Poglavlje 8
Određeni integral

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Teorem (8.1)

Neka je $f : [a, b] \to \mathbb{R}$ neprekidna funkcija na segmentu $[a, b]$. Tada vrijedi **Newton-Leibnizova formula**:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

gdje je F bilo koja primitivna funkcija od f.

Svojstva određenog integrala:

- $$\int_a^a f(x)\,dx = 0$$
- $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$
- $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx, \forall b \in \langle a, c\rangle$$
- $$\int_a^b \left(f(x) \pm g(x)\right)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

- $\displaystyle\int\limits_a^b (c \cdot f(x))\ dx = c \cdot \int\limits_a^b f(x)\,dx$
- $\displaystyle\int\limits_{-a}^a f(x)\,dx = 0$, ako je f neparna na $[-a, a]$
- $\displaystyle\int\limits_{-a}^a f(x)\,dx = 2\int\limits_0^a f(x)\,dx$, ako je f parna na $[-a, a]$

8.1. Metoda supstitucije u određenom integralu

Zadatak (8.1)

Odredite sljedeće integrale:

a) $$\int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{1 + \sin^2 x} \quad =$$

Zadatak (8.1)

Odredite sljedeće integrale:

a) $$\int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{1+\sin^2 x} = \left\{ \begin{matrix} t = \sin x & x = 0 & \mapsto & t = 0 \\ dt = \cos x \, dx & x = \frac{\pi}{2} & \mapsto & t = 1 \end{matrix} \right\}$$

$$= \int_0^1 \frac{dt}{1+t^2}$$

$$= \operatorname{arctg} t \Big|_0^1$$

$$= \operatorname{arctg} 1 - \operatorname{arctg} 0$$

$$= \frac{\pi}{4}$$

b) $$\int_{3}^{29} \frac{\sqrt[3]{(x-2)^2}}{\sqrt[3]{(x-2)^2}+3}\,dx \quad =$$

$$\text{b)} \int_3^{29} \frac{\sqrt[3]{(x-2)^2}}{\sqrt[3]{(x-2)^2}+3}\,dx = \left\{ \begin{array}{lcl} t^3 = x-2 & \Longrightarrow & t = \sqrt[3]{x-2} \\ x = 3 & \mapsto & t = 1 \\ x = 29 & \mapsto & t = 3 \\ 3t^2\,dt = dx & & \end{array} \right\}$$

$$= \int_1^3 \frac{t^2}{t^2+3} \cdot 3t^2\,dt$$

$$= 3\int_1^3 \frac{t^4}{t^2+3}\,dt = (*)$$

$$
\begin{array}{l}
\quad t^4 \qquad\qquad\qquad : (t^2+3) = t^2 - 3 \\
-(t^4 + 3t^2) \\
\hline
\qquad -3t^2 \\
\quad -\left(-3t^2 - 9\right) \\
\hline
\qquad\qquad\quad 9
\end{array}
$$

$$
\begin{aligned}
(*) \quad &= 3\int_{1}^{3}\left(t^2-3+\frac{9}{t^2+3}\right)\,dt \\
&= 3\left(\frac{1}{3}t^3-3t+\frac{9}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}\right)\Bigg|_1^3 \\
&= 3\left(\frac{27}{3}-9+\frac{9}{\sqrt{3}}\operatorname{arctg}\frac{3}{\sqrt{3}}\right)-3\left(\frac{1}{3}-3+\frac{9}{\sqrt{3}}\operatorname{arctg}\frac{1}{\sqrt{3}}\right) \\
&= 3\left(3\sqrt{3}\cdot\frac{\pi}{3}\right)-3\left(\frac{1}{3}-3+3\sqrt{3}\cdot\frac{\pi}{6}\right) \\
&= 3\sqrt{3}\pi+8-\frac{3\sqrt{3}\pi}{2} \\
&= 8+\frac{3\sqrt{3}\pi}{2}
\end{aligned}
$$

c) $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\, dx \quad =$

c) $$\int_0^{\ln 2} \sqrt{e^x - 1}\,dx = \left\{ \begin{array}{lcl} t^2 = e^x - 1 & \Rightarrow & e^x = t^2 + 1 \\ x = 0 & \mapsto & t = 0 \\ x = \ln 2 & \mapsto & t = 1 \\ 2t\,dt = e^x\,dx & \Rightarrow & \dfrac{2t\,dt}{t^2+1} = dx \end{array} \right\}$$

$$= \int_0^1 t \cdot \frac{2t\,dt}{t^2+1}$$

$$= 2\int_0^1 \frac{t^2\,dt}{t^2+1}$$

$$= 2\int_0^1 \frac{t^2+1-1}{t^2+1}\,dt$$

$$
\begin{aligned}
&= 2\int_0^1 \left(1 - \frac{1}{t^2+1}\right)\, dt \\
&= 2\left(t - \operatorname{arctg} t\right)\Big|_0^1 \\
&= 2\left(1 - \operatorname{arctg} 1\right) - 2\left(0 - \operatorname{arctg} 0\right) \\
&= 2 - \frac{\pi}{2}
\end{aligned}
$$

d) $\displaystyle\int_1^{e^4} \frac{dx}{x\sqrt{9-2\ln x}} \quad =$

$$\text{d) } \int_1^{e^4} \frac{dx}{x\sqrt{9-2\ln x}} = \left\{ \begin{array}{lcl} t^2 = 9 - 2\ln x & \Rightarrow & t = \sqrt{9-2\ln x} \\ x = 1 & \mapsto & t = 3 \\ x = e^4 & \mapsto & t = 1 \\ \cancel{2}t\,dt = -\frac{\cancel{2}^1}{x}\,dx & & \end{array} \right\}$$

$$= -\int_3^1 \frac{\cancel{t}\,dt}{\cancel{t}}$$

$$= \int_1^3 dt$$

$$= t\Big|_1^3$$

$$= 2$$

8.2 Metoda parcijalne integracije u određenom integralu

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Zadatak (8.2)

Odredite sljedeće integrale:

a) $\int_0^1 xe^x \, dx \quad =$

Zadatak (8.2)

Odredite sljedeće integrale:

$$
\text{a) } \int_0^1 xe^x\,dx = \begin{Bmatrix} u = x & \Rightarrow & du = dx \\ dv = e^x\,dx & \Rightarrow & v = e^x \end{Bmatrix}
$$

$$
\begin{aligned}
&= xe^x\Big|_0^1 - \int_0^1 e^x\,dx \\
&= (1 \cdot e^1 - 0 \cdot e^0) - e^x\Big|_0^1 \\
&= e - (e^1 - e^0) \\
&= 1
\end{aligned}
$$

b) $\int\limits_0^1 x^2 e^{2x}\,dx \quad =$

$$
\begin{aligned}
\text{b)} \int_0^1 x^2 e^{2x}\,dx &= \left\{\begin{matrix} u = x^2 & \Rightarrow & du = 2x\,dx \\ dv = e^{2x}\,dx & \Rightarrow & v = \frac{1}{2}e^{2x} \end{matrix}\right\} \\
&= \frac{1}{2}x^2 e^{2x}\Big|_0^1 - \int_0^1 \frac{1}{\cancel{2}} \cdot \cancel{2}x e^{2x}\,dx \\
&= \left(\frac{1}{2} \cdot 1^2 \cdot e^2 - 0^2 \cdot e^0\right) - \int_0^1 x e^{2x}\,dx \\
&= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = e^{2x}\,dx & \Rightarrow & v = \frac{1}{2}e^{2x} \end{matrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\cdot e^2 - \left(\frac{1}{2}xe^{2x}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx\right)\\
&= \cancel{\frac{1}{2}\cdot e^2} - \cancel{\frac{1}{2}\cdot e^2} + \frac{1}{4}\left(e^2 - e^0\right)\\
&= \frac{1}{4}\left(e^2 - 1\right)
\end{aligned}
$$

c) $\displaystyle\int_0^1 x^3 e^{2x}\,dx \quad =$

$$\begin{aligned}
\text{c) } \int_0^1 x^3 e^{2x}\, dx &= \left\{ \begin{array}{lcl} u = x^3 & \Rightarrow & du = 3x^2\, dx \\ dv = e^{2x}\, dx & \Rightarrow & v = \dfrac{1}{2} e^{2x} \end{array} \right\} \\
&= \frac{1}{2} x^3 e^{2x} \Big|_0^1 - \frac{3}{2} \int_0^1 x^2 e^{2x}\, dx \\
&= \left(\frac{1}{2} \cdot 1^3 \cdot e^2 - 0^3 \cdot e^0 \right) - \frac{3}{2} \cdot \frac{1}{4} (e^2 - 1) \\
&= \frac{1}{8} \left(e^2 + 3 \right)
\end{aligned}$$

d) $\int\limits_1^e \ln x \, dx \quad =$

$$
\begin{aligned}
\text{d) } \int_1^e \ln x \, dx &= \left\{ \begin{matrix} u = \ln x & \Rightarrow & du = \dfrac{dx}{x} \\ dv = dx & \Rightarrow & v = x \end{matrix} \right\} \\
&= x \ln x \Big|_1^e - \int_1^e \cancel{x} \cdot \frac{dx}{\cancel{x}} \\
&= (e \cdot \ln e - 1 \cdot \ln 1) - x \Big|_1^e \\
&= e - (e - 1) \\
&= 1
\end{aligned}
$$

e) $\displaystyle\int_0^1 \ln(1+x^2)\,dx \quad =$

$$
\begin{aligned}
\text{e) } \int_0^1 \ln(1+x^2)\,dx &= \left\{\begin{matrix} u = \ln(1+x^2) & \Rightarrow & du = \dfrac{2x\,dx}{1+x^2} \\ dv = dx & \Rightarrow & v = x \end{matrix}\right\} \\
&= x\ln(1+x^2)\Big|_0^1 - \int_0^1 x \cdot \frac{2x\,dx}{1+x^2} \\
&= \ln 2 - 2\int_0^1 \frac{x^2}{1+x^2}\,dx \\
&= \ln 2 - 2\int_0^1 \frac{1+x^2-1}{1+x^2}\,dx \\
&= \ln 2 - 2\int_0^1 \left(1 - \frac{1}{1+x^2}\right)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \ln 2 - 2\left(x - \operatorname{arctg} x\right)\Big|_0^1 \\
&= \ln 2 - 2\left(1 - \frac{\pi}{4}\right) \\
&= \ln 2 - 2 + \frac{\pi}{2}
\end{aligned}
$$

f) $\displaystyle\int_{0}^{\frac{\pi}{2}} x \cos x \, dx \quad =$

f) $$\int_0^{\frac{\pi}{2}} x\cos x\,dx = \left\{\begin{array}{lcl} u = x & \Rightarrow & du = dx \\ dv = \cos x\,dx & \Rightarrow & v = \sin x \end{array}\right\}$$

$$= x\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{\pi}{2}\cdot\sin\frac{\pi}{2} + \cos x\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 1$$

g) $\int\limits_0^{\pi} e^x \sin x \, dx \quad =$

$$
\begin{aligned}
\text{g)} \int_0^{\pi} e^x \sin x \, dx &= \left\{\begin{matrix} u = \sin x & \Rightarrow & du = \cos x \, dx \\ dv = e^x \, dx & \Rightarrow & v = e^x \end{matrix}\right\} \\
&= \underbrace{e^x \sin x \Big|_0^{\pi}}_{0} - \int_0^{\pi} e^x \cos x \, dx \\
&= \left\{\begin{matrix} u = \cos x & \Rightarrow & du = -\sin x \, dx \\ dv = e^x \, dx & \Rightarrow & v = e^x \end{matrix}\right\} \\
&= -\left(e^x \cos x \Big|_0^{\pi} - \int_0^{\pi} e^x (-\sin x) \, dx \right) \\
&= -e^{\pi} \underbrace{\cos \pi}_{-1} + \underbrace{e^0 \cos 0}_{1} - \int_0^{\pi} e^x \sin x \, dx
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\pi} e^x \sin x \, dx &= 1 + e^{\pi} - \int_0^{\pi} e^x \sin x \, dx \\
2\int_0^{\pi} e^x \sin x \, dx &= 1 + e^{\pi} \\
\int_0^{\pi} e^x \sin x \, dx &= \frac{1 + e^{\pi}}{2}
\end{aligned}
$$

h) $\int_0^1 xe^{-x}\,dx \quad =$

$$
\begin{aligned}
\text{h) } \int_0^1 xe^{-x}\,dx &= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = e^{-x}\,dx & \Rightarrow & v = -e^{x} \end{matrix}\right\} \\
&= -xe^{-x}\Big|_0^1 - \int_0^1 -e^{-x}\,dx \\
&= -e^{-1} + \int_0^1 e^{-x}\,dx \\
&= -e^{-1} + (-e^{-x})\Big|_0^1 \\
&= -e^{-1} - e^{-1} + e^{0} \\
&= 1 - \frac{2}{e}
\end{aligned}
$$

8.3 Nepravi integral

1. tip: Integrand f nije ograničen na području integracije. Neka je $f : [a, b\rangle \to \mathbb{R}$ neprekidna i $\lim\limits_{x \to b^-} f(x) = +\infty$. Ako postoji $L = \lim\limits_{c \to b^-} \int\limits_a^c f(x)\,dx$ kažemo da nepravi integral **konvergira** i pišemo:

$$\int\limits_a^b f(x)\,dx = \lim_{c \to b^-} \int\limits_a^c f(x)\,dx$$

Inače, kažemo da divergira. Slično postupamo i u slučajevima kada $f(x) \to -\infty$ i za slučajeve kada f nije ograničena u a.

Zadatak (8.3)

Odredite sljedeće integrale:

a) $$\int_0^1 \frac{dx}{\sqrt{1-x^2}} =$$

Zadatak (8.3)

Odredite sljedeće integrale:

$$\text{a)}\ \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$
$$= \lim_{b\to 1^-} \left(\arcsin x \Big|_0^b \right)$$
$$= \lim_{b\to 1^-} (\arcsin b - \arcsin 0)$$
$$= \frac{\pi}{2}$$

Nepravi integral jer $\dfrac{1}{\sqrt{1-x^2}} \to +\infty$ kad $x \to 1^-$.

b) $\int_0^1 \frac{dx}{\sqrt[3]{x}} =$

$$
\text{b)} \int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt[3]{x}}
$$

$$
= \lim_{a\to 0^+} \left(\frac{3}{2} \cdot x^{\frac{2}{3}} \Big|_a^1 \right)
$$

$$
= \lim_{a\to 0^+} \left(\frac{3}{2} \cdot 1^{\frac{2}{3}} - \frac{3}{2} \cdot a^{\frac{2}{3}} \right)
$$

$$
= \frac{3}{2}
$$

c) $\displaystyle\int_0^1 \frac{dx}{x} \quad =$

c) $$\int_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x}$$
$$= \lim_{a\to 0^+} \left(\ln|x| \Big|_a^1 \right)$$
$$= \lim_{a\to 0^+} (\ln 1 - \ln a)$$
$$= 0 - (-\infty)$$
$$= +\infty \Longrightarrow \text{integral divergira}$$

Napomena: Ako se točka u kojoj funkcija nije definirana ne nalazi na rubu nego unutar segmenta integracije, onda se interval rastavi na dva intervala s tom točkom kao granicom.

d) $\displaystyle\int_{-1}^{1} \frac{dx}{x^2} \quad =$

$$
\begin{aligned}
\text{d) } \int_{-1}^{1} \frac{dx}{x^2} &= \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2} \\
&= \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^2} + \lim_{a\to 0^+} \int_{a}^{1} \frac{dx}{x^2} \\
&= \lim_{b\to 0^-} \left(-\frac{1}{x}\Big|_{-1}^{b} \right) + \lim_{a\to 0^+} \left(-\frac{1}{x}\Big|_{a}^{1} \right) \\
&= \underbrace{\lim_{b\to 0^-} \frac{-1}{b}}_{+\infty} -1 + \underbrace{\lim_{a\to 0^+} \frac{1}{a}}_{+\infty} -1 \\
&= +\infty \Longrightarrow \text{integral divergira}
\end{aligned}
$$

2. tip: Područje integracije je neograničeno. Neka je $f : [a, +\infty\rangle \to \mathbb{R}$ neprekidna. Ako postoji $L = \lim\limits_{b \to +\infty} \int\limits_a^b f(x)\, dx$ kažemo da nepravi integral **konvergira** i pišemo:

$$\int\limits_a^{+\infty} f(x)\, dx = \lim_{b \to +\infty} \int\limits_a^b f(x)\, dx$$

Inače, kažemo da divergira. Slično postupamo i u slučajevima kada $f : \langle -\infty, b] \to \mathbb{R}$ i $f : \langle -\infty, +\infty\rangle \to \mathbb{R}$.

Zadatak (8.4)

Odredite sljedeće integrale:

a) $\int_{0}^{+\infty} e^{-x}\, dx \quad =$

Zadatak (8.4)

Odredite sljedeće integrale:

$$\text{a)} \int_0^{+\infty} e^{-x}\,dx = \lim_{b\to+\infty} \int_0^{b} e^{-x}\,dx = \lim_{b\to+\infty} \left(-e^{-x} \Big|_0^b \right) = \lim_{b\to+\infty} \left(-e^{-b} + e^{0} \right) = 1$$

b) $\displaystyle\int_{1}^{+\infty} \frac{dx}{x} \quad =$

b) $$\int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to+\infty} \left(\ln x \Big|_1^b \right) = \lim_{b\to+\infty} \ln b = +\infty$$

c) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} \quad =$

$$\text{c)} \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to+\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \left(\operatorname{arctg} x \Big|_{a}^{0} \right) + \lim_{b\to+\infty} \left(\operatorname{arctg} x \Big|_{0}^{b} \right)$$

$$= \lim_{a\to-\infty} (\underbrace{\operatorname{arctg} 0}_{0} - \underbrace{\operatorname{arctg} a}_{-\frac{\pi}{2}}) + \lim_{b\to+\infty} (\underbrace{\operatorname{arctg} b}_{\frac{\pi}{2}} - \underbrace{\operatorname{arctg} 0}_{0})$$

$$= \pi$$