Integriranje racionalnih funkcija

Zadatak (7.6)

Odredite sljedeće integrale:

a) $\displaystyle\int \frac{1}{2x+1}\,dx$

Zadatak (7.6)

Odredite sljedeće integrale:

$$\text{a)} \int \frac{1}{2x+1}\,dx = \begin{Bmatrix} t = 2x+1 \\ dt = 2\,dx \end{Bmatrix}$$

$$= \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}\ln|t| + c$$

$$= \frac{1}{2}\ln|2x+1| + c$$

b) $\int \frac{3x+1}{x^2+1}\,dx$

$$
\begin{aligned}
\text{b) } \int \frac{3x+1}{x^2+1}\,dx &= \int \frac{3x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\
&= \left\{ \begin{matrix} t = x^2+1 \\ dt = 2x\,dx \end{matrix} \right\} \\
&= \frac{3}{2}\int \frac{dt}{t} + \operatorname{arctg} x + c \\
&= \frac{3}{2}\ln|t| + \operatorname{arctg} x + c \\
&= \frac{3}{2}\ln(x^2+1) + \operatorname{arctg} x + c
\end{aligned}
$$

c)

$$\int \frac{dx}{x^2 + x + 1}$$

c)

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$= \int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\,\text{arctg}\,\frac{x}{a}+c$$

$$= \left\{ \begin{array}{l} t = x+\dfrac{1}{2} \\ dt = dx \end{array} \right\}$$

$$= \int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\,\text{arctg}\,\frac{2t}{\sqrt{3}}+c$$

$$= \frac{2}{\sqrt{3}}\,\text{arctg}\,\frac{2x+1}{\sqrt{3}}+c$$

d) $\int \frac{x\,dx}{2x^2 - 3}$

$$\text{d)} \int \frac{x\,dx}{2x^2-3} = \begin{Bmatrix} t = 2x^2-3 \\ dt = 4x\,dx \end{Bmatrix}$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\ln|t| + c$$

$$= \frac{1}{4}\ln|2x^2-3| + c$$

e) $\int \frac{dx}{x^2 - 4}$

$$\text{e)} \int \frac{dx}{x^2-4} = \int \frac{dx}{(x-2)(x+2)} = (*)$$

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \quad / \cdot (x^2-4)$$

$$\begin{aligned} 1 &= A(x+2) + B(x-2) \\ 1 &= x(A+B) + 2A - 2B \end{aligned}$$

$$\begin{aligned} 0 &= A+B \\ 1 &= 2A - 2B \\ 1 &= 4A \end{aligned}$$

$$A = \frac{1}{4} \quad \text{i} \quad B = -\frac{1}{4}$$

$$\begin{aligned} (*) &= \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2} \\ &= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + c \end{aligned}$$

f) $\int \frac{dx}{(x-a)^p}$

$$\text{f) } \int \frac{dx}{(x-a)^p} = \left\{\begin{matrix} t = x - a \\ dt = dx \end{matrix}\right\}$$

$$= \frac{1}{4} \int \frac{dt}{t^p}$$

$$= \frac{t^{-p+1}}{-p+1} + c$$

$$= \frac{1}{-p+1} \cdot \frac{1}{(x-a)^{(p-1)}} + c$$

Napomena: Integral $\int \frac{dx}{ax^2 + bx + c}$ rješavamo rastavom na parcijalne razlomke, ako je $D = b^2 - 4ac \geq 0$. Ako je $D < 0$, zapisujemo ga u obliku $\int \frac{dt}{t^2 + a^2}$.

Zadatak (7.7)

Odredite sljedeće integrale:

a) $\int \frac{-x^2 + x + 1}{(x-1)^3} dx$

$$
\begin{aligned}
\text{a) } \int \frac{-x^2+x+1}{(x-1)^3}dx &= (*) \\
\frac{-x^2+x+1}{(x-1)^3} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \quad \Big/ \cdot (x-1)^3 \\
-x^2+x+1 &= A(x^2-2x+1) + B(x-1) + C \\
-x^2+x+1 &= Ax^2 + x(-2A+B) + A - B + C
\end{aligned}
$$

$$
\begin{aligned}
A &= -1 \\
-2A+B &= 1 \Longrightarrow B = -1 \\
A-B+C &= 1 \Longrightarrow C = 1
\end{aligned}
$$

$$
\begin{aligned}
(*) &= -\int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x-1)^3} \\
&= -\ln|x-1| + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + c
\end{aligned}
$$

b) $\int \frac{x^2+1}{x^4-x^3}dx$

$$\text{b)} \int \frac{x^2+1}{x^4-x^3}dx = \int \frac{x^2+1}{x^3(x-1)}dx = (*)$$

$$\frac{x^2+1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} \quad \Big/ \cdot x^3(x-1)$$

$$x^2+1 = A(x^3-x^2) + B(x^2-x) + C(x-1) + Dx^3$$

$$x^2+1 = x^3(A+D) + x^2(-A+B) + x(-B+C) - C$$

$$\begin{aligned} -C &= 1 \implies C = -1 \\ -B+C &= 0 \implies B = -1 \\ -A+B &= 1 \implies A = -2 \\ A+D &= 0 \implies D = 2 \end{aligned}$$

$$\begin{aligned} (*) &= -2\int \frac{dx}{x} - \int \frac{dx}{x^2} - \int \frac{dx}{x^3} + 2\int \frac{dx}{x-1} \\ &= -2\ln|x| + \frac{1}{x} + \frac{1}{2x^2} + 2\ln|x-1| + c \end{aligned}$$

c) $\displaystyle\int \frac{x^2 - 12}{x(x^2-4)}dx$

c) $$\int \frac{x^2-12}{x(x^2-4)}dx = \int \frac{x^2-12}{x(x-2)(x+2)}dx = (*)$$

$$\frac{x^2-12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \quad \Big/ \cdot x(x-2)(x+2)$$

$$x^2-12 = A(x^2-4) + B(x^2+2x) + C(x^2-2x)$$

$$x^2-12 = x^2(A+B+C) + x(2B-2C) - 4A$$

$$-4A = -12 \implies A = 3$$

$$2B-2C = 0$$

$$A+B+C = 1$$

$$B-C = 0$$

$$B+C = -2$$

$$B = -1 \implies C = -1$$

$$
\begin{aligned}
(*) \quad &= 3\int \frac{dx}{x} - \int \frac{dx}{x-2} - \int \frac{dx}{x+2} \\
&= 3\ln|x| - \ln|x-2| - \ln|x+2| + c \\
&= \ln\left|\frac{x^3}{x^2-4}\right| + c
\end{aligned}
$$

d)
$$\int \frac{dx}{x^4 - 16}$$

d)

$$\int \frac{dx}{x^4-16} = \int \frac{dx}{(x-2)(x+2)(x^2+4)} dx = (*)$$

$$\frac{1}{x^4-16} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4} \Big/ \cdot (x-2)(x+2)(x^2+4)$$

$$1 = A(x^3+2x^2+4x+8) + B(x^3-2x^2+4x-8) + (Cx+D)(x^2-4)$$
$$1 = A(x^3+2x^2+4x+8) + B(x^3-2x^2+4x-8) + Cx^3 - 4Cx + Dx^2 - 4D$$
$$1 = x^3(A+B+C) + x^2(2A-2B+D) + x(4A+4B-4C) + 8A-8B-4D$$

$$\begin{aligned} A+B+C &= 0 \\ 2A-2B+D &= 0 \\ 4A+4B-4C &= 0 \\ 8A-8B-4D &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ 4 & 4 & -4 & 0 & 0 \\ 8 & -8 & 0 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \\ / : 4 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 8 & -8 & 0 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \text{II}-2\text{I} \\ \text{III}-\text{I} \\ \text{IV}-8\text{I} \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -16 & -8 & -4 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \text{IV}-4\text{II} \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -8 & 1 \end{array}\right]$$

$$D = -\frac{1}{8} \qquad C = 0 \quad -4B - 2C + D = 0 \quad A + B + C = 0$$

$$-4B = \frac{1}{8} \qquad A = \frac{1}{32}$$

$$B = -\frac{1}{32}$$

$$\begin{aligned}(*) \quad &= \quad \frac{1}{32}\int \frac{dx}{x-2} - \frac{1}{32}\int \frac{dx}{x+2} - \frac{1}{8}\int \frac{dx}{x^2+4} \\ &= \quad \frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{16}\text{arctg}\,\frac{x}{2} + c\end{aligned}$$

e) $\int \frac{x^3}{x^2+x+1}dx =$

e) $\int \frac{x^3}{x^2+x+1}\,dx = (*)$

$$
\begin{array}{l}
x^3 \qquad : (x^2+x+1) = x-1 \\
-(x^3+x^2+x) \\
\hline
-x^2-x \\
-(-x^2-x-1) \\
\hline
1
\end{array}
$$

$$
\begin{aligned}
(*) &= \int \left(x - 1 + \frac{1}{x^2+x+1} \right) dx \\
&= \int (x-1)\,dx + \int \frac{1}{x^2+x+1} dx \\
&= \frac{x^2}{2} - x + \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\
&= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) + c
\end{aligned}
$$

f) $\int \frac{2x^3 + 3x^2 + x - 1}{(x^2 + 2x + 2)^2} dx = (*)$

f) $\displaystyle\int \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}dx = (*)$

$$\begin{aligned}
\frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2} &= \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2} \Big/ \cdot (x^2+2x+2)^2 \\
2x^3+3x^2+x-1 &= (Ax+B)(x^2+2x+2) + Cx + D \\
2x^3+3x^2+x-1 &= Ax^3 + 2Ax^2 + 2Ax + Bx^2 + 2Bx + 2B + Cx + D \\
2x^3+3x^2+x-1 &= Ax^3 + x^2(2A+B) + x(2A+2B+C) + 2B + D
\end{aligned}$$

$$\begin{aligned}
A &= 2 \\
2A+B &= 3 \implies B = -1 \\
2A+2B+C &= 1 \implies C = -1 \\
2B+D &= -1 \implies D = 1
\end{aligned}$$

$$
\begin{aligned}
(*) &= \int \frac{2x-1}{x^2+2x+2}dx + \int \frac{-x+1}{(x^2+2x+2)^2}dx \\
&= \int \frac{2x-1}{(x+1)^2+1}dx + \int \frac{-x+1}{\left((x+1)^2+1\right)^2}dx \\
&= \left\{ \begin{array}{ll} t = x+1 & \Rightarrow x = t-1 \\ dx = dt & \end{array} \right\} \\
&= \int \frac{2t-3}{t^2+1}dt + \int \frac{-t+2}{\left(t^2+1\right)^2}dt \\
&= \underbrace{\int \frac{2t}{t^2+1}dt}_{\Delta} - 3\int \frac{1}{t^2+1}dt - \underbrace{\int \frac{t}{\left(t^2+1\right)^2}dt}_{\square} + 2\underbrace{\int \frac{1}{\left(t^2+1\right)^2}dt}_{\blacksquare} \\
&= (*)
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \int \frac{2t}{t^2+1}\,dt \\
&= \left\{ \begin{aligned} w &= t^2+1 \\ dw &= 2t\,dt \end{aligned} \right\} \\
&= \int \frac{dw}{w} \\
&= \ln|w| + c \\
&= \ln(t^2+1) + c
\end{aligned}
$$

$$
\begin{aligned}
\square &= -\int \frac{t}{(t^2+1)^2}\,dt \\
&= \left\{ \begin{aligned} & w = t^2+1 \\ & dw = 2t\,dt \Rightarrow \frac{dw}{2} = t\,dt \end{aligned} \right\} \\
&= -\frac{1}{2}\int \frac{dw}{w^2} \\
&= \frac{1}{2}\cdot\frac{1}{w} + c \\
&= \frac{1}{2(t^2+1)} + c
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare \quad &= \int \frac{1}{\left(t^2+1\right)^2}\,dt \\
&= \int \frac{t^2+1-t^2}{\left(t^2+1\right)^2}\,dt \\
&= \int \frac{t^2+1}{\left(t^2+1\right)^2}\,dt - \int \frac{t^2}{\left(t^2+1\right)^2}\,dt \\
&= \int \frac{1}{t^2+1}\,dt - \int \frac{t\cdot t}{\left(t^2+1\right)^2}\,dt \\
&= \left\{ \begin{array}{lcl} u = t & \Rightarrow & du = dt \\ dv = \underbrace{\dfrac{t}{(t^2+1)^2}\,dt}_{-\Box} & \Rightarrow & v = -\dfrac{1}{2(t^2+1)} \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname{arctg} t + \frac{t}{2(t^2+1)} - \frac{1}{2}\operatorname{arctg} t + c \\
&= \frac{1}{2}\operatorname{arctg} t + \frac{t}{2(t^2+1)} + c
\end{aligned}
$$

$$
\begin{aligned}
(*) \quad &= \ln(t^2+1) - 3\operatorname{arctg} t + \frac{1}{2(t^2+1)} + 2\left(\frac{1}{2}\operatorname{arctg} t + \frac{t}{2(t^2+1)}\right) + c \\
&= \ln(t^2+1) - 3\operatorname{arctg} t + \frac{1}{2(t^2+1)} + \operatorname{arctg} t + \frac{t}{(t^2+1)} + c \\
&= \ln(t^2+1) - 2\operatorname{arctg} t + \frac{1+2t}{2(t^2+1)} + c \\
&= \ln(x^2+2x+2) - 2\operatorname{arctg}(x+1) + \frac{2x+3}{2(x^2+2x+2)} + c
\end{aligned}
$$

g) $\int \frac{e^x}{e^{2x} - 2e^x + 10} dx$

$$
\text{g)} \int \frac{e^x}{e^{2x} - 2e^x + 10} dx = \left\{ \begin{matrix} t = e^x \\ dt = e^x \, dx \end{matrix} \right\}
$$

$$
= \int \frac{dt}{t^2 - 2t + 10}
$$

$$
= \int \frac{dt}{(t-1)^2 + 3^2}
$$

$$
= \left\{ \begin{matrix} w = t - 1 \\ dw = dt \end{matrix} \right\}
$$

$$
= \int \frac{dw}{w^2 + 3^2}
$$

$$
\begin{aligned}
&= \frac{1}{3} \operatorname{arctg} \frac{w}{3} + c \\
&= \frac{1}{3} \operatorname{arctg} \frac{t-1}{3} + c \\
&= \frac{1}{3} \operatorname{arctg} \frac{e^x - 1}{3} + c
\end{aligned}
$$

7.4 Integriranje iracionalnih funkcija

Zadatak (7.8)

Odredite sljedeće integrale:

a) $\displaystyle\int \frac{dx}{2\sqrt{x}+\sqrt[3]{x}}$

Zadatak (7.8)

Odredite sljedeće integrale:

$$\text{a)} \int \frac{dx}{2\sqrt{x} + \sqrt[3]{x}} = \begin{Bmatrix} x = t^6 \\ dx = 6t^5\,dt \end{Bmatrix}$$

$$= \int \frac{6t^5\,dt}{2t^3 + t^2}$$

$$= \int \frac{6t^{\cancel{5}3}\,dt}{\cancel{t^2}(2t+1)}$$

$$= \int \frac{6t^3\,dt}{2t+1} = (*)$$

$$
\begin{array}{c}
6t^3 \\
-(6t^3 + 3t^2) \\
\hline
-3t^2 \\
-\left(-3t^2 - \frac{3}{2}t\right) \\
\hline
\frac{3}{2}t \\
-\left(\frac{3}{2}t + \frac{3}{4}\right) \\
\hline
-\frac{3}{4}
\end{array}
\quad : (2t + 1) = 3t^2 - \frac{3}{2}t + \frac{3}{4}
$$

$$
\begin{aligned}
(*) &= \int \left(3t^2 - \frac{3}{2}t + \frac{3}{4} - \frac{\frac{3}{4}}{2t+1} \right) dt \\
&= t^3 - \frac{3}{4}t^2 + \frac{3}{4}t - \frac{3}{4}\int \frac{1}{2t+1} dt \\
&= t^3 - \frac{3}{4}t^2 + \frac{3}{4}t - \frac{3}{8}\ln|2t+1| + c \\
&= \sqrt{x} - \frac{3}{4}\sqrt[3]{x} + \frac{3}{4}\sqrt[6]{x} - \frac{3}{8}\ln|2\sqrt[6]{x}+1| + c
\end{aligned}
$$

b) $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$

$$
\begin{aligned}
\text{b) } \int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{4-x^2}}\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx \\
&= \left\{ \begin{array}{l} t = 4 - x^2 \\ dt = -2x\,dx \Rightarrow -\dfrac{dt}{2} = x\,dx \end{array} \right\} \\
&= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} + 2\int \frac{1}{\sqrt{4-x^2}}\,dx \\
&= -\frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{t} + 2\arcsin\frac{x}{2} + c \\
&= -\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + c
\end{aligned}
$$

c) $\int \sqrt{e^x - 1}\, dx$

$$
\begin{aligned}
\text{c) } \int \sqrt{e^x - 1}\, dx &= \left\{ \begin{matrix} t^2 = e^x - 1 & \Rightarrow & e^x = t^2 + 1 \\ 2t\, dt = e^x\, dx & \Rightarrow & \dfrac{2t\, dt}{t^2+1} = dx \end{matrix} \right\} \\
&= \int t \cdot \frac{2t\, dt}{t^2+1} \\
&= 2 \int \frac{t^2}{t^2+1}\, dt \\
&= 2 \int \frac{t^2+1-1}{t^2+1}\, dt \\
&= 2 \int dt - 2 \int \frac{1}{t^2+1}\, dt \\
&= 2t - 2 \operatorname{arctg} t + c \\
&= 2\sqrt{e^x - 1} - 2 \operatorname{arctg} \sqrt{e^x - 1} + c
\end{aligned}
$$

d) $\displaystyle\int \ln(x + \sqrt{x^2+1})\, dx$

d) $\int \ln(x+\sqrt{x^2+1})\,dx$

$$= \left\{ \begin{array}{ll} u = \ln(x+\sqrt{x^2+1}) & dv = dx \\ du = \dfrac{1}{x+\sqrt{x^2+1}}\left(1+\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right)dx & v = x \\ \quad = \dfrac{1}{\cancel{x+\sqrt{x^2+1}}}\left(\dfrac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right)dx & \\ \quad = \dfrac{1}{\sqrt{x^2+1}}dx & \end{array} \right\}$$

$$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}\,dx$$

$$
\begin{aligned}
&= \begin{Bmatrix} t = x^2 + 1 \\ dt = 2x\,dx \end{Bmatrix} \\
&= x\ln(x + \sqrt{x^2+1}) - \frac{1}{2}\int \frac{dt}{\sqrt{t}}\,dx \\
&= x\ln(x + \sqrt{x^2+1}) - \frac{1}{\not 2} \cdot \not 2\sqrt{t} + c \\
&= x\ln(x + \sqrt{x^2+1}) - \sqrt{x^2+1} + c
\end{aligned}
$$

e) $\int e^{\sqrt{2x+1}}\, dx$

$$
\begin{aligned}
\text{e) } \int e^{\sqrt{2x+1}}\,dx &= \left\{\begin{matrix} t^2 = 2x+1 \\ \not{2}t\,dt = \not{2}\,dx \end{matrix}\right\} \\
&= \int e^t \cdot t\,dt \\
&= \left\{\begin{matrix} u = t & \Rightarrow & du = dt \\ dv = e^t\,dt & \Rightarrow & v = e^t \end{matrix}\right\} \\
&= te^t - \int e^t\,dt \\
&= te^t - e^t + c \\
&= \sqrt{2x+1}\,e^{\sqrt{2x+1}} - e^{\sqrt{2x+1}} + c
\end{aligned}
$$

f) $\int \frac{\ln \sqrt{x+3}}{\sqrt{x+3}}\, dx$

$$
\begin{aligned}
\text{f) } \int \frac{\ln\sqrt{x+3}}{\sqrt{x+3}}\,dx &= \begin{Bmatrix} t^2 = x+3 \\ 2t\,dt = dx \end{Bmatrix} \\
&= \int \frac{\ln t}{\cancel{t}} \cdot 2\cancel{t}\,dt \\
&= 2\int \ln t\,dt \\
&= \begin{Bmatrix} u = \ln t & \Rightarrow & du = \dfrac{dt}{t} \\ dv = dt & \Rightarrow & v = t \end{Bmatrix} \\
&= 2t\ln t - 2\int \cancel{t} \cdot \frac{dt}{\cancel{t}} \\
&= 2t\ln t - 2t + c \\
&= 2\sqrt{x+3}\cdot\ln\sqrt{x+3} - 2\sqrt{x+3} + c
\end{aligned}
$$

g) $\int \frac{(\sqrt{x}+1)\sin\sqrt{x}}{\sqrt{x}}\,dx$

$$
\begin{aligned}
\text{g) } \int \frac{(\sqrt{x}+1)\sin\sqrt{x}}{\sqrt{x}}\,dx &= \left\{\begin{aligned} &t=\sqrt{x}\\ &dt=\frac{dx}{2\sqrt{x}} \Rightarrow 2dt=\frac{dx}{\sqrt{x}} \end{aligned}\right\}\\
&= 2\int (t+1)\sin t\,dt\\
&= \left\{\begin{aligned} &u=t+1 && \Rightarrow && du=dt\\ &dv=\sin t\,dt && \Rightarrow && v=-\cos t \end{aligned}\right\}\\
&= -2(t+1)\cos t + 2\int \cos t\,dt\\
&= -2(t+1)\cos t + 2\sin t + c\\
&= -2(\sqrt{x}+1)\cos\sqrt{x} + 2\sin\sqrt{x} + c
\end{aligned}
$$

h) $\displaystyle\int \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx$

$$
\begin{aligned}
\text{h) } \int \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx &= \left\{\begin{matrix} t^2 = e^x - 1 & \Rightarrow & e^x = t^2+1 \\ 2t\,dt = e^x\,dx & \Rightarrow & \dfrac{2t}{t^2+1}\,dt = dx \end{matrix}\right\} \\
&= \int \frac{\cancel{(t^2+1)}\cdot t}{t^2+4}\cdot\frac{2t}{\cancel{t^2+1}}\,dt \\
&= 2\int \frac{t^2}{t^2+4}\,dt \\
&= 2\int \frac{t^2+4-4}{t^2+4}\,dt \\
&= 2\int \left(1-\frac{4}{t^2+4}\right)\,dt \\
&= 2\left(t - \cancel{4}^2\cdot\frac{1}{\cancel{2}}\operatorname{arctg}\frac{t}{2}\right)+c \\
&= 2\left(\sqrt{e^x-1} - 2\operatorname{arctg}\frac{\sqrt{e^x-1}}{2}\right)+c
\end{aligned}
$$

7.5 Integriranje trigonometrijskih funkcija

Prisjetimo se nekih jednakosti koje vrijede za trigonometrijske funkcije:

i) $\sin^2 x + \cos^2 x = 1$

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iii) $\cos(2x) = \cos^2 x - \sin^2 x$

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vi) $\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)]$

Ako zbrojimo jednakosti i) i iii) dobijemo:

vii) $\cos^2 x = \frac{1+\cos(2x)}{2}$,

Ako zbrojimo jednakosti i) i iii) dobijemo:

vii) $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$,

Ako ih oduzmemo dobijemo:

viii) $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$

Zadatak (7.9)

Odredite sljedeće integrale:

a) $\int \sin^2 x \, dx$

Zadatak (7.9)

Odredite sljedeće integrale:

$$
\begin{aligned}
\text{a) } \int \sin^2 x \, dx &= \int \frac{1-\cos(2x)}{2}\, dx \\
&= \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\, dx \\
&= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c
\end{aligned}
$$

Zadatak (7.9)

Odredite sljedeće integrale:

a)
$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$$

b)
$$\int \cos^2 x\,dx$$

Zadatak (7.9)

Odredite sljedeće integrale:

a) $$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$$

b) $$\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$$

c) $\int \sin(kx)\, dx$

$$
\begin{aligned}
\text{c) } \int \sin(kx)\,dx &= \left\{ \begin{array}{l} t = kx \\ dt = k\,dx \quad \Rightarrow \quad \dfrac{dt}{k} = dx \end{array} \right\} \\
&= \frac{1}{k} \int \sin t\,dt \\
&= -\frac{1}{k} \cos t + c \\
&= -\frac{1}{k} \cos(kx) + c
\end{aligned}
$$

c) $$\int \sin(kx)\,dx = \left\{\begin{array}{l} t = kx \\ dt = k\,dx \quad \Rightarrow \quad \dfrac{dt}{k} = dx \end{array}\right\}$$

$$= \frac{1}{k}\int \sin t\,dt$$

$$= -\frac{1}{k}\cos t + c$$

$$= -\frac{1}{k}\cos(kx) + c$$

Napomena: Slično se dobije i $\displaystyle\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c$

d) $\int \sin(3x)\cos(2x)\,dx$

d)
$$\begin{aligned}\int \sin(3x)\cos(2x)\,dx &= \frac{1}{2}\int [\sin(3x-2x)+\sin(3x+2x)]\ dx \\ &= \frac{1}{2}\int [\sin x+\sin(5x)]\ dx \\ &= -\frac{1}{2}\cos x-\frac{1}{10}\cos(5x)+c\end{aligned}$$

e) $\int \sin^2 x \cos^2 x \, dx$

$$\text{e)} \int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int (2\sin x \cos x)^2 \, dx$$

$$= \frac{1}{4}\int \sin^2(2x)\, dx$$

$$= \left\{ \begin{array}{l} t = 2x \\ dt = 2\, dx \quad \Rightarrow \quad \dfrac{dt}{2} = dx \end{array} \right\}$$

$$= \frac{1}{8}\int \sin^2(t)\, dt$$

$$= \frac{1}{8}\left(\frac{1}{2}t - \frac{1}{4}\sin(2t)\right) + c$$

$$= \frac{\cancel{2}x}{\cancel{16}8} - \frac{1}{32}\sin(4x) + c$$

$$= \frac{x}{8} - \frac{1}{32}\sin(4x) + c$$

f) $\int \cos^4 x \, dx$

$$
\begin{aligned}
\text{f) } \int \cos^4 x \, dx &= \int \left(\frac{1+\cos(2x)}{2} \right)^2 dx \\
&= \frac{1}{4} \int \left(1 + 2\cos(2x) + \cos^2(2x) \right) dx \\
&= \frac{1}{4}x + \frac{1}{\cancel{4}2} \cdot \cancel{2} \cdot \frac{1}{2}\sin(2x) + \frac{1}{4} \int \cos^2(2x) \, dx \\
&= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4} \int \frac{1+\cos(4x)}{2} \, dx \\
&= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + c \\
&= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c
\end{aligned}
$$

g) $\int \sin^5 x \, dx$

$$
\begin{aligned}
\text{g) } \int \sin^5 x\, dx &= \int \sin x \sin^4 x\, dx \\
&= \int \sin x (1-\cos^2 x)^2\, dx \\
&= \left\{ \begin{aligned} t &= \cos x \\ dt &= -\sin x\, dx \end{aligned} \right\} \\
&= -\int (1-t^2)^2\, dt \\
&= -\int (1-2t^2+t^4)\, dt \\
&= -t + \frac{2}{3}t^3 - \frac{1}{5}t^5 + c \\
&= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c
\end{aligned}
$$

h) $\int \sqrt{\sin^3 x} \cos^3 x \, dx$

$$
\begin{aligned}
\text{h)} \int \sqrt{\sin^3 x}\cos^3 x\, dx &= \int \sqrt{\sin^3 x}\cos^2 x \cos x\, dx \\
&= \int \sqrt{\sin^3 x}(1-\sin^2 x)\cos x\, dx \\
&= \left\{ \begin{array}{l} t = \sin x \\ dt = \cos x\, dx \end{array} \right\} \\
&= \int \sqrt{t^3}(1-t^2)\, dt \\
&= \int \left(t^{\frac{3}{2}} - t^{\frac{7}{2}} \right)\, dt \\
&= \frac{2}{5} t^{\frac{5}{2}} - \frac{2}{9} t^{\frac{9}{2}} + c \\
&= \frac{2}{5}\sqrt{\sin^5 x} - \frac{2}{9}\sqrt{\sin^9 x} + c
\end{aligned}
$$

i) $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

$$
\text{i)} \int \frac{\sin(\ln x)}{x}\,dx = \begin{Bmatrix} t = \ln x \\ dt = \dfrac{dx}{x} \end{Bmatrix} = \int \sin t\,dt = -\cos t + c = -\cos(\ln x) + c
$$

Napomena: Integrale oblika $\int \sqrt{a^2 - x^2}\, dx$ rješavamo supstitucijom $x = a \cdot \sin t$, a integrale oblika $\int \sqrt{a^2 + x^2}\, dx$ supstitucijom $x = a \cdot \text{sh}\, t$.

j) $\int \sqrt{a^2 - x^2}\, dx$

$$\text{j)} \int \sqrt{a^2 - x^2}\, dx = \begin{Bmatrix} x = a \cdot \sin t \\ dx = a \cdot \cos t\, dt \end{Bmatrix}$$

$$= a^2 \int \sqrt{1 - \sin^2 t} \cdot \cos t\, dt$$

$$= a^2 \int \sqrt{\cos^2 t} \cdot \cos t\, dt$$

$$= a^2 \int \cos^2 t\, dt$$

$$= a^2 \int \left(\frac{1 + \cos(2t)}{2} \right)\, dt$$

$$= \frac{a^2}{2} \int (1 + \cos(2t))\, dt$$

$$= \frac{a^2}{2} \left(t - \frac{1}{2} \sin(2t) \right) + c$$

k) $\int \sqrt{a^2 + x^2}\, dx$

$$
\begin{aligned}
\text{k) } \int \sqrt{a^2 + x^2}\, dx &= \begin{Bmatrix} x = a \cdot \operatorname{sh} t \\ dx = a \cdot \operatorname{ch} t\, dt \end{Bmatrix} \\
&= a^2 \int \sqrt{1 + \operatorname{sh}^2 t} \cdot \operatorname{ch} t\, dt \\
&= a^2 \int \sqrt{\operatorname{ch}^2 t} \cdot \operatorname{ch} t\, dt \\
&= a^2 \int \operatorname{ch}^2 t\, dt \\
&= a^2 \int \left(\frac{1 + \operatorname{ch}(2t)}{2} \right) dt \\
&= \frac{a^2}{2} \int \left(1 + \operatorname{ch}(2t)\right) dt \\
&= \frac{a^2}{2} \left(t + \frac{1}{2} \operatorname{sh} t \right) + c
\end{aligned}
$$

Promotrimo supstituciju $t = \operatorname{tg}\dfrac{x}{2}$. Tada je:

$$\begin{aligned}
dt &= \frac{1}{\cos^2 \frac{x}{2}} \cdot \frac{1}{2}\, dx \\
&= \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}\, dx \qquad : \frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \\
&= \frac{\operatorname{tg}^2 \frac{x}{2} + 1}{2}\, dx \\
&= \frac{t^2 + 1}{2}\, dx \\
dx &= \frac{2\, dt}{t^2 + 1}
\end{aligned}$$

$$
\begin{aligned}
\sin x &= \sin\left(2\cdot\frac{x}{2}\right) \\
&= 2\cdot\sin\frac{x}{2}\cos\frac{x}{2} \\
&= 2\cdot\frac{\sin\frac{x}{2}\cos\frac{x}{2}}{1} \\
&= 2\cdot\frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \quad : \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} \\
&= 2\cdot\frac{\mathrm{tg}\frac{x}{2}}{\mathrm{tg}^2\frac{x}{2}+1} \\
&= \frac{2t}{t^2+1}
\end{aligned}
$$

$$
\begin{aligned}
\cos x &= \cos\left(2\cdot\frac{x}{2}\right)\\
&= \cos^2\frac{x}{2}-\sin^2\frac{x}{2}\\
&= \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1}\\
&= \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \quad : \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}\\
&= \frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}\\
&= \frac{1-t^2}{t^2+1}
\end{aligned}
$$

Dakle, kod supstitucije $t = \mathrm{tg}\frac{x}{2}$ vrijedi:

$$dx = \frac{2\,dt}{t^2+1} \quad \sin x = \frac{2t}{t^2+1} \quad \cos x = \frac{1-t^2}{t^2+1}$$

Zadatak (7.10)

Odredite sljedeće integrale:

a)

$$\int \frac{\cos x}{1 + \cos x}\, dx$$

Zadatak (7.10)

Odredite sljedeće integrale:

a)

$$\int \frac{\cos x}{1+\cos x}\,dx = \left\{\begin{matrix} t = \operatorname{tg}\frac{x}{2} \\ dx = \frac{2\,dt}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{matrix}\right\}$$

$$= \int \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$$

$$
\begin{aligned}
&= \int \frac{\frac{1-t^2}{\cancel{1+t^2}}}{\frac{\cancel{2}}{\cancel{1+t^2}}} \cdot \frac{\cancel{2}\,dt}{1+t^2} \\
&= \int \frac{1-t^2}{1+t^2}\,dt \\
&= \int \frac{2-1-t^2}{1+t^2}\,dt \\
&= \int \left(-1+\frac{2}{1+t^2}\right)\,dt \\
&= -t + 2\operatorname{arctg} t + c \\
&= -\operatorname{tg}\frac{x}{2} + 2\operatorname{arctg}\left(\operatorname{tg}\frac{x}{2}\right) + c \\
&= -\operatorname{tg}\frac{x}{2} + x + c
\end{aligned}
$$

b)

$$\int \frac{dx}{8 - 4\sin x + 7\cos x}$$

b)

$$\begin{aligned}\int \frac{dx}{8-4\sin x+7\cos x} &= \left\{t=\operatorname{tg}\frac{x}{2}\right\}\\ &= \int \frac{\frac{2\,dt}{1+t^2}}{8-\frac{8t}{1+t^2}+\frac{7-7t^2}{1+t^2}}\cdot\\ &= \int \frac{\frac{2\,dt}{\cancel{1+t^2}}}{\frac{8+8t^2-8t+7-7t^2}{\cancel{1+t^2}}}\\ &= 2\int \frac{dt}{t^2-8t+15}\\ &= 2\int \frac{dt}{(t-3)(t-5)}=(*)\end{aligned}$$

$$
\begin{aligned}
\frac{1}{(t-3)(t-5)} &= \frac{A}{t-3} + \frac{B}{t-5} \quad \Big/ \cdot (t-3)(t-5) \\
1 &= A(t-5) + B(t-3) \\
1 &= t(A+B) - 5A - 3B \\
\\
0 &= A + B \\
1 &= -5A - 3B \\
\\
A &= -\frac{1}{2} \\
B &= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
(*) \quad &= -\int \frac{dt}{t-3} + \int \frac{dt}{t-5} \\
&= -\ln|t-3| + \ln|t-5| + c \\
&= \ln\left|\frac{t-5}{t-3}\right| + c \\
&= \ln\left|\frac{\operatorname{tg}\frac{x}{2} - 5}{\operatorname{tg}\frac{x}{2} - 3}\right| + c
\end{aligned}
$$