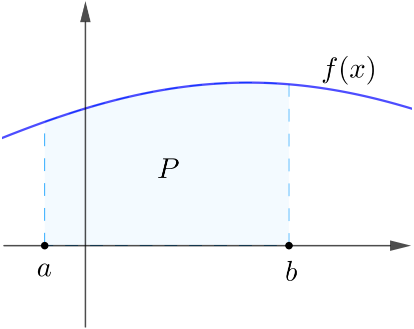
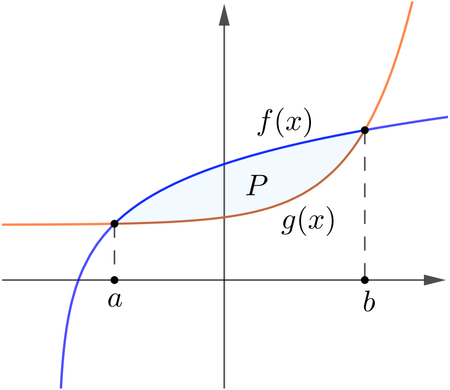
Poglavlje 9
Primjena integrala

ak. god. 2021./2022.

9.1 Površine ravninskih likova



$$P = \int_a^b f(x)\,dx$$



$$P = \int_a^b \left(f(x) - g(x)\right)\,dx$$

Zadatak (9.1.)

Odredite površinu lika omeđenog krivuljom:

a) $y = \sqrt{1 - x^2}$ i osi x.

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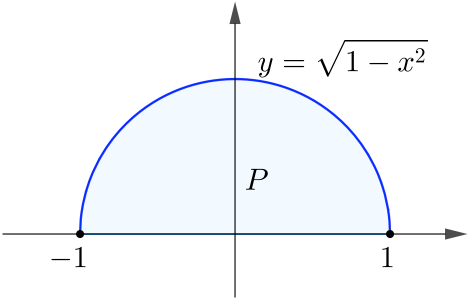

Odredimo sjecišta krivulja:

$$\begin{aligned} \sqrt{1-x^2} &= 0 \\ 1-x^2 &= 0 \\ x^2 &= 1 \end{aligned}$$

$x_1 = -1$ i $x_2 = 1$

$$
\begin{aligned}
P &= \int_{-1}^{1} \sqrt{1-x^2}\,dx \\
&= \left\{ \begin{matrix} x = \sin t & & x = -1 & \mapsto & t = -\dfrac{\pi}{2} \\ dx = \cos t\,dt & & x = 1 & \mapsto & t = \dfrac{\pi}{2} \end{matrix} \right\} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - sin^2 t}\ \cos t\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt \\
&= \left(\frac{1}{2} t - \frac{1}{4} \sin(2t) \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \left(\frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{4} \sin \pi \right) - \left(\frac{1}{2} \cdot \frac{-\pi}{2} - \frac{1}{4} \sin(-\pi) \right) \\
&= \frac{\pi}{4} - 0 + \frac{\pi}{4} + 0 \\
&= \frac{\pi}{2}
\end{aligned}
$$

b) $y = x(x-1)(x-2)$ i osi x.

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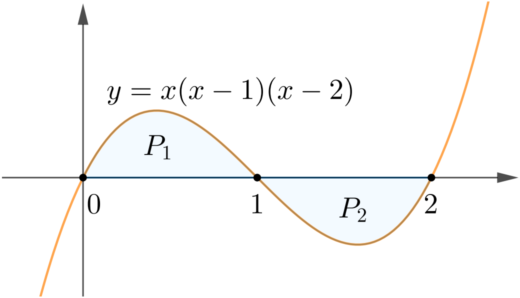

$$
\begin{aligned}
y &= x(x-1)(x-2) \\
&= (x^2-x)(x-2) \\
&= x^3-2x^2-x^2+2x \\
&= x^3-3x^2+2x
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= \int\limits_0^1 \left(x^3 - 3x^2 + 2x\right)\, dx & P_2 &= -\int\limits_1^2 \left(x^3 - 3x^2 + 2x\right)\, dx \\
&= \left(\frac{1}{4}x^4 - x^3 + x^2\right)\Big|_0^1 & &= -\left(\frac{1}{4}x^4 - x^3 + x^2\right)\Big|_1^2 \\
&= \frac{1}{4} & &= -(4 - 8 + 4) + \left(\frac{1}{4} - 1 + 1\right) \\
& & &= \frac{1}{4}
\end{aligned}
$$

$$
P = P_1 + P_2 = \frac{1}{2}
$$

Zadatak (9.2.)

Odredite površinu lika omeđenog krivuljama:

a) $y = \ln x$ i $y = \ln^2 x$

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Odredite površinu lika omeđenog krivuljama:

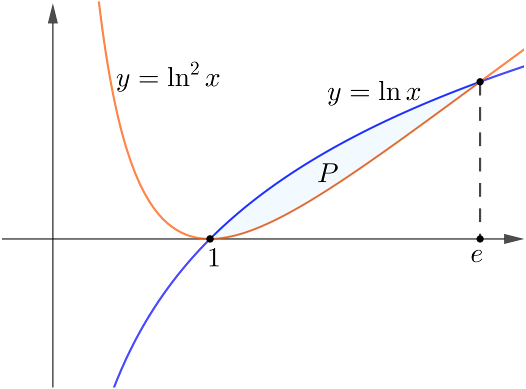
a) $y = \ln x$ i $y = \ln^2 x$

Odredimo sjecišta:

$$
\begin{aligned}
\ln^2 x &= \ln x \\
\ln^2 x - \ln x &= 0 \\
\ln x(\ln x - 1) &= 0
\end{aligned}
$$

$$\ln x = 0 \text{ i } \ln x = 1$$

$$x_1 = 1 \text{ i } x_2 = e$$



$$
\begin{aligned}
P &= \int_1^e \left(\ln x - \ln^2 x\right)\, dx \\
&= \underbrace{\int_1^e \ln x \, dx}_{\Delta} - \underbrace{\int_1^e \ln^2 x \, dx}_{\square}
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \int_1^e \ln x \, dx \\
&= \left\{ \begin{array}{lcl} u = \ln x & \Rightarrow & du = \dfrac{dx}{x} \\ dv = dx & \Rightarrow & v = x \end{array} \right\} \\
&= x \ln x \Big|_1^e - \int_1^e \cancel{x} \cdot \frac{dx}{\cancel{x}} \\
&= e - (e - 1) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\square &= \int_1^e \ln^2 x \, dx \\
&= \left\{ \begin{array}{lll} u = \ln^2 x & \Rightarrow & du = \dfrac{2 \ln x \, dx}{x} \\ dv = dx & \Rightarrow & v = x \end{array} \right\} \\
&= x \ln^2 x \Big|_1^e - \int_1^e \cancel{x} \cdot \frac{2 \ln x \, dx}{\cancel{x}} \\
&= e - 2 \underbrace{\int_1^e \ln x \, dx}_{\Delta} \\
&= e - 2
\end{aligned}
$$

$$
\begin{aligned}
P &= 1 - (e - 2) \\
P &= 3 - e
\end{aligned}
$$

b) $y = -x^2 - 2x + 1$ i $y = -\frac{1}{2}x$

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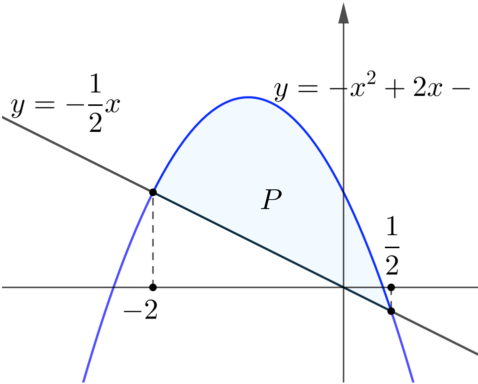
Odredimo sjecišta krivulja:

$$\begin{aligned} -x^2 - 2x + 1 &= -\frac{1}{2}x \\ -x^2 - \frac{3}{2}x + 1 &= 0 \quad / \cdot (-2) \\ 2x^2 + 3x - 2 &= 0 \end{aligned}$$

$$x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot (-2)}}{4}$$

$$x_{1,2} = \frac{-3 \pm \sqrt{25}}{4}$$

$$x_1 = -2 \text{ i } x_2 = \frac{1}{2}$$



$$
\begin{aligned}
P &= \int_{-2}^{\frac{1}{2}} \left(-x^2 - 2x + 1 + \frac{1}{2}x\right)\, dx \\
&= \int_{-2}^{\frac{1}{2}} \left(-x^2 - \frac{3}{2}x + 1\right)\, dx \\
&= \left(-\frac{1}{3}x^3 - \frac{3}{4}x^2 + x\right)\Big|_{-2}^{\frac{1}{2}} \\
&= \left(-\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{3}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\right) - \left(-\frac{1}{3}(-2)^3 - \frac{3}{4}(-2)^2 - 2\right) \\
&= \left(-\frac{1}{24} - \frac{3}{16} + \frac{1}{2}\right) - \left(\frac{8}{3} - 5\right) \\
&= \frac{-2 - 9 + 24}{48} + \frac{7}{3} \\
&= \frac{125}{48}
\end{aligned}
$$

c) $y = \dfrac{8}{x^2+4}$ i $y = \dfrac{x^2}{4}$

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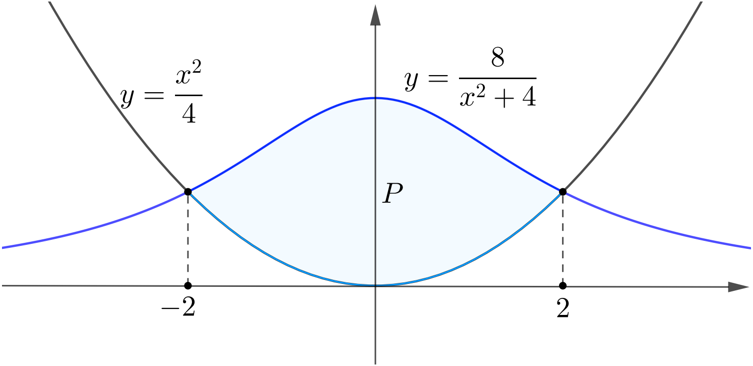
Odredimo sjecišta krivulja:

$$
\begin{aligned}
\frac{8}{x^2+4} &\xcancel{=} \frac{x^2}{4} \\
x^4 + 4x^2 - 32 &= 0 \\
x^4 + 8x^2 - 4x^2 - 32 &= 0 \\
x^2(x^2+8) - 4(x^2+8) &= 0 \\
(x^2-4)(x^2+8) &= 0 \\
(x-2)(x+2)(x^2+8) &= 0
\end{aligned}
$$

$$x_1 = -2 \text{ i } x_2 = 2$$


$y = \frac{x^2}{4}$
$y = \frac{8}{x^2+4}$
P
-2
2

$$
\begin{aligned}
P &= \int_{-2}^{2} \left(\frac{8}{x^2+4} - \frac{x^2}{4} \right) dx \\
&= \left(\cancel{8}^4 \cdot \frac{1}{\cancel{2}} \operatorname{arctg} \frac{x}{2} - \frac{1}{4} \cdot \frac{1}{3} x^3 \right) \Big|_{-2}^{2} \\
&= \left(4\operatorname{arctg} 1 - \frac{1}{12} \cdot 2^3 \right) - \left(4\operatorname{arctg}(-1) - \frac{1}{12} \cdot (-2)^3 \right) \\
&= \left(\cancel{4} \cdot \frac{\pi}{\cancel{4}} - \frac{\cancel{8}^2}{\cancel{12}^3} \right) - \left(\cancel{4} \cdot \frac{-\pi}{\cancel{4}} - \frac{\cancel{-8}^{-2}}{\cancel{12}^3} \right) \\
&= 2\pi - \frac{4}{3}
\end{aligned}
$$

d) $y = x^2 + 2x - 3$ i $y = -2x - 3$

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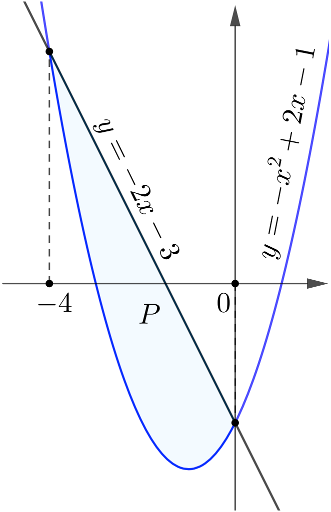
Odredimo sjecišta krivulja:

$$\begin{aligned} x^2 + 2x - 3 &= -2x - 3 \\ x^2 + 4x &= 0 \\ x(x+4) &= 0 \end{aligned}$$

$$x_1 = -4 \text{ i } x_2 = 0$$



$$
\begin{aligned}
P &= \int_{-4}^{0} \left(-2x - 3 - x^2 - 2x + 3\right)\, dx \\
&= \int_{-4}^{0} \left(-x^2 - 4x\right)\, dx \\
&= \left(-\frac{1}{3}x^3 - 2x^2\right)\Big|_{-4}^{0} \\
&= \left(-\frac{1}{3}\cdot 0^3 - 2\cdot 0^2\right) - \left(-\frac{1}{3}\cdot(-4)^3 - 2\cdot(-4)^2\right) \\
&= -\frac{64}{3} + 32 \\
&= \frac{32}{3}
\end{aligned}
$$

e) $y = e^x$, $y = e^{-x}$ i $x = 1$

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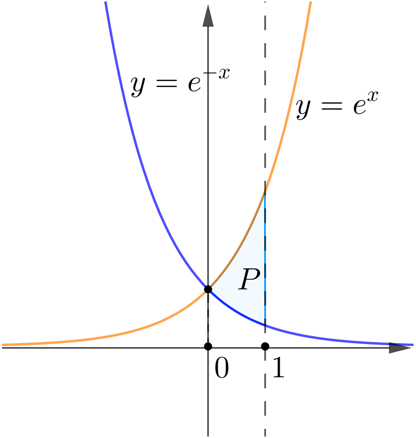
Odredimo sjecišta krivulja:

$$\begin{aligned} e^x &= e^{-x} \\ x &= 0 \end{aligned}$$



$$\begin{aligned} P &= \int_0^1 \left(e^x - e^{-x}\right)\, dx \\ &= \left(e^x + e^{-x}\right)\Big|_0^1 \\ &= \left(e^1 + e^{-1}\right) - \left(e^0 + e^0\right) \\ &= e + \frac{1}{e} - 2 \end{aligned}$$

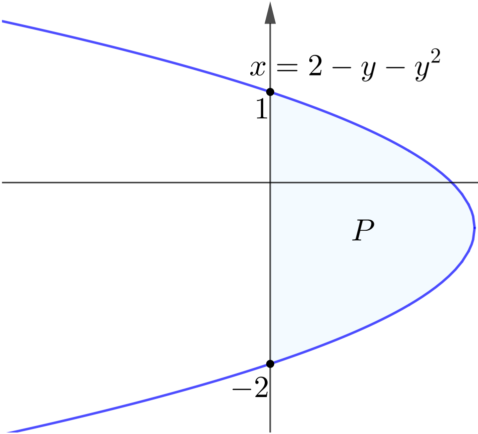
f) $x = 2 - y - y^2$ i osi y

f) $x = 2 - y - y^2$ i osi y

Odredimo sjecišta krivulje s osi y:

$$\begin{aligned} 2 - y - y^2 &= 0 \\ y^2 + y - 2 &= 0 \\ (y+2)(y-1) &= 0 \end{aligned}$$

$y_1 = -2$ i $y_2 = 1$



$$
\begin{aligned}
P &= \int_{-2}^{1} \left(2 - y - y^2\right)\, dy \\
&= \left(2y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_{-2}^{1} \\
&= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) \\
&= \frac{9}{2}
\end{aligned}
$$

Zadatak (9.3.)

U presječnim točkama pravca $y = 0$ i parabole $y = x^2 - 4$ povučene su normale na parabolu. Odredite površinu određenu normalama i parabolom.

Zadatak (9.3.)

U presječnim točkama pravca $y = 0$ i parabole $y = x^2 - 4$ povučene su normale na parabolu. Odredite površinu određenu normalama i parabolom.

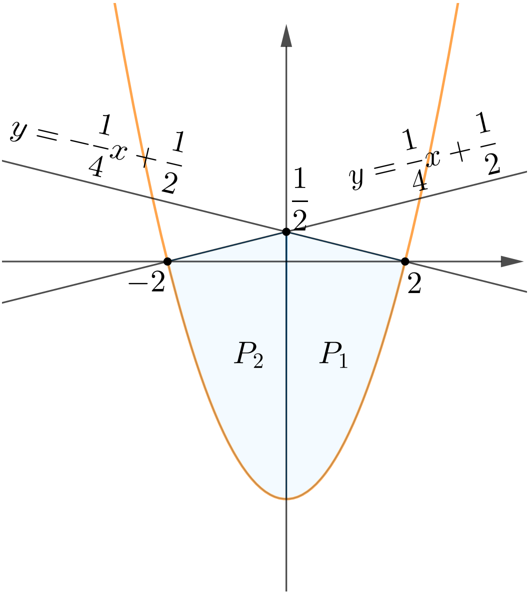
$y' = 2x$, pa je $y'(2) = 4$, a $y'(-2) = -4$.

Normala u točki $T_1(-2, 0)$ je:

$$\begin{aligned} y - 0 &= \frac{1}{4}(x + 2) \\ n_1 \ldots y &= \frac{1}{4}x + \frac{1}{2} \end{aligned}$$

Normala u točki $T_2(2, 0)$ je:

$$\begin{aligned} y - 0 &= -\frac{1}{4}(x - 2) \\ n_2 \ldots y &= -\frac{1}{4}x + \frac{1}{2} \end{aligned}$$



$y = -\frac{1}{4}x + \frac{1}{2}$
$y = \frac{1}{4}x + \frac{1}{2}$
$\frac{1}{2}$
-2
2
P_2
P_1

Kako je $P_1 = P_2$, dovoljno je računati samo P_1.

$$
\begin{aligned}
P &= 2 \cdot P_1 \\
&= 2 \int_0^2 \left(-\frac{1}{4}x + \frac{1}{2} - x^2 + 4 \right) dx \\
&= 2 \int_0^2 \left(-x^2 - \frac{1}{4}x + \frac{9}{2} \right) dx \\
&= 2 \left(-\frac{1}{3}x^3 - \frac{1}{8}x^2 + \frac{9}{2}x \right) \Big|_0^2 \\
&= 2 \left(-\frac{1}{3} \cdot 2^3 - \frac{1}{8} \cdot 2^2 + \frac{9}{2} \cdot 2 \right) - 0 \\
&= \frac{35}{3}
\end{aligned}
$$

Zadatak (9.4.)

Odredite površinu omeđenu krivuljama $k_1 ... y = -x^2 + 6x - 5$, $k_2 ... y = -x^2 + 4x - 3$ i osi x.

Zadatak (9.4.)

Odredite površinu omeđenu krivuljama $k_1 \ldots y = -x^2 + 6x - 5$, $k_2 \ldots y = -x^2 + 4x - 3$ i osi x.

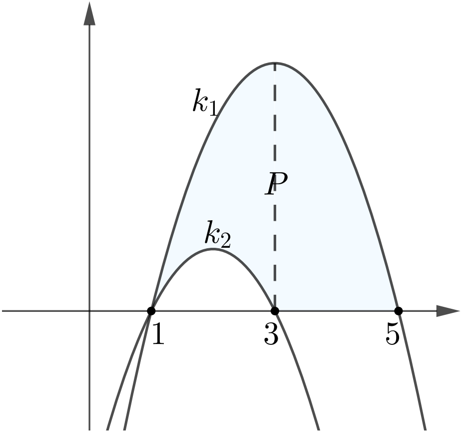
Odredimo nultočke krivulja:

$$\begin{aligned} -x^2 + 6x - 5 &= 0 \\ -x^2 + 5x + x - 5 &= 0 \\ -x(x-5) + (x-5) &= 0 \\ (-x+1)(x-5) &= 0 \end{aligned}$$

$$x_1 = 1 \text{ i } x_2 = 5$$

$$\begin{aligned} -x^2 + 4x - 3 &= 0 \\ -x^2 + 3x + x - 3 &= 0 \\ -x(x-3) + (x-3) &= 0 \\ (-x+1)(x-3) &= 0 \end{aligned}$$

$$x_1 = 1 \text{ i } x_2 = 3$$


k_1
P
k_2
1
3
5

$$
\begin{aligned}
P &= \int_1^3 \left(-x^2+6x-5-(-x^2+4x-3)\right)\,dx + \int_3^5 \left(-x^2+6x-5\right)\,dx \\
&= \int_1^3 (2x-2)\,dx + \int_3^5 \left(-x^2+6x-5\right)\,dx \\
&= \left(x^2-2x\right)\Big|_1^3 + \left(-\frac{1}{3}x^3+3x^2-5x\right)\Big|_3^5 \\
&= (9-6-1+2) + \left(-\frac{125}{3}+75-25+\frac{27}{3}-27+15\right) \\
&= 4 + \left(-\frac{98}{3}+38\right) \\
&= 4 + \frac{16}{3} \\
&= \frac{28}{3}
\end{aligned}
$$

Zadatak (9.5.)

Odredite površinu omeđenu krivuljom $k \ldots y = 2x - x^2$, tangentom na krivulju u točki $T\left(\frac{1}{2}, \frac{3}{4}\right)$ i osi x.

Zadatak (9.5.)

Odredite površinu omeđenu krivuljom $k \ldots y = 2x - x^2$, tangentom na krivulju u točki $T\left(\frac{1}{2}, \frac{3}{4}\right)$ i osi x.

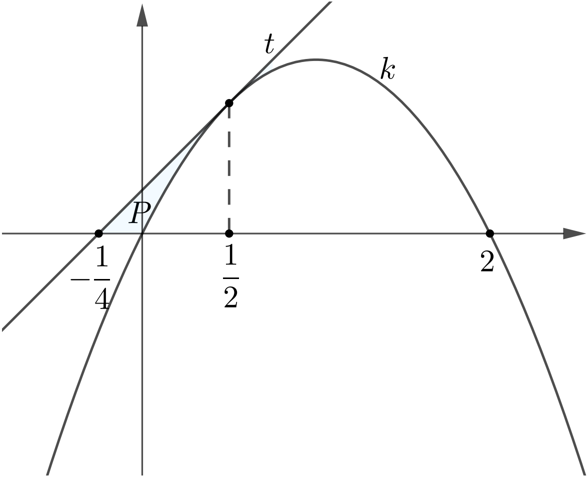
Odredimo nultočke krivulje i jednadžbu tangente:

$$\begin{aligned} 2x - x^2 &= 0 \\ x(2-x) &= 0 \end{aligned}$$

$$x_1 = 0 \text{ i } x_2 = 2$$

$$\begin{aligned} y' &= 2 - 2x \\ y'\left(\frac{1}{2}\right) &= 1 = k \\ y - \frac{3}{4} &= 1\left(x - \frac{1}{2}\right) \end{aligned}$$

$$t \ldots y = x + \frac{1}{4}$$



t
k
P
$-\frac{1}{4}$
$\frac{1}{2}$
2

$$
\begin{aligned}
P &= \int_{-\frac{1}{4}}^{0} \left(x + \frac{1}{4}\right)\, dx + \int_{0}^{\frac{1}{2}} \left(x + \frac{1}{4} - 2x + x^2\right)\, dx \\
&= \int_{-\frac{1}{4}}^{0} \left(x + \frac{1}{4}\right)\, dx + \int_{0}^{\frac{1}{2}} \left(x^2 - x + \frac{1}{4}\right)\, dx \\
&= \left(\frac{1}{2}x^2 + \frac{1}{4}x\right)\Big|_{-\frac{1}{4}}^{0} + \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right)\Big|_{0}^{\frac{1}{2}} \\
&= 0 - \left(\frac{1}{32} - \frac{1}{16}\right) + \left(\frac{1}{24} - \frac{1}{8} + \frac{1}{8}\right) - 0 \\
&= \frac{1}{32} + \frac{1}{24} \\
&= \frac{7}{96}
\end{aligned}
$$

Zadatak (9.6.)

Odredite površinu omeđenu krivuljama $k_1 \ldots y = 6x^2 - 5x - 1$, $k_2 \ldots y = \cos(\pi\, x)$, $x = 0$ i $x = \dfrac{1}{2}$.

Zadatak (9.6.)

Odredite površinu omeđenu krivuljama $k_1 ... y = 6x^2 - 5x - 1$, $k_2 ... y = \cos(\pi x)$, $x = 0$ i $x = \frac{1}{2}$.

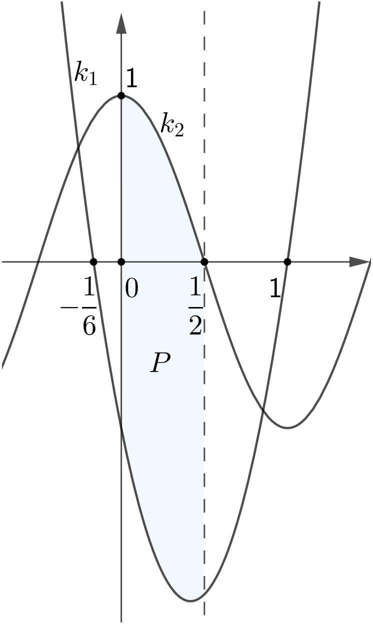
Za $x = 0$ i $x = \frac{1}{2}$ imamo:

$$\begin{aligned} \cos 0 &= 1 \\ \cos\frac{\pi}{2} &= 0 \end{aligned}$$

Odredimo nultočke parabole:

$$\begin{aligned} 6x^2 - 5x - 1 &= 0 \\ 6x^2 - 6x + x - 1 &= 0 \\ 6x(x-1) + (x-1) &= 0 \\ (6x+1)(x-1) &= 0 \end{aligned}$$

$$x_1 = -\frac{1}{6} \text{ i } x_2 = 0$$

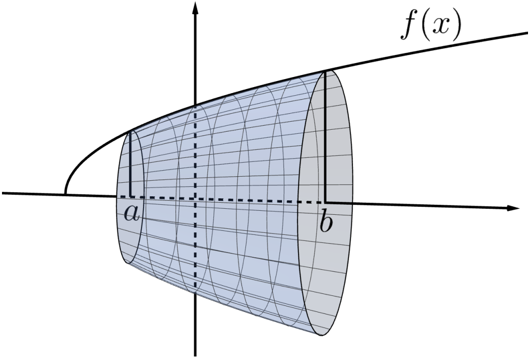


k_1
1
k_2
$-\frac{1}{6}$
0
$\frac{1}{2}$
1
P

$$
\begin{aligned}
P &= \int_0^{\frac{1}{2}} \left(\cos(\pi\, x) - 6x^2 + 5x + 1\right)\, dx \\
&= \left(\frac{1}{\pi}\sin(\pi\, x) - 2x^3 + \frac{5}{2}x^2 + x\right)\Big|_0^{\frac{1}{2}} \\
&= \left(\frac{1}{\pi}\sin\frac{\pi}{2} - \frac{1}{4} + \frac{5}{8} + \frac{1}{2}\right) - 0 \\
&= \frac{1}{\pi} + \frac{7}{8}
\end{aligned}
$$

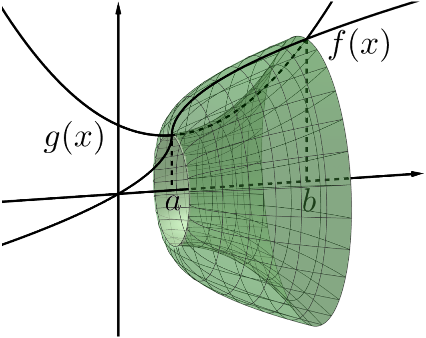
9.2 Volumen rotacijskih tijela

Volumen tijela dobivenih rotacijom oko osi x:



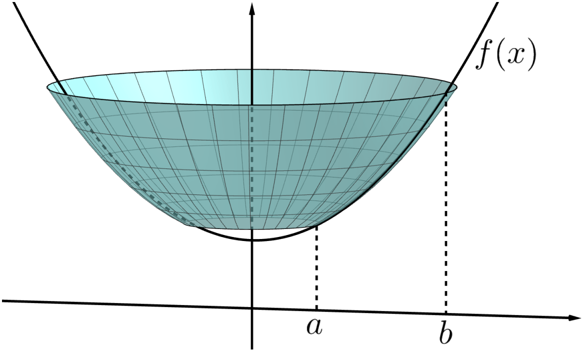
$$V_x = \pi \int_a^b f(x)^2 \, dx$$

Volumen tijela dobivenih rotacijom oko osi x:



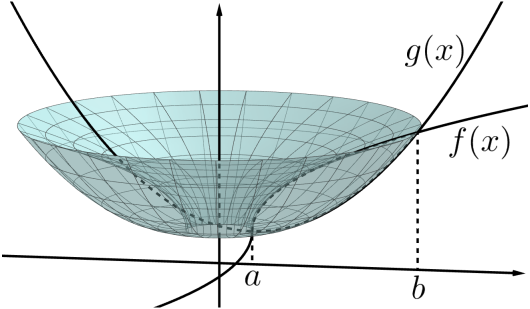
$$V_x = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) \, dx$$

Volumen tijela dobivenih rotacijom oko osi y:



$$V_y = 2\pi \int_a^b x \cdot f(x)\, dx$$

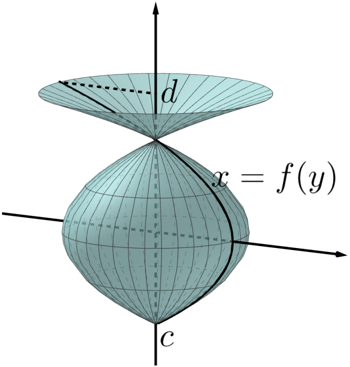
Volumen tijela dobivenih rotacijom oko osi y:



$$V_y = 2\pi \int_a^b x\,(f(x) - g(x))\ dx$$

Ako je $x = f(y)$ i tijelo je dobiveno rotacijom oko osi y na segmentu $[c, d]$, onda je

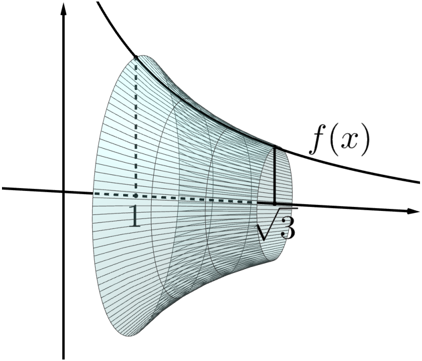
$$V = \pi \int_c^d f(y)^2 \, dy$$



Zadatak (9.7.)

Odredite volumen tijela koje nastaje rotacijom funkcije $f(x) = \dfrac{1}{x\sqrt{1+x^2}}$ oko osi x na segmentu $\left[1, \sqrt{3}\right]$

$$
\begin{aligned}
V_x &= \pi \int_1^{\sqrt{3}} f(x)^2 \, dx \\
&= \pi \int_1^{\sqrt{3}} \frac{1}{x^2(1+x^2)} \, dx = (*)
\end{aligned}
$$



$$
\begin{aligned}
\frac{1}{x^2(1+x^2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2} \quad \Big/ \cdot x^2(1+x^2) \\
1 &= A(x^3+x) + B(1+x^2) + Cx^3 + Dx^2 \\
1 &= x^3(A+C) + x^2(B+D) + Ax + B
\end{aligned}
$$

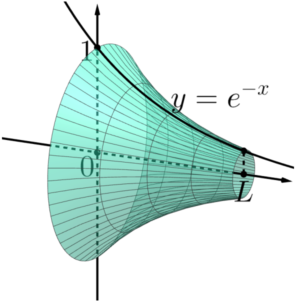
$$
\begin{aligned}
B &= 1 \\
A &= 0 \\
A+C &= 0 \quad \Longrightarrow \quad C = 0 \\
B+D &= 0 \quad \Longrightarrow \quad D = -1
\end{aligned}
$$

$$
\begin{aligned}
(*) &= \pi \int\limits_{1}^{\sqrt{3}} \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) \, dx \\
&= \pi \left(-\frac{1}{x} - \operatorname{arctg} x \right) \Big|_{1}^{\sqrt{3}} \\
&= \pi \left(-\frac{1}{\sqrt{3}} - \underbrace{\operatorname{arctg}\sqrt{3}}_{\frac{\pi}{3}} + \frac{1}{1} + \underbrace{\operatorname{arctg} 1}_{\frac{\pi}{4}} \right) \\
&= \pi \left(1 - \frac{\sqrt{3}}{3} - \frac{\pi}{12} \right)
\end{aligned}
$$

Zadatak (9.8.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y = e^{-x}$, pravcem $x = L$ i koordinatnim osima x i y, oko osi x.

$$
\begin{aligned}
V_x &= \pi \int_0^L e^{-2x}\, dx \\
&= -\frac{\pi}{2} \cdot e^{-2x} \Big|_0^L \\
&= -\frac{\pi}{2}\left(e^{-2L} - 1\right) \\
&= \frac{\pi}{2} - \frac{\pi}{2} e^{-2L}
\end{aligned}
$$



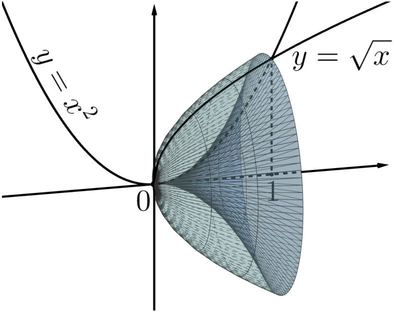
Zadatak (9.9.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2$ i $y = \sqrt{x}$, oko osi x.

Odredimo granice integriranja:

$$\begin{aligned} x^2 &= \sqrt{x} \quad \Big/^2 \\ x^4 &= x \\ x^4 - x &= 0 \\ x(x^3-1) &= 0 \\ x(x-1)(x^2+x+1) &= 0 \end{aligned}$$

$$x_1 = 0 \text{ i } x_2 = 1$$



$$
\begin{aligned}
V_x &= \pi \int_0^1 \left(x - x^4\right)\, dx \\
&= \pi \left(\frac{1}{2}x^2 - \frac{1}{5}x^5 \right) \Bigg|_0^1 \\
&= \pi \left(\frac{1}{2} - \frac{1}{5} \right) \\
&= \frac{3\pi}{10}
\end{aligned}
$$

Zadatak (9.10.)

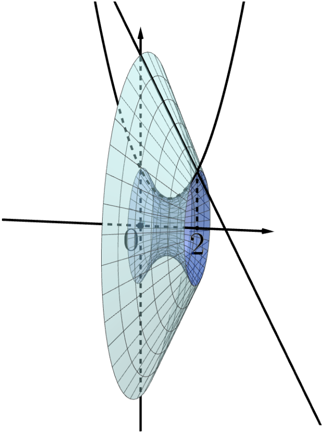
Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2 - 2x + 2$, $y = -2x + 6$ i osi y, oko osi x.

Odredimo sjecišta krivulja:

$$x^2 - 2x + 2 = -2x + 6$$

$$x^4 - 4 = 0$$

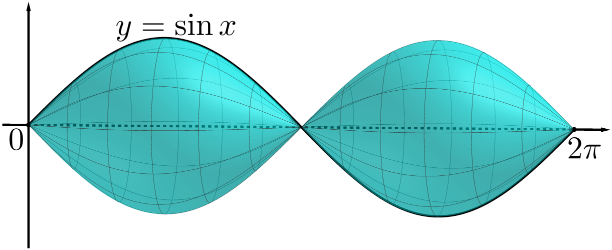
$$x_1 = -2 \text{ i } x_2 = 2$$



$$
\begin{aligned}
V_x &= \pi \int_0^2 \left[(-2x+6)^2 - \left(x^2-2x+2\right)^2\right] dx \\
&= \pi \int_0^2 \left[\cancel{4x^2} - 24x + 36 - x^4 - \cancel{4x^2} - 4 + 4x^3 - 4x^2 + 8x\right] dx \\
&= \pi \int_0^2 \left[-x^4 + 4x^3 - 4x^2 - 16x + 32\right] dx \\
&= \pi \left[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 - 8x^2 + 32x\right]\Bigg|_0^2 \\
&= \pi \left[-\frac{1}{5}\cdot 32 + 16 - \frac{4}{3}\cdot 8 - 32 + 64 - 0\right] \\
&= \frac{464\pi}{15}
\end{aligned}
$$

Zadatak (9.11.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y = \sin x$ i osi x, na segmentu $[0, 2\pi]$, oko osi x.

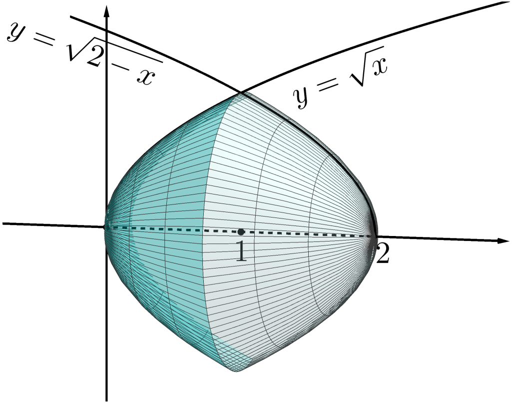


Kako imamo dva jednaka volumena, dovoljno je računati jedan.

$$
\begin{aligned}
V_x &= 2\left(\pi\int_0^{\pi}\sin^2 x\,dx\right)\\
&= 2\pi\int_0^{\pi}\frac{1-\cos(2x)}{2}\,dx\\
&= \pi\left(x-\frac{1}{2}\sin(2x)\right)\Bigg|_0^{\pi}\\
&= \pi\left(\pi-\frac{1}{2}\sin(2\pi)-0+\frac{1}{2}\sin 0\right)\\
&= \pi^2
\end{aligned}
$$

Zadatak (9.12.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = \sqrt{x}$, $y = \sqrt{2-x}$ i osi x, oko osi x.



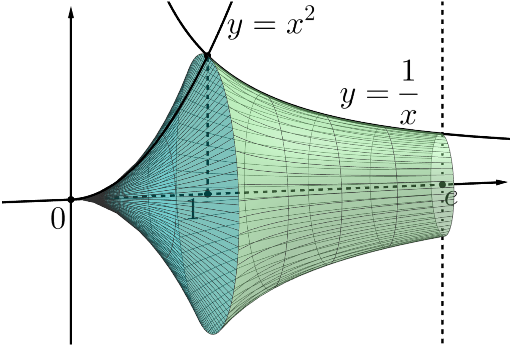
$$
\begin{aligned}
V_x &= \pi \int_0^1 \sqrt{x}^2 \, dx + \pi \int_1^2 \sqrt{2-x}^2 \, dx \\
&= \pi \int_0^1 x \, dx + \pi \int_1^2 (2-x) \, dx \\
&= \pi \cdot \frac{1}{2} x^2 \Big|_0^1 + \pi \left(2x - \frac{x^2}{2} \right) \Big|_1^2 \\
&= \frac{\pi}{2} + \pi \left(4 - 2 - 2 + \frac{1}{2} \right) \\
&= \pi
\end{aligned}
$$

Zadatak (9.13.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2$, $y = \dfrac{1}{x}$, $x = e$ i osi x, oko osi x.

Odredimo sjecišta:

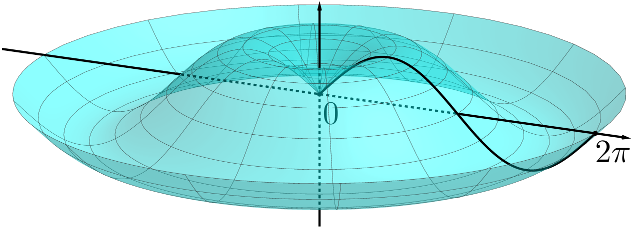
$$\begin{aligned} x^2 &= \frac{1}{x} \\ x^3 &= 1 \\ x &= 1 \end{aligned}$$



$$
\begin{aligned}
V_y &= \pi \int_0^1 x^4 \, dx + \pi \int_1^e \frac{1}{x^2} \, dx \\
&= \pi \frac{x^5}{5} \Big|_0^1 - \pi \frac{1}{x} \Big|_1^e \\
&= \frac{\pi}{5} - \frac{\pi}{e} + \pi \\
&= \pi \left(\frac{6}{5} - \frac{1}{e} \right)
\end{aligned}
$$

Zadatak (9.14.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog kruvuljom $y = \sin x$ i osi x, na segmentu $[0, 2\pi]$, oko osi y.

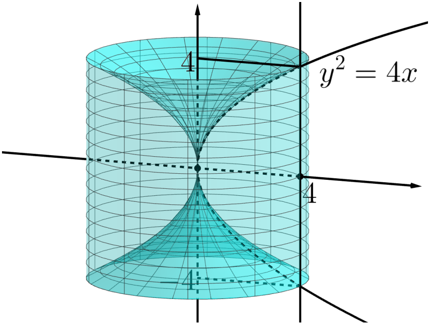


Kako imamo dva jednaka volumena, dovoljno je računati jedan.

$$
\begin{aligned}
V_y &= 2\left(2\pi\int\limits_0^{\pi} x\sin x\,dx\right)\\
&= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = \sin x\,dx & \Rightarrow & v = -\cos x \end{matrix}\right\}\\
&= 4\pi\left(-x\cos x\right)\Big|_0^{\pi} + \int\limits_0^{\pi}\cos x\,dx\\
&= 4\pi\left(-\pi\cos(\pi) - 0\right) + \sin x\Big|_0^{\pi}\\
&= 4\pi^2
\end{aligned}
$$

Zadatak (9.15.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljom $y^2 = 4x$ i $x = 4$, oko osi y.



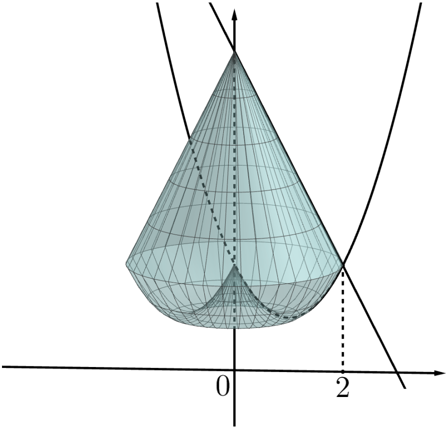
$$
\begin{aligned}
V &= \pi \int_{-4}^{4} \left[4^2 - \left(\frac{y^2}{4} \right)^2 \right] dy \\
&= \pi \int_{-4}^{4} \left[16 - \frac{y^4}{16} \right] dy \\
&= \pi \left(16y - \frac{y^5}{80} \right) \Bigg|_{-4}^{4} \\
&= \pi \left(64 - \frac{1024}{80} + 64 - \frac{1024}{80} \right) \\
&= \frac{512}{5} \pi
\end{aligned}
$$

Zadatak (9.16.)

Odredite volumen tijela koje nastaje rotacijom lika omeđenog krivuljama $y = x^2 - 2x + 2$, $y = -2x + 6$ i osi y, oko osi y.

Odredimo sjecišta krivulja:

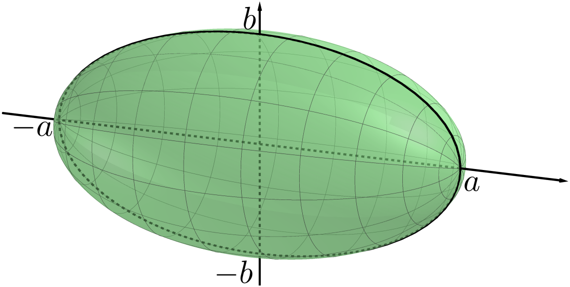
$$
\begin{aligned}
x^2 - 2x + 2 &= -2x + 6 \\
x^4 - 4 &= 0 \\
x_1 = -2 \text{ i } x_2 &= 2
\end{aligned}
$$



$$
\begin{aligned}
V_y &= 2\pi \int_0^2 x\left[-2x+6-\left(x^2-2x+2\right)^2\right]\, dx \\
&= 2\pi \int_0^2 x\left(-x^2+4\right)\, dx \\
&= 2\pi \int_0^2 \left(-x^3+4x\right)\, dx \\
&= 2\pi \left(-\frac{1}{4}x^4+2x^2\right)\Bigg|_0^2 \\
&= 2\pi\left(-4+8\right) \\
&= 8\pi
\end{aligned}
$$

Zadatak (9.17.)

Odredite volumen tijela koje nastaje rotacijom elipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ oko osi x.



$$
\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \quad \Big/ \cdot b^2 \\
\frac{b^2 x^2}{a^2} + y^2 &= b^2 \\
y^2 &= b^2 - \frac{b^2 x^2}{a^2} \\
y^2 &= b^2 \left(1 - \frac{x^2}{a^2}\right)
\end{aligned}
$$

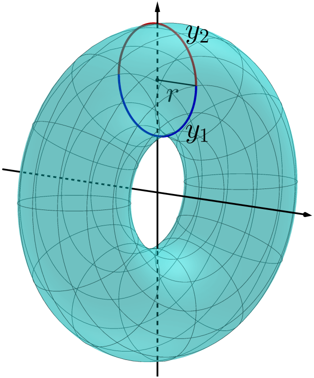
$$
\begin{aligned}
V_x &= 2\pi \int_0^a \left(b^2 \left(1 - \frac{x^2}{a^2} \right) \right) \, dx \\
&= 2b^2\pi \int_0^a \left(1 - \frac{x^2}{a^2} \right) \, dx \\
&= 2b^2\pi \left(x - \frac{x^3}{3a^2} \right) \Big|_0^a \\
&= 2b^2\pi \left(a - \frac{a^{\cancel{3}}}{3\cancel{a^2}} \right) - 0 \\
&= \frac{4}{3} ab^2\pi
\end{aligned}
$$

Uočimo da, ako je $a = b = r$, promatrano tijelo postaje kugla, a rezultat zadatka nam daje da je $V = \dfrac{4}{3} r^3 \pi$.

Zadatak (9.18.)

Odredite volumen tijela koje nastaje rotacijom kruga $x^2 + (y - b)^2 = r^2$, gdje je $r \leq b$, oko osi x.

Rotacijom kruga dobije se torus.



$$
\begin{aligned}
x^2 + (y-b)^2 &= r^2 \\
(y-b)^2 &= r^2 - x^2 \qquad /\sqrt{} \\
y_1 &= b + \sqrt{r^2 - x^2} \\
y_2 &= b - \sqrt{r^2 - x^2}
\end{aligned}
$$

$$
\begin{aligned}
V_x &= \pi \int_{-r}^{r} \left(y_1^2 - y_2^2\right)\, dx \\
&= \pi \int_{-r}^{r} \left(\left(b + \sqrt{r^2 - x^2}\right)^2 - \left(b - \sqrt{r^2 - x^2}\right)^2 \right) dx \\
&= \pi \int_{-r}^{r} \left(\cancel{b^2} + 2b\sqrt{r^2 - x^2} + \cancel{r^2} - \cancel{x^2} - \cancel{b^2} + 2b\sqrt{r^2 - x^2} - \cancel{r^2} + \cancel{x^2} \right) dx \\
&= 4b\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx \\
&= \left\{ \begin{array}{lll} x = r \sin t & x = -r \mapsto & t = -\dfrac{\pi}{2} \\ dt = r \cos t \, dt & x = r \mapsto & t = \dfrac{\pi}{2} \end{array} \right\}
\end{aligned}
$$

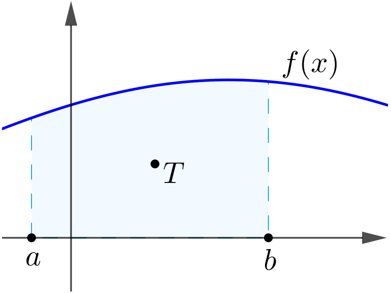
$$
\begin{aligned}
&= 4b\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 t} \cdot r \cos t \, dt \\
&= 4r^2 b\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t} \cdot \cos t \, dt \\
&= 4r^2 b\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt \\
&= 4r^2 b\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2t)}{2} \, dt \\
&= 2r^2 b\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + \cos(2t)\right) \, dt
\end{aligned}
$$

$$
\begin{aligned}
&= 2r^2 b\pi \left(t + \frac{1}{2}\sin(2t) \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 2r^2 b\pi \left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi) + \frac{\pi}{2} - \frac{1}{2}\sin(-\pi) \right) \\
&= 2r^2 b\pi^2
\end{aligned}
$$

9.3 Težište

Neka je $f : [a, b] \to \mathbb{R}$ neprekidna. Koordinate težišta T **lika** omeđenog grafom ove funkcije i osi x dane su formulama:

$$x_T = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} \qquad \text{i} \qquad y_T = \frac{\int_a^b f(x)^2\,dx}{2\int_a^b f(x)\,dx}$$



Zadatak (9.11.)

Odredite težište homogenog lika gustoće 1 omeđenog grafom funkcije $f(x) = \sin x$ na $[0, \pi]$ i osi x.

$$x_T = \frac{\int\limits_0^\pi x \sin x \, dx}{\int\limits_0^\pi \sin x \, dx}$$

$$
\begin{aligned}
\int_0^{\pi} x\sin x\,dx &= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = \sin x\,dx & \Rightarrow & v = -\cos x \end{matrix}\right\} \\
&= -x\cos x\Big|_0^{\pi} + \int_0^{\pi} \cos x\,dx \\
&= \pi + \sin x\Big|_0^{\pi} \\
&= \pi
\end{aligned}
$$

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi}$$

$$= 1 - (-1)$$

$$= 2$$

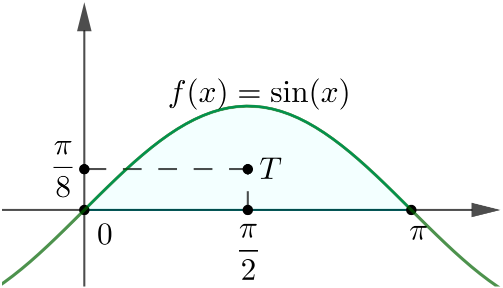
$$x_T = \frac{\pi}{2}$$

$$y_T = \frac{\int_0^{\pi} \sin^2 x \, dx}{2\int_0^{\pi} \sin x \, dx}$$

$$\begin{aligned}
\int_0^{\pi} \sin^2 x \, dx &= \int_0^{\pi} \frac{1-\cos 2x}{2} \, dx \\
&= \left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\Big|_0^{\pi} \\
&= \frac{1}{2}\pi
\end{aligned}$$

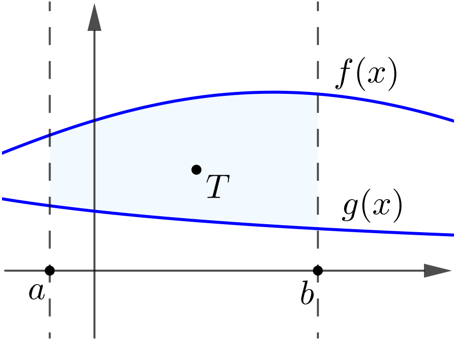
$$y_T = \frac{\pi}{8}$$

Imamo da su koordinate težišta $T = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$.



Koordinate težišta lika omeđenog grafovima neprekidnih funkcija f i g na segmentu $[a, b]$ i pravcima $x = a$ i $x = b$ dano je formulama:

$$x_T = \frac{\int\limits_a^b x\,(f(x) - g(x))\ dx}{\int\limits_a^b (f(x) - g(x))\ dx} \qquad \text{i} \qquad y_T = \frac{\int\limits_a^b \left(f(x)^2 - g(x)^2\right)\ dx}{2\int\limits_a^b (f(x) - g(x))\ dx}$$



Zadatak (9.12.)

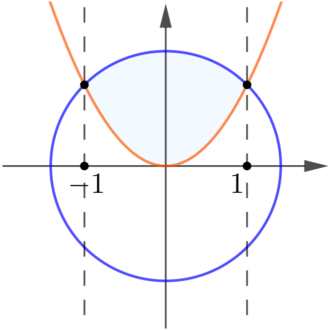
Odredite težište homogenog lika gustoće 1 omeđenog kružnicom $x^2 + y^2 = 2$ i parabolom $y = x^2$.

Odredimo koordinate sjecišta krivulja:

$$\begin{aligned} x^2 + y^2 &= 2 \\ y &= \sqrt{2 - x^2} \\ x^2 &= \sqrt{2 - x^2} \quad \Big/^2 \\ x^4 + x^2 - 2 &= 0 \end{aligned}$$

$x_1^2 = 1,\ \cancel{x_2^2 = -2}$

$x_1 = -1,\ x_2 = 1$



$$x_T = \frac{\int\limits_{-1}^{1} x\left(\sqrt{2-x^2}-x^2\right)\,dx}{\int\limits_{-1}^{1}\left(\sqrt{2-x^2}-x^2\right)\,dx}$$

$$\int\limits_{-1}^{1} x\left(\sqrt{2-x^2}-x^2\right)\,dx = 0 \qquad \text{(funkcija neparna)}$$

$$x_T = 0$$

$$y_T = \frac{\int\limits_{-1}^{1} \left(\left(\sqrt{2-x^2}\right)^2 - \left(x^2\right)^2 \right)\, dx}{2\int\limits_{-1}^{1} \left(\sqrt{2-x^2} - x^2 \right)\, dx}$$

$$\begin{aligned}
\int\limits_{-1}^{1} \left(\left(\sqrt{2-x^2}\right)^2 - \left(x^2\right)^2 \right)\, dx &= \int\limits_{-1}^{1} \left(2 - x^2 - x^4\right)\, dx \\
&= \left(2x - \frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Bigg|_{-1}^{1} \\
&= \left(2 - \frac{1}{3} - \frac{1}{5} + 2 - \frac{1}{3} - \frac{1}{5}\right) \\
&= \frac{44}{15}
\end{aligned}$$

$$\int_{-1}^{1}\left(\sqrt{2-x^2}-x^2\right)\,dx = \left\{\begin{array}{lll} x=\sqrt{2}\sin t & \Rightarrow & dx=\sqrt{2}\cos t dt \\ x=-1 & \mapsto & t=-\frac{\pi}{4} \\ x=1 & \mapsto & t=\frac{\pi}{4} \end{array}\right\}$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2 t\,dt - \frac{1}{3}x^3\Big|_{-1}^{1}$$

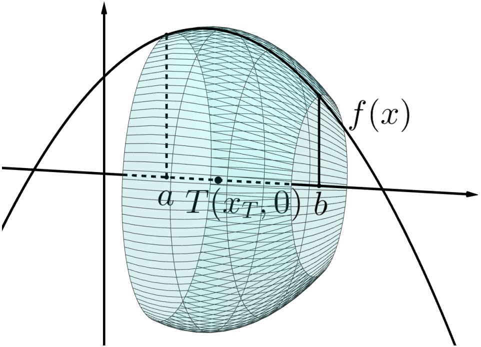
$$= 2\left(\frac{1}{2}t+\frac{1}{4}\sin(2t)\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \frac{2}{3}$$

$$= \frac{3\pi+2}{6}$$

$$y_T = \frac{\frac{44}{15}}{2\cdot\frac{3\pi+2}{6}} \implies T=\left(0,\frac{44}{15\pi+10}\right)$$

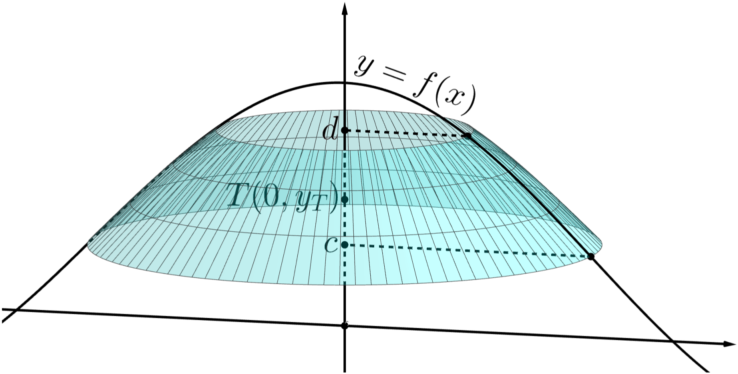
Neka je $f : [a, b] \to \mathbb{R}$ neprekidna. Koordinate težišta T **tijela** dobivenog rotacijom grafa funkcije f oko osi x dane su formulama:

$$x_T = \frac{\int_a^b x f(x)^2 \, dx}{\int_a^b f(x)^2 \, dx} \quad \text{i} \quad y_T = 0 \text{ zbog simetrije.}$$



Tijelo dobiveno rotacijom oko osi y ima koordinate težišta:

$$x_T = 0 \text{ zbog simetrije} \quad \text{i} \quad y_T = \frac{\int\limits_c^d y f^{-1}(y)^2 \, dy}{\int\limits_c^d f^{-1}(y)^2 \, dy}$$



Zadatak (9.13.)

Odredite težište homogenog tijela gustoće 1 koje se dobije rotacijom funkcije $f(x) = e^x$ na segmentu $[0, 1]$:

a) oko osi x

$$x_T = \frac{\int_0^1 xe^{2x}\, dx}{\int_0^1 e^{2x}\, dx}$$

$$
\begin{aligned}
\int_0^1 xe^{2x}\,dx &= \left\{\begin{matrix} u = x & \Rightarrow & du = dx \\ dv = e^{2x}\,dx & \Rightarrow & v = \frac{1}{2}e^{2x} \end{matrix}\right\} \\
&= \frac{1}{2}xe^{2x}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx \\
&= \frac{1}{2}\cdot 1\cdot e^2 - 0 - \frac{1}{4}e^{2x}\Big|_0^1 \\
&= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \\
&= \frac{e^2+1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 e^{2x}\,dx &= \frac{1}{2}e^{2x}\Big|_0^1 \\
&= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \\
&= \frac{e^2-1}{2}
\end{aligned}
$$

Imamo da su koordinate težišta $T = \left(\dfrac{e^2+1}{2e^2-2}, 0\right)$.

b) oko osi y

$$
\begin{aligned}
y &= f(x) \\
y &= e^x \quad \Big/ \ln & x = 0 &\mapsto y = 1 \\
\ln y &= x & x = 1 &\mapsto y = e \\
f^{-1}(y) &= \ln y
\end{aligned}
$$

$$
y_T = \frac{\int_1^e y \ln^2 y \, dy}{\int_1^e \ln^2 y \, dy}
$$

$$
\begin{aligned}
\int_1^e y \ln^2 y \, dy &= \left\{ \begin{array}{lcl} u = \ln^2 y & \Rightarrow & du = 2 \ln y \cdot \dfrac{1}{y} \, dy \\ dv = y \, dy & \Rightarrow & v = \dfrac{1}{2} y^2 \end{array} \right\} \\
&= \frac{1}{2} y^2 \ln^2 y \Big|_1^e - \int_1^e y \ln y \, dy \\
&= \frac{e^2}{2} - \int_1^e y \ln y \, dy \\
&= \left\{ \begin{array}{lcl} u = \ln y & \Rightarrow & du = \dfrac{1}{y} \, dy \\ dv = y \, dy & \Rightarrow & v = \dfrac{1}{2} y^2 \end{array} \right\} \\
&= \frac{e^2}{2} - \left(\frac{1}{2} y^2 \ln y \Big|_1^e - \frac{1}{2} \int_1^e y \, dy \right)
\end{aligned}
$$

$$= \frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{4} y^2 \Big|_1^e \right)$$

$$= \frac{e^2 - 1}{4}$$

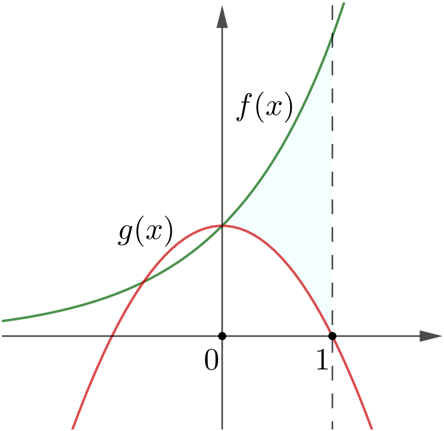
$$
\begin{aligned}
\int_1^e \ln^2 y \, dy &= \left\{ \begin{matrix} u = \ln^2 y & \Rightarrow & du = 2 \ln y \cdot \dfrac{1}{y} \, dy \\ dv = dy & \Rightarrow & v = y \end{matrix} \right\} \\
&= y \ln^2 y \Big|_1^e - 2 \int_1^e \ln y \, dy \\
&= e - 2 \int_1^e \ln y \, dy \\
&= \left\{ \begin{matrix} u = \ln y & \Rightarrow & du = \dfrac{1}{y} \, dy \\ dv = dy & \Rightarrow & v = y \end{matrix} \right\} \\
&= e - 2 \left(y \ln y \Big|_1^e - \int_1^e dy \right)
\end{aligned}
$$

$$
\begin{aligned}
&= e - 2\left(e - y\Big|_1^e\right)\\
&= e - 2\left(e - e + 1\right)\\
&= e - 2
\end{aligned}
$$

Imamo da su koordinate težišta $T = \left(0, \dfrac{e^2 - 1}{4e - 8}\right)$.

Zadatak (9.14.)

Odredite težište homogenog lika gustoće 1 omeđenog grafovima funkcija $f(x) = e^x$ i $g(x) = 1 - x^2$ na segmentu $[0, 1]$.



$$x_T = \frac{\int_0^1 x\left(e^x - 1 + x^2\right)\, dx}{\int_0^1 \left(e^x - 1 + x^2\right)\, dx}$$

$$y_T = \frac{\int_0^1 \left(e^{2x} - 1 + 2x^2 - x^4\right)\, dx}{2\int_0^1 \left(e^x - 1 + x^2\right)\, dx}$$

$$
\begin{aligned}
\int_0^1 x\left(e^x - 1 + x^2\right)\, dx &= \int_0^1 xe^x\, dx + \int_0^1 \left(-x + x^3\right)\, dx \\
&= \left\{\begin{array}{lcl} u = x & \Rightarrow & du = dx \\ dv = e^x\, dx & \Rightarrow & v = e^x \end{array}\right\} \\
&= xe^x\Big|_0^1 - \int_0^1 e^x\, dx + \left(-\frac{1}{2}x^2 + \frac{1}{4}x^4\right)\Big|_0^1 \\
&= e - e^x\Big|_0^1 - \frac{1}{4} \\
&= e - e + 1 - \frac{1}{4} \\
&= \frac{3}{4}
\end{aligned}
$$

$$\int_0^1 \left(e^{2x} - 1 + 2x^2 - x^4\right)\, dx = \left(\frac{1}{2}e^{2x} - x + \frac{2}{3}x^3 - \frac{1}{5}x^5\right)\Bigg|_0^1$$

$$= \left(\frac{1}{2}e^2 - 1 + \frac{2}{3} - \frac{1}{5}\right) - \frac{1}{2}$$

$$= \frac{15e^2 - 31}{30}$$

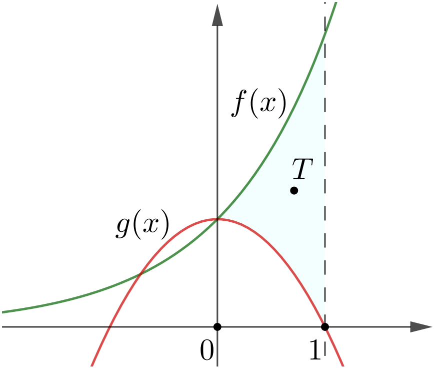
$$\int_0^1 \left(e^x - 1 + x^2\right)\, dx = \left(e^x - x + \frac{1}{3}x^3\right)\Bigg|_0^1$$

$$= \left(e - 1 + \frac{1}{3}\right) - 1$$

$$= \frac{3e - 5}{3}$$

Koordinate težišta:

$$T = \left(\frac{9}{12e - 20}, \frac{15e^2 - 31}{60e - 100} \right)$$



Zadatak (9.15.)

Odredite težište homogenog tijela gustoće 1 dobivenog rotacijom područja omeđenog krivuljama $f(x) = x^2$ i $g(x) = \frac{x^2}{2}$ na segmentu $[0, 1]$.

a) oko osi x

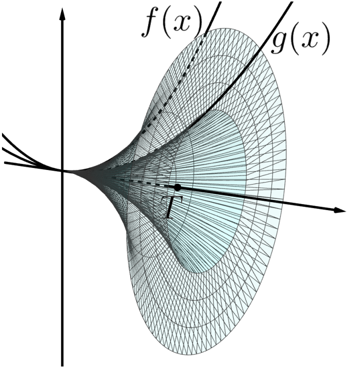
$$x_T = \frac{\int\limits_0^1 x\left(x^4 - \frac{x^4}{4}\right)\,dx}{\int\limits_0^1 \left(x^4 - \frac{x^4}{4}\right)\,dx}$$

$$y_T = 0$$

$$\int_0^1 x\left(x^4 - \frac{x^4}{4}\right)\,dx = \frac{3}{4}\int_0^1 x^5\,dx = \frac{3}{4}\left(\frac{1}{6}x^6\right)\Big|_0^1 = \frac{1}{8}$$

$$\int_0^1 \left(x^4 - \frac{x^4}{4}\right)\,dx = \frac{3}{4}\int_0^1 x^4\,dx = \frac{3}{4}\left(\frac{1}{5}x^5\right)\Big|_0^1 = \frac{3}{20}$$

Koordinate težišta: $T = \left(\dfrac{5}{6}, 0\right)$



b) oko osi y

$$\begin{aligned} y &= f(x) \\ y &= x^2 \quad \Big/ \sqrt{\ } \\ \sqrt{y} &= x \\ f^{-1}(y) &= \sqrt{y} \end{aligned} \qquad \begin{aligned} y &= g(x) \\ y &= \frac{x^2}{2} \quad \Big/ \cdot 2 \\ 2y &= x^2 \quad \Big/ \sqrt{\ } \\ g^{-1}(y) &= \sqrt{2y} \end{aligned}$$

$$\begin{aligned} x = 0 &\quad \mapsto \quad y = 0 \\ x = 1 &\quad \mapsto \quad y = 1 \end{aligned}$$

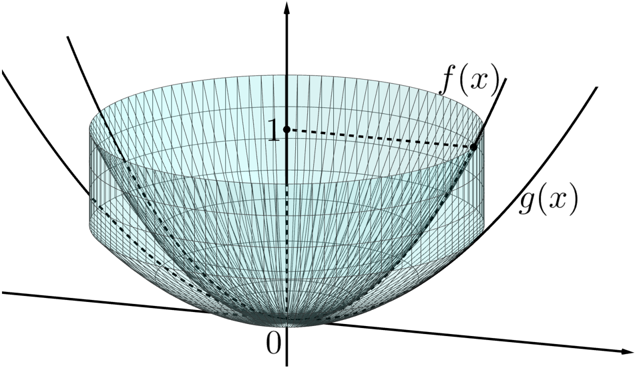
$$x_T = 0$$

$$y_T = \frac{\int\limits_0^1 y\,(2y - y)\ dy}{\int\limits_0^1 (2y - y)\ dy}$$

$$
\begin{aligned}
\int_0^1 y\,(2y-y)\;dy &= \int_0^1 y^2\,dy \\
&= \left.\frac{1}{3}y^3\right|_0^1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 (2y-y)\;dy &= \int_0^1 y\,dy \\
&= \left.\frac{1}{2}y^2\right|_0^1 \\
&= 1
\end{aligned}
$$

Koordinate težišta: $T = (0, 1)$



9.4 Duljina luka ravninske krivulje

Duljinu luka krivulje $y = y(x)$, $x \in [a, b]$ određujemo po formuli:

$$s = \int_a^b \sqrt{1 + y'(x)^2}\, dx$$

Zadatak (9.16.)

Odredite duljinu luka krivulje $y = \text{ch}\, x$ na segmentu $[0, 1]$.

$$
\begin{aligned}
y' &= \text{sh}\, x \\
s &= \int_0^1 \sqrt{1 + \text{sh}^2 x}\, dx \\
&= \int_0^1 \sqrt{\text{ch}^2 x}\, dx \\
&= \text{sh}\, x \Big|_0^1 \\
&= \text{sh}\, 1 - \text{sh}\, 0 \\
&= \frac{e - e^{-1}}{2}
\end{aligned}
$$

Zadatak (9.17.)

Odredite duljinu luka krivulje $y^3 = x^2$ na segmentu $[0, 1]$.

$$\begin{aligned} y &= \sqrt[3]{x^2} = x^{\frac{2}{3}} \\ y' &= \frac{2}{3}x^{-\frac{1}{3}} \\ s &= \underbrace{\int_0^1 \sqrt{1 + \frac{4}{9}x^{-\frac{2}{3}}}\, dx}_{\text{Nije jednostavno integrirati!}} \end{aligned}$$

Promatrat ćemo: $x = x(y)$

$$\begin{aligned} y^3 &= x^2 \\ x &= \sqrt{y^3} = y^{\frac{3}{2}} \\ x' &= \frac{3}{2}y^{\frac{1}{2}} \end{aligned}$$

$$x = 0 \quad \mapsto \quad y = 0$$
$$x = 1 \quad \mapsto \quad y = 1$$

$$
\begin{aligned}
s &= \int_0^1 \sqrt{1+\frac{9}{4}y}\,dy \\
&= \left\{ \begin{matrix} t = 1+\frac{9}{4}y & dt = \frac{9}{4}\,dy \Rightarrow \quad \frac{4}{9}\,dt = dy \\ y = 0 \mapsto t = 1 & y = 1 \mapsto t = \frac{13}{4} \end{matrix} \right\} \\
&= \frac{4}{9}\int_1^{\frac{13}{4}} \sqrt{t}\,dt \\
&= \frac{4}{9}\left(\frac{2}{3}t^{\frac{3}{2}}\right)\Bigg|_1^{\frac{13}{4}} \\
&= \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right)
\end{aligned}
$$

Zadatak (9.18.)

Odredite duljinu luka krivulje $y = \ln x$ na segmentu $\left[\sqrt{3}, \sqrt{8}\right]$.

$$
\begin{aligned}
y &= \ln x \\
y' &= \frac{1}{x} \\
s &= \int\limits_{\sqrt{3}}^{\sqrt{8}} \sqrt{1 + \frac{1}{x^2}}\, dx \\
&= \int\limits_{\sqrt{3}}^{\sqrt{8}} \sqrt{\frac{x^2+1}{x^2}}\, dx
\end{aligned}
$$

$$= \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{x^2+1}}{x}\, dx$$

$$= \left\{ \begin{array}{lllll} t^2 = 1 + x^2 & \Rightarrow & x = \sqrt{t^2-1} & x = \sqrt{3} \mapsto & t = 2 \\ \cancel{2}t\, dt = \cancel{2}x\, dx & \Rightarrow & \dfrac{t\, dt}{\sqrt{t^2-1}} = dx & x = \sqrt{8} \mapsto & t = 3 \end{array} \right\}$$

$$= \int_2^3 \frac{t}{\sqrt{t^2-1}} \cdot \frac{t\, dt}{\sqrt{t^2-1}}$$

$$= \int_2^3 \frac{t^2}{t^2-1}\, dt$$

$$= \int_2^3 \left(\frac{t^2-1+1}{t^2-1} \right) dt$$

$$
\begin{aligned}
&= \int_2^3 dt + \int_2^3 \frac{1}{(t-1)(t+1)}\,dt = (*) \\
\frac{1}{t^2-1} &= \frac{A}{t-1} + \frac{B}{t+1} \quad \Big/ \cdot (t^2-1) \\
1 &= A(t+1) + B(t-1) \\
1 &= t(A+B) + A - B \\
0 &= A + B \\
1 &= A - B \\
A &= \frac{1}{2} \\
B &= -\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
(*) \quad &= \int_2^3 dt + \frac{1}{2}\int_2^3 \frac{dt}{t-1}\,dt - \frac{1}{2}\int_2^3 \frac{dt}{t+1}\,dt \\
&= t\Big|_2^3 + \frac{1}{2}\ln|t-1|\Big|_2^3 - \frac{1}{2}\ln|t+1|\Big|_2^3 \\
&= 3-2+\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 3 \\
&= 1+\frac{1}{2}\ln\frac{2\cdot 3}{4} \\
&= 1+\frac{1}{2}\ln\frac{3}{2}
\end{aligned}
$$

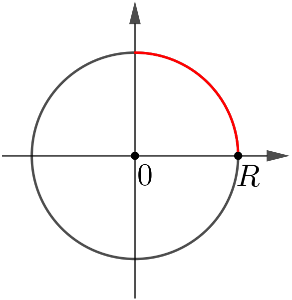
Zadatak (9.19.)

Odredite opseg kružnice radijusa R.

$$x^2 + y^2 = R^2$$

$$y = \sqrt{R^2 - x^2}$$

$$y' = \frac{-\cancel{2}x}{\cancel{2}\sqrt{R^2 - x^2}}$$



Zbog simetričnosti je dovoljno računati na segmentu $[0, R]$.

$$
\begin{aligned}
s &= \int_0^R \sqrt{1+\frac{x^2}{R^2-x^2}}\,dx \\
&= \int_0^R \sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}\,dx \\
&= R\int_0^R \frac{1}{\sqrt{R^2-x^2}}\,dx \\
&= R\cdot \arcsin\frac{x}{R}\Big|_0^R \\
&= R\cdot\left(\arcsin\frac{R}{R}-\arcsin\frac{0}{R}\right) \\
&= \frac{R\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
o_\circ &= 4 \cdot \frac{R\pi}{2} \\
o_\circ &= 2R\pi
\end{aligned}
$$

Zadatak (9.20.)

Odredite duljinu krivulje $y = e^x$ na segmentu $\left[\ln\sqrt{3}, \ln\sqrt{15}\,\right]$.

$$y = e^x$$
$$y' = e^x$$

$$
\begin{aligned}
s &= \int\limits_{\ln\sqrt{3}}^{\ln\sqrt{15}} \sqrt{1+e^{2x}}\,dx \\
&= \left\{
\begin{array}{lll}
1+e^{2x} = t^2 & \Rightarrow & e^{2x} = t^2 - 1 \\
\cancel{2}e^{2x}\,dx = \cancel{2}t\,dt & \Rightarrow & dx = \dfrac{t\,dt}{t^2-1} \\
x = \ln\sqrt{3} & \mapsto & t = \sqrt{1+e^{\ln\sqrt{3}^2}} = 2 \\
x = \ln\sqrt{15} & \mapsto & t = \sqrt{1+e^{\ln\sqrt{15}^2}} = 4
\end{array}
\right\}
\end{aligned}
$$

$$= \int_2^4 \frac{t^2}{t^2-1}\,dt$$

$$= \int_2^4 \frac{t^2-1+1}{t^2-1}\,dt$$

$$= \int_2^4 dt + \int_2^4 \frac{1}{(t-1)(t+1)}\,dt = (*)$$

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} \quad \Big/ \cdot (t^2-1)$$

$$1 = A(t+1) + B(t-1)$$

$$1 = t(A+B) + A - B$$

$$0 = A + B$$

$$1 = A - B$$

$$A = \frac{1}{2} \text{ i } B = -\frac{1}{2}$$

$$
\begin{aligned}
(*) &= t\Big|_2^4 + \frac{1}{2}\int_2^4 \frac{1}{t-1}\,dt - \frac{1}{2}\int_2^4 \frac{1}{t+1}\,dt \\
&= (4-2) + \frac{1}{2}(\ln 3 - \ln 1) - \frac{1}{2}(\ln 5 - \ln 3) \\
&= 2 + \ln 3 - \frac{\ln 5}{2}
\end{aligned}
$$

Zadatak (9.21.)

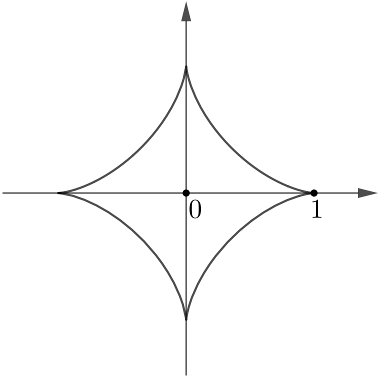
Odredite duljinu krivulje $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \quad \Big/'$$

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot y' = 0 \quad \Big/ \cdot \frac{3}{2}$$

$$y^{-\frac{1}{3}} \cdot y' = -x^{-\frac{1}{3}} \quad \Big/ : y^{-\frac{1}{3}}$$

$$y' = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$



Zbog simetričnosti je dovoljno računati na segmentu [0, 1].

$$
\begin{aligned}
s &= \int_0^1 \sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx \\
&= \int_0^1 \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx \\
&= \int_0^1 \sqrt{\frac{1}{x^{\frac{2}{3}}}}\, dx \\
&= \int_0^1 x^{-\frac{1}{3}}\, dx \\
&= \frac{3}{2} x^{\frac{2}{3}} \Big|_0^1\, dx \\
&= \frac{3}{2}
\end{aligned}
$$

2. način:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

$$y^{\frac{2}{3}} = 1 - x^{\frac{2}{3}}$$

$$y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$y' = \frac{3}{2}\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot \frac{-2}{3} x^{-\frac{1}{3}}$$

$$y' = -\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}$$

$$
\begin{aligned}
s &= \int_0^1 \sqrt{1+\left(1-x^{\frac{2}{3}}\right)\cdot x^{-\frac{2}{3}}}\, dx \\
&= \int_0^1 \sqrt{x^{-\frac{2}{3}}}\, dx \\
&= \int_0^1 x^{-\frac{1}{3}}\, dx \\
&= \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1\, dx \\
&= \frac{3}{2} \\
o &= 4\cdot\frac{3}{2} = 6
\end{aligned}
$$