Prisilne oscilacije sustava s dva stupnja slobode (bez prigušenja) s primjerima

SADRŽAJ:

Prisilne oscilacije sustava s više stupnjeva slobode (bez prigušenja)

Diferencijalna jednadžba gibanja u općem obliku

- s prigušenjem: $$[m]\cdot\{\ddot{u}(t)\}+[c]\cdot\{\dot{u}(t)\}+[k]\cdot\{u(t)\}=\{p(t)\} \tag{1}$$
- bez prigušenja: $$[m]\cdot\{\ddot{u}(t)\}+[k]\cdot\{u(t)\}=\{p(t)\} \tag{2}$$

Za konstrukciju s n stupnjeva slobode sustav se sastoji od N međusobno <u>zavisnih</u> običnih diferencijalnih jednadžbi (3).

$$\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N \end{bmatrix}\cdot\begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \vdots \\ \ddot{u}_N(t) \end{Bmatrix}+\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N1} & k_{N2} & \cdots & k_{NN} \end{bmatrix}\cdot\begin{Bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{Bmatrix}=\begin{Bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_N(t) \end{Bmatrix} \tag{3}$$

$$m_i\cdot\ddot{u}_i(t)+k_{i1}u_1(t)+k_{i2}u_2(t)+\ldots+k_{iN}u_N(t)=p_i(t),\ i=1...N \tag{4}$$

Za linearne sustave (s klasičnim prigušenjem) rješenje se može naći postupkom ***modalne analize***, transformacijom (7) sustava jednadžbi iz izvornih koordinata (5a) u modalne koordinate (5b), pomoću matrice (6) čiji stupci odgovaraju prirodnim oblicima titranja (vlastitim vektorima) sustava. Tada sustav prelazi u niz neovisnih jednadžbi, jednu za svaki stupanj slobode, s ukupno N nepoznanica.

$$\{u(t)\}=\begin{Bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{Bmatrix},\quad \{q(t)\}=\begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{Bmatrix}=\begin{Bmatrix} A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t) \\ A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t) \\ \vdots \\ A_N\cos(\omega_N t)+B_N\sin(\omega_N t) \end{Bmatrix} \tag{5a,b}$$

$$[\phi]=\left[\{\phi_1\}\{\phi_2\}\cdots\{\phi_N\}\right]=\begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \tag{6}$$

$$\{u(t)\}=\begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix}\{q(t)\}=\begin{Bmatrix} \phi_{11}q_1(t)+\phi_{12}q_2(t)+...+\phi_{1N}q_N(t) \\ \phi_{21}q_1(t)+\phi_{22}q_2(t)+...+\phi_{2N}q_N(t) \\ \vdots \\ \phi_{N1}q_1(t)+\phi_{N2}q_2(t)+...+\phi_{NN}q_N(t) \end{Bmatrix}=\begin{Bmatrix} u_{11}+u_{12}+...+u_{1N} \\ u_{21}+u_{22}+...+u_{2N} \\ \vdots \\ u_{N1}+u_{N2}+...+u_{NN} \end{Bmatrix} \tag{7}$$

Prirodne frekvencije i oblici titranja mogu se dobiti iz homogenog sustava jednadžbi (8). Doprinosi n-tog oblika titranja prikazani su izrazom (9), odnosno odgovarajuća brzina i ubrzanje izrazima (10) i (11).

- bez prigušenja: $$[m]\cdot\{\ddot{u}(t)\}+[k]\cdot\{u(t)\}=\{0\} \tag{8}$$

$$\left\{u_n(t)\right\} = \begin{Bmatrix} u_{1n}(t) \\ u_{2n}(t) \\ \vdots \\ u_{Nn}(t) \end{Bmatrix} = \left\{\phi_n\right\} q_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{Nn} \end{Bmatrix} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right] \tag{9}$$

$$\left\{\dot{u}_n(t)\right\} = \begin{Bmatrix} \dot{u}_{1n}(t) \\ \dot{u}_{2n}(t) \\ \vdots \\ \dot{u}_{Nn}(t) \end{Bmatrix} = \left\{\phi_n\right\} \dot{q}_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{Nn} \end{Bmatrix} \left[-\omega_n A_n \sin(\omega_n t) + \omega_n B_n \cos(\omega_n t)\right] \tag{10}$$

$$\left\{\ddot{u}_n(t)\right\} = \begin{Bmatrix} \ddot{u}_{1n}(t) \\ \ddot{u}_{2n}(t) \\ \vdots \\ \ddot{u}_{Nn}(t) \end{Bmatrix} = \left\{\phi_n\right\} \ddot{q}_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{Nn} \end{Bmatrix} \left[-\omega_n{}^2 A_n \cos(\omega_n t) - \omega_n{}^2 B_n \sin(\omega_n t)\right] \tag{11}$$

Iz jednadžbe (8) uvrštavanjem izraza (9) i (11), uz sređivanje izraza (12) dobije se problem vlastitih vrijednosti (13) iz kojeg slijede prirodne frekvencije i oblici titranja.

$$\left(-\left[m\right]\left\{\phi_n\right\}\omega_n^2 + \left[k\right]\left\{\phi_n\right\}\right)\cdot\left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right] = \left\{0\right\} \tag{12}$$

$$\left[k\right]\left\{\phi_n\right\} = \omega_n^2 \left[m\right]\left\{\phi_n\right\} \tag{13}$$

Za određivanje odziva sustava na proizvoljnu prisilnu pobudu, diferencijalna jednadžba (8) se može napisati u modalnim koordinatama prema izrazu (14). Nakon množenja s transponiranim vlastitim vektorom n-tog oblika osciliranja, te uz primjenu svojstva ortogonalnosti vektora (15), slijedi konačan oblik diferencijalne jednadžbe u modalnim koordinatama (16), odnosno (17), uvođenjem pojma generalizirane mase, krutosti i sile prema izrazima (18a,b i c).

$$\begin{gathered}
\left[m\right]\cdot\left\{\ddot{u}(t)\right\} + \left[k\right]\cdot\left\{u(t)\right\} = \left\{p(t)\right\} \qquad \left\{u(t)\right\} = \left[\phi\right]\cdot\left\{q(t)\right\} = \sum_{r=1}^{N}\left\{\phi_r\right\} q_r(t) \\
\left[m\right]\cdot\left[\phi\right]\cdot\left\{\ddot{q}(t)\right\} + \left[k\right]\cdot\left[\phi\right]\cdot\left\{q(t)\right\} = \left\{p(t)\right\} \\
\sum_{r=1}^{N}\left[m\right]\cdot\left\{\phi_r\right\}\cdot\ddot{q}_r(t) + \sum_{r=1}^{N}\left[k\right]\cdot\left\{\phi_r\right\}\cdot q_r(t) = \left\{p(t)\right\} \qquad /\cdot\left\{\phi_n\right\}^T \\
\sum_{r=1}^{N}\left\{\phi_n\right\}^T\cdot\left[m\right]\cdot\left\{\phi_r\right\}\cdot\ddot{q}_r(t) + \sum_{r=1}^{N}\left\{\phi_n\right\}^T\cdot\left[k\right]\cdot\left\{\phi_r\right\}\cdot q_r(t) = \left\{\phi_n\right\}^T\cdot\left\{p(t)\right\}
\end{gathered} \tag{14}$$

$$\left\{\phi_n\right\}^T\cdot\left[m\right]\cdot\left\{\phi_r\right\} = 0 \quad \text{i} \quad \left\{\phi_n\right\}^T\cdot\left[k\right]\cdot\left\{\phi_r\right\} = 0 \quad \forall \; n \neq r \tag{15}$$

(ostaju članovi samo za $r = n$)

$$\left\{\phi_n\right\}^T\cdot\left[m\right]\cdot\left\{\phi_n\right\}\cdot\ddot{q}_n(t) + \left\{\phi_n\right\}^T\cdot\left[k\right]\cdot\left\{\phi_n\right\}\cdot q_n(t) = \left\{\phi_n\right\}^T\cdot\left\{p(t)\right\} \tag{16}$$

$$M_n\cdot\ddot{q}_n(t) + K_n\cdot q_n(t) = P_n(t) \tag{17}$$

$$M_n = \left\{\phi_n\right\}^T\cdot\left[m\right]\cdot\left\{\phi_n\right\}, \quad K_n = \left\{\phi_n\right\}^T\cdot\left[k\right]\cdot\left\{\phi_n\right\}, \quad P_n(t) = \left\{\phi_n\right\}^T\cdot\left\{p(t)\right\} \tag{18a,b,c}$$

Za titranje sustava **s dva stupnja slobode** (n = 1 … N, N = 2), dano jednadžbom (19), prirodne frekvencije i oblici titranja mogu se odrediti iz zakona slobodnih oscilacija (20). Prema izrazu (21), odnosno (22), nakon sređivanja (23), iz kvadratne jednadžbe (24) mogu se izračunati vlastite vrijednosti $\omega_n^2 = \lambda_{1,2}$, odnosno $\omega_1^2 = \lambda_1$ i $\omega_2^2 = \lambda_2$ (25).

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \cdot \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{19}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \cdot \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{20}$$

$$\left(\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_n^2 \cdot \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right) \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{21}$$

$$\det \left\| \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_n^2 \cdot \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\| = \begin{vmatrix} k_{11} - \omega_n^2 m_1 & k_{12} \\ k_{12} & k_{22} - \omega_n^2 m_2 \end{vmatrix} = 0 \tag{22}$$

$$\begin{gathered} \left(k_{11} - \omega_n^2 \cdot m_1\right) \cdot \left(k_{22} - \omega_n^2 \cdot m_2\right) - k_{12} \cdot k_{21} = 0 \\ k_{11} \cdot k_{22} - k_{12}^2 - \omega_n^2 \left(k_{11} \cdot m_2 + k_{22} \cdot m_1\right) + \omega_n^4 m_1 m_2 = 0 \end{gathered} \tag{23}$$

$$m_1 \cdot m_2 \cdot \lambda^2 - \left(k_{11} \cdot m_2 + k_{22} \cdot m_1\right) \cdot \lambda + k_{11} \cdot k_{22} - k_{12}^2 = 0 \tag{24}$$

$$\lambda_{1,2} = \frac{k_{11} \cdot m_2 + k_{22} \cdot m_1 \pm \sqrt{\left(k_{11} \cdot m_2 + k_{22} \cdot m_1\right)^2 - 4 \cdot m_1 \cdot m_2 \cdot \left(k_{11} \cdot k_{22} - k_{12}^2\right)}}{2 \cdot m_1 \cdot m_2} \tag{25}$$

$$\begin{gathered} \omega_1 = \sqrt{\lambda_1} \;\rightarrow\; \mathrm{T}_1 = \frac{2\pi}{\omega_1} \\ \omega_2 = \sqrt{\lambda_2} \;\rightarrow\; \mathrm{T}_2 = \frac{2\pi}{\omega_2} \end{gathered} \tag{26}$$

Vratimo se na početnu jednadžbu problema vlastitih vrijednosti da bismo odredili vlastite oblike

$$\left(\left[k\right] - \omega_n^2 \left[m\right]\right) \cdot \left\{\phi_n\right\} = \left\{0\right\} \qquad \left(\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_n^2 \cdot \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right) \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{27}$$

raspisano: $k_{11}\phi_{1n} + k_{12}\phi_{2n} - \omega_n^2 \cdot m_1 \phi_{1n} = 0$ $\qquad n$=1,2

$k_{21}\phi_{1n} + k_{22}\phi_{2n} - \omega_n^2 \cdot m_2 \phi_{2n} = 0$ $\qquad n$=1,2

Vlastiti oblici su bezdimenzionalne veličine i određene su do na konstantu. Poznati su nam samo omjeri amplituda pa je potrebno pretpostaviti pojedine veličine.

npr. za $\phi_{1n} = 1{,}0$ $\qquad \phi_{2n} = -\dfrac{k_{11} - \omega_n^2 m_1}{k_{12}}$ $\qquad n$=1,2

Konačno, vlastiti oblici sustava s dva stupnja slobode su:

za ω_1 , T_1 $\quad \phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix}$

za ω_2 , T_2 $\quad \phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix}$

ili matrično $\left[\phi\right] = \left[\{\phi_1\}\{\phi_2\}\right] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$

Za određivanje odziva na prisilnu pobudu, diferencijalna jednadžba (19) može se napisati u modalnim koordinatama prema izrazu (28), odnosno (29), primjenom svojstva ortogonalnosti vlastitih vektora (15). Uvođenjem generalizirane mase, krutosti i sile prema izrazima (18a,b i c), odnosno (31a,b i c), može se dobiti sustav od dvije nezavisne jednadžbe s ukupno dvije nepoznanice.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \cdot \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \cdot \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \cdot \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \qquad / \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \tag{28}$$

$$\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \cdot \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix} \cdot \ddot{q}_n(t) + \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \cdot \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix} \cdot q_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \cdot \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{29}$$

$$\begin{aligned} M_1 \cdot \ddot{q}_1(t) + K_1 \cdot q_1(t) &= P_1(t) \\ M_2 \cdot \ddot{q}_2(t) + K_2 \cdot q_2(t) &= P_2(t) \end{aligned} \tag{30a, b}$$

$$M_n = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \cdot [m] \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}, \quad K_n = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \cdot [k] \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}, \quad P_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T \cdot \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{31a,b,c}$$

POČETNI UVJETI

Za određivanje slobodnog odziva sustava uz početne uvjete, opće rješenje diferencijalne jednadžbe (8) se može raspisati kao superpozicija odziva za pojedini oblik:

$$\begin{aligned} \{u(t)\} &= \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11}q_1(t) + \phi_{12}q_2(t) + \ldots + \phi_{1N}q_N(t) \\ \phi_{21}q_1(t) + \phi_{22}q_2(t) + \ldots + \phi_{2N}q_N(t) \\ \vdots \\ \phi_{N1}q_1(t) + \phi_{N2}q_2(t) + \ldots + \phi_{NN}q_N(t) \end{Bmatrix} = \\ &= \{\phi_1\} q_1(t) + \ldots + \{\phi_N\} q_N(t) \\ &= \sum_{n=1}^{N} \{\phi_n\} q_n(t) = \sum_{n=1}^{N} \{\phi_n\} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \end{aligned} \tag{32}$$

a brzina slobodnog titranja kao:

$$\{\dot{u}(t)\} = \sum_{n=1}^{N} \{\phi_n\} \dot{q}_n(t) = \sum_{n=1}^{N} \{\phi_n\} \omega_n \left[-A_n \sin(\omega_n t) + B_n \cos(\omega_n t) \right] \tag{33}$$

Raspišimo početne uvjete:

za t=0;
$$\{u(t=0)\}=\{u(0)\}=\sum_{n=1}^{N}\{\phi_n\}A_n \quad /\cdot \phi_n^T[m] \tag{34}$$

$$\{\dot{u}(t=0)\}=\{\dot{u}(0)\}=\sum_{n=1}^{N}\{\phi_n\}\omega_n B_n \quad /\cdot \phi_n^T[m] \tag{35}$$

Kad izrazi (34) i (35) pomnože s lijeva s umnoškom $\phi_n^T[m]$ i koristeći uvjete ortogonalnosti vlastitih vektora (izraz (15)) dobiju se konstante A_n i B_n:

$$\begin{aligned}\{\phi_n\}^T[m]\{u(0)\}&=\{\phi_n\}^T[m]\{\phi_n\}A_n\\ A_n&=\frac{\{\phi_n\}^T[m]\{u(0)\}}{\{\phi_n\}^T[m]\{\phi_n\}}\end{aligned} \tag{36}$$

$$\begin{aligned}\{\phi_n\}^T[m]\{\dot{u}(0)\}&=\{\phi_n\}^T[m]\{\phi_n\}\omega_n B_n\\ B_n&=\frac{1}{\omega_n}\frac{\{\phi_n\}^T[m]\{\dot{u}(0)\}}{\{\phi_n\}^T[m]\{\phi_n\}}\end{aligned} \tag{37}$$

Za slobodno titranje uz početne uvjete sustava **s dva stupnja slobode** ($n = 1 \ldots N, N = 2$), prema izrazima (36) i (37) konstante A_n i B_n su:

$$A_1=\frac{\{\phi_1\}^T[m]\{u(0)\}}{\{\phi_1\}^T[m]\{\phi_1\}} \quad i \quad A_2=\frac{\{\phi_2\}^T[m]\{u(0)\}}{\{\phi_2\}^T[m]\{\phi_2\}} \tag{38 a,b}$$

$$B_1=\frac{1}{\omega_1}\frac{\{\phi_1\}^T[m]\{\dot{u}(0)\}}{\{\phi_1\}^T[m]\{\phi_1\}} \quad i \quad B_2=\frac{1}{\omega_2}\frac{\{\phi_2\}^T[m]\{\dot{u}(0)\}}{\{\phi_2\}^T[m]\{\phi_2\}} \tag{39 a,b}$$

a ukupni odziv:

$$\begin{Bmatrix}u_1(t)\\u_2(t)\end{Bmatrix}=\begin{bmatrix}\phi_{11}&\phi_{12}\\\phi_{21}&\phi_{22}\end{bmatrix}\cdot\begin{Bmatrix}q_1(t)\\q_2(t)\end{Bmatrix}=\begin{Bmatrix}\phi_{11}\left[A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t)\right]+\phi_{12}\left[A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t)\right]\\\phi_{21}\left[A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t)\right]+\phi_{22}\left[A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t)\right]\end{Bmatrix} \tag{38}$$

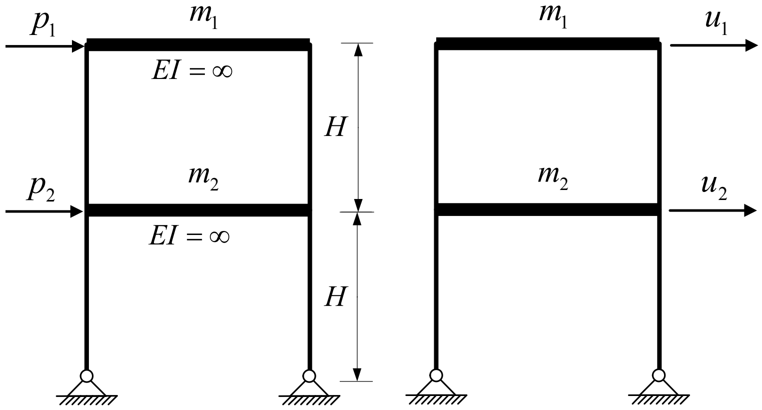
Zadatak 1. Okvir

Odrediti:

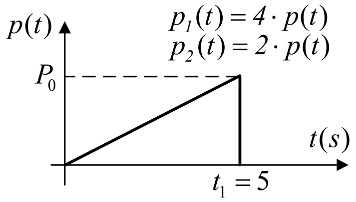
- dinamičke stupnjeve slobode,
- vlastite frekvencije i forme (forme prikazati grafički)
- te oscilacije prikazane konstrukcije nastale uslijed djelovanja pobude.

Konstrukcija je prije djelovanja pobude mirovala.

I. Određivanje dinamičkih stupnjeva slobode sustava



$H = 3,0$ m
$m_1 = m_2 = 2$ t
$E = 3,0 \times 10^7$ kN/m^2
$b / h = 40 / 40$ cm
$P_o = 50$ *kN*
$EI = 64000$ kNm^2



Homogena diferencijalna jednadžba gibanja bez prigušenja u općem obliku

$$[m]\{\ddot{u}(t)\} + [k]\{u(t)\} = \{0\}$$

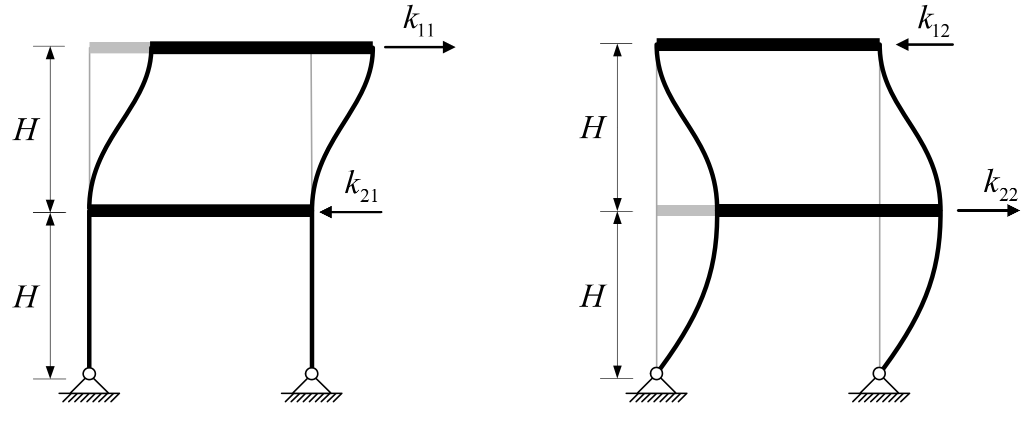
Za konstrukciju s 2 stupnja slobode sustav se sastoji od 2 međusobno zavisne diferencijalne jednadžbe:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

II. a) Određivanje matrice masa sustava

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

b) Određivanje koeficijenata krutosti sustava



$$k_{11}=2\frac{12EI}{H^3}=56888,9\text{ kN/m} \qquad k_{21}=-k_{11}=-2\frac{12EI}{H^3}=-56888,9\text{ kN/m}$$

$$k_{22}=2\frac{12EI}{H^3}+2\frac{3EI}{H^3}=71111,1\text{ kN/m} \qquad k_{12}=k_{21}=-56888,9\text{ kN/m}$$

Globalna matrica krutosti sustava:

$$[k]=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}=\frac{EI}{H^3}\begin{bmatrix} 24 & -24 \\ -24 & 30 \end{bmatrix}=\begin{bmatrix} 56888,9 & -56888,9 \\ -56888,9 & 71111,1 \end{bmatrix}$$

c) Određivanje vlastitih frekvencija i oblika osciliranje

Pretpostavljeno rješenje za funkcije pomaka

$$\{u(t)\}=[\phi]\{q(t)\}=\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}=\begin{Bmatrix} \phi_{11}q_1(t)+\phi_{12}q_2(t) \\ \phi_{21}q_1(t)+\phi_{22}q_2(t) \end{Bmatrix}=\begin{Bmatrix} u_{11}(t)+u_{12}(t) \\ u_{21}(t)+u_{22}(t) \end{Bmatrix}$$

$$\{u(t)\}=\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}, \quad \{q(t)\}=\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}=\begin{Bmatrix} A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t) \\ A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t) \end{Bmatrix}=A_n\cos(\omega_n t)+B_n\sin(\omega_n t)$$

$$\{u_n(t)\}=\{\phi_n\}q_n(t)=\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\left[A_n\cos(\omega_n t)+B_n\sin(\omega_n t)\right]$$

$$\{\dot{u}_n(t)\}=\{\phi_n\}\dot{q}_n(t)=\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\left[-\omega_n A_n\sin(\omega_n t)+\omega_n B_n\cos(\omega_n t)\right]$$

$$\{\ddot{u}_n(t)\}=\{\phi_n\}\ddot{q}_n(t)=\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\left[-\omega_n^2 A_n\cos(\omega_n t)-\omega_n^2 B_n\sin(\omega_n t)\right]$$

uvrštavanjem slijede vlastite frekvencije i oblici titranja.

$$\left(-[m]\{\phi_n\}\omega_n^2+[k]\{\phi_n\}\right)\cdot\left[A_n\cos(\omega_n t)+B_n\sin(\omega_n t)\right]=\{0\}$$

Slijedi da je:

$[k]\{\phi_n\}=\omega_n^2[m]\{\phi_n\}$ - problem vlastitih vrijednosti

$$\left([k]-\omega_n^2[m]\right)\{\phi_n\}=\{0\}$$

$$\det\left|[k]-\omega_n^2[m]\right|=0$$

Raspisivanjem determinante imamo:

$$\left(k_{11}-\omega_n^2\cdot m_1\right)\cdot\left(k_{22}-\omega_n^2\cdot m_2\right)-k_{12}\cdot k_{21}=0$$

$$k_{11}\cdot k_{22}-k_{12}^2-\omega_n^2\left(k_{11}\cdot m_2+k_{22}\cdot m_1\right)+\omega_n^4 m_1 m_2=0 \qquad \text{uz} \qquad \lambda=\omega_n^2$$

$$m_1 \cdot m_2 \cdot \lambda^2 - \left(k_{11} \cdot m_2 + k_{22} \cdot m_1\right) \cdot \lambda + k_{11} \cdot k_{22} - k_{12}{}^2 = 0$$

$$\lambda_{1,2} = \frac{k_{11} \cdot m_2 + k_{22} \cdot m_1 \pm \sqrt{\left(k_{11} \cdot m_2 + k_{22} \cdot m_1\right)^2 - 4 \cdot m_1 \cdot m_2 \cdot \left(k_{11} \cdot k_{22} - k_{12}{}^2\right)}}{2 \cdot m_1 \cdot m_2}$$

Za zadane vrijednosti imamo vrijednosti:

$$\lambda_{1,2} = \frac{256000 \pm 229326,5}{8}$$

$$\lambda_1 = \omega_1^2 = 3334,2 \quad \rightarrow \quad \omega_1 = 57,7\ s^{-1} \quad \rightarrow \quad \mathrm{T}_1 = 0,109\ s$$

$$\lambda_2 = \omega_2^2 = 60665,8 \quad \rightarrow \quad \omega_2 = 246,3\ s^{-1} \quad \rightarrow \quad \mathrm{T}_2 = 0,026\ s$$

Vratimo se na početnu jednadžbu problema vlastitih vrijednosti da bismo odredili vlastite oblike

$$\left([k] - \omega_n{}^2 [m]\right) \cdot \{\phi_n\} = \{0\} \qquad \left(\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_n{}^2 \cdot \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right) \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_{11}\phi_{1n} + k_{12}\phi_{2n} - \omega_n{}^2 \cdot m_1\phi_{1n} = 0$$

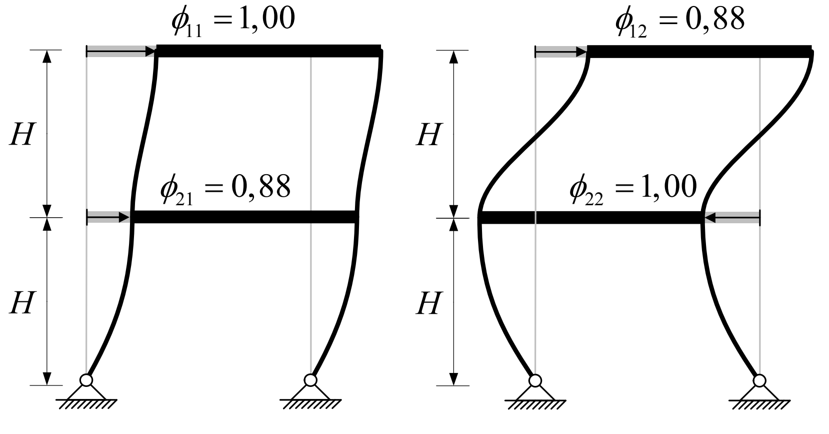
$$k_{21}\phi_{1n} + k_{22}\phi_{2n} - \omega_n{}^2 \cdot m_2\phi_{2n} = 0$$

Vlastiti oblici su bezdimenzionalne veličine i određene su do na konstantu. Poznati su nam samo omjeri amplituda pa je potrebno pretpostaviti pojedine veličine.

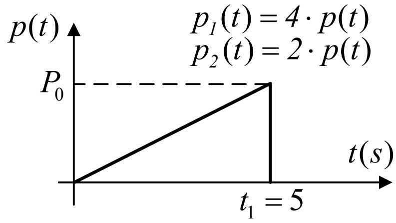
$$\phi_{1n} = 1,0 \qquad \phi_{2n} = -\frac{k_{11} - \omega_n^2 m_1}{k_{12}}$$

$$\phi_{21} = -\frac{56888,9 - 3334,2 \cdot 2}{-56888,9} = 0,883 \qquad \phi_{22} = -\frac{56888,9 - 60665,8 \cdot 2}{-56888,9} = -1,133$$

$$\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1,000 \\ 0,883 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1,000 \\ -1,133 \end{Bmatrix} \cdot \frac{1}{1,133} = \begin{Bmatrix} 0,883 \\ -1,000 \end{Bmatrix}$$



III. Određivanje odziva konstrukcije na zadanu pobudu



$$\{p(t)\} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} = \begin{Bmatrix} 40 \cdot t \\ 20 \cdot t \end{Bmatrix}$$

Modalna analiza

Diferencijalna jednadžba uz pretpostavljeno rješenje

$$[m]\cdot[\phi]\cdot\{\ddot{q}(t)\} + [k]\cdot[\phi]\cdot\{q(t)\} = \{p(t)\} \qquad /\cdot\{\phi_n\}^T$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\cdot\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}\cdot\begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\cdot\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}\cdot\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \qquad /\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T$$

Uvjet ortogonalnosti: $\{\phi_i\}^T\cdot[m]\cdot\{\phi_j\} = 0, \quad \{\phi_i\}^T\cdot[k]\cdot\{\phi_j\} = 0 \quad \forall\ i \neq j$

Dobivamo sustav neovisnih diferencijalnih jednadžbi

$$\{\phi_n\}^T\cdot[m]\cdot\{\phi_n\}\cdot\ddot{q}_n(t) + \{\phi_n\}^T\cdot[k]\cdot\{\phi_n\}\cdot q_n(t) = \{\phi_n\}^T\cdot\{p(t)\}$$

$$\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\cdot\ddot{q}_n(t) + \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\cdot q_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

$M_n\cdot\ddot{q}_n(t) + K_n\cdot q_n(t) = P_n(t)$ gdje su modalni koeficijenti:

$$M_n = \{\phi_n\}^T\cdot[m]\cdot\{\phi_n\}, \quad K_n = \{\phi_n\}^T\cdot[k]\cdot\{\phi_n\}, \quad P_n(t) = \{\phi_n\}^T\cdot\{p(t)\}$$

Kod dva stupnja slobode nam ostaju dvije neovisne diferencijalne jednadžbe

$$M_1\cdot\ddot{q}_1(t) + K_1\cdot q_1(t) = P_1(t)$$
$$M_2\cdot\ddot{q}_2(t) + K_2\cdot q_2(t) = P_2(t)$$

$$M_n = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot[m]\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}, \quad K_n = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot[k]\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}, \quad P_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

Za zadane vrijednosti imamo:

$$M_1 = \{\phi_1\}^T[m]\{\phi_1\} = \{1 \quad 0{,}883\}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 0{,}883 \end{Bmatrix} = 3{,}559$$

$$M_2 = \{\phi_2\}^T [m] \{\phi_2\} = \{0,883 \quad -1,000\} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 0,883 \\ -1,000 \end{Bmatrix} = 3,559$$

$$P_1 = \{\phi_1\}^T \{p(t)\} = \{1 \quad 0,833\} \begin{Bmatrix} 40 \cdot t \\ 20 \cdot t \end{Bmatrix} = 57,66 \cdot t \qquad K_1 = \omega_1^2 M_1 = 11865,1$$

$$P_2 = \{\phi_2\}^T \{p(t)\} = \{0,883 \quad -1,000\} \begin{Bmatrix} 40 \cdot t \\ 20 \cdot t \end{Bmatrix} = 15,31 \cdot t \qquad K_2 = \omega_2^2 M_2 = 215885,9$$

Potrebno je riješiti dvije neovisne diferencijalne jednadžbe s konstantnim koeficijentima (pogledati prethodno izvedena rješenja za traženi oblik pobude kod jednog stupnja slobode).

1. Diferencijalna jednadžba

$$3,559 \cdot \ddot{q}_1(t) + 11865,1 \cdot q_1(t) = 57,66 \cdot t$$

...izvesti odziv!!! (pogledati u separatu *Prisilne oscilacije sustava s jednim stupnjem slobode (bez prigušenja)*)

Rješenje: $$q_1(t) = \frac{57,66}{11865,1} \cdot \left[t - \frac{1}{57,7} \cdot \sin(57,7 \cdot t) \right]$$

2. Diferencijalna jednadžba

$$3,559 \cdot \ddot{q}_2(t) + 215885,9 \cdot q_2(t) = 15,31 \cdot t$$

...izvesti odziv!!! (pogledati u separatu *Prisilne oscilacije sustava s jednim stupnjem slobode (bez prigušenja)*)

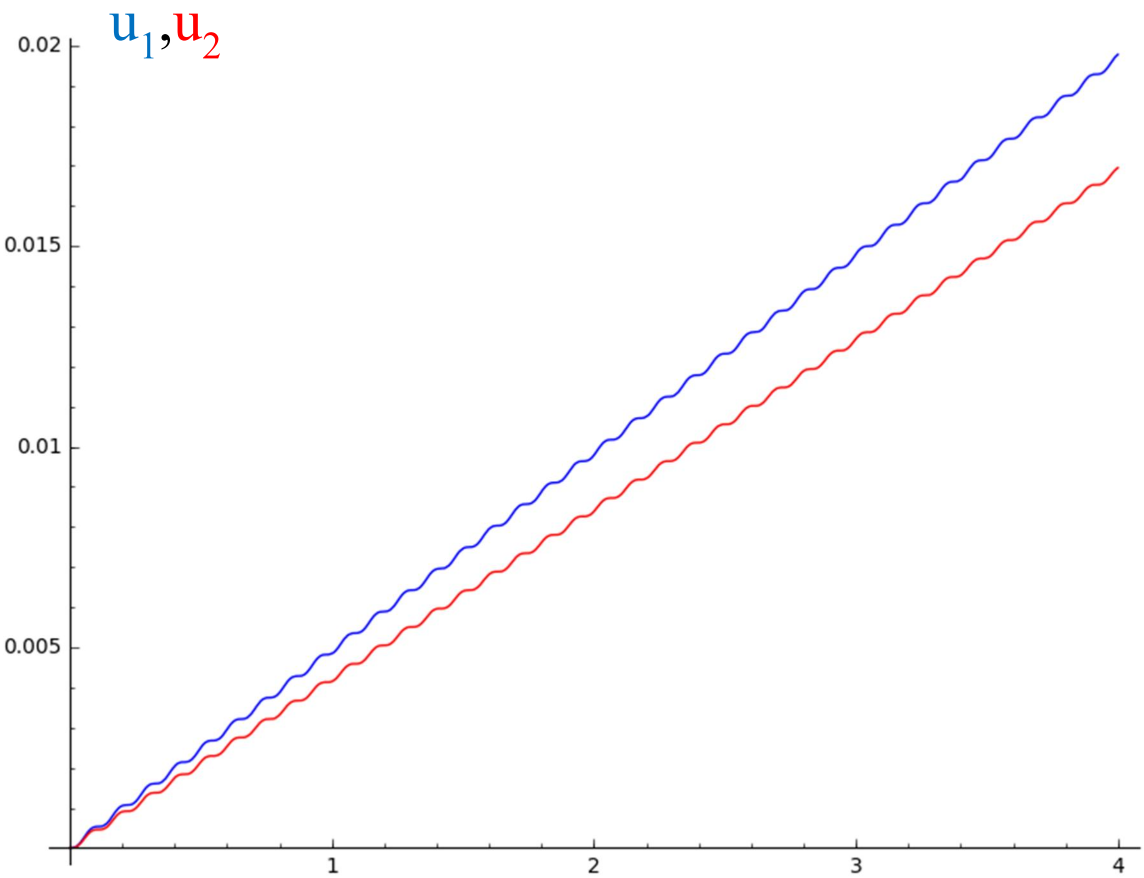
Rješenje: $$q_2(t) = \frac{15,31}{215885,9} \cdot \left[t - \frac{1}{246,3} \cdot \sin(246,3 \cdot t) \right]$$

Ukupni odziv se dobije linearnom kombinacijom dva dobivena rješenja.

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 0,883 \\ 0,883 & -1,000 \end{bmatrix} \begin{Bmatrix} \frac{57,66}{11865,1} \cdot \left[t - \frac{1}{57,7} \cdot \sin(57,7 \cdot t) \right] \\ \frac{15,31}{215885,9} \cdot \left[t - \frac{1}{246,3} \cdot \sin(246,3 \cdot t) \right] \end{Bmatrix}$$

$$u_1(t) = \phi_{11} \cdot q_1(t) + \phi_{12} \cdot q_2(t) = 1 \cdot \frac{57,66}{11865,1} \cdot \left[t - \frac{1}{57,7} \cdot \sin(57,7 \cdot t) \right] + 0,883 \cdot \frac{15,31}{215885,9} \cdot \left[t - \frac{1}{246,3} \cdot \sin(246,3 \cdot t) \right]$$

$$u_2(t) = \phi_{21} \cdot q_1(t) + \phi_{22} \cdot q_2(t) = 0,883 \cdot \frac{57,66}{11865,1} \cdot \left[t - \frac{1}{57,7} \cdot \sin(57,7 \cdot t) \right] - \\ -1,0 \cdot \frac{15,31}{215885,9} \cdot \left[t - \frac{1}{246,3} \cdot \sin(246,3 \cdot t) \right]$$


u_1,u_2
0.02
0.015
0.01
0.005
1
2
3
4

Zadatak 2. Stup

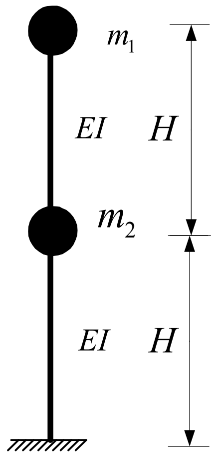
Zadan je konzolni stup visine 8 m s dvije koncentrirane mase.

Odrediti:

- dinamičke stupnjeve slobode,
- vlastite frekvencije i forme (forme prikazati grafički)
- te oscilacije prikazane konstrukcije nastale uslijed početnog pomaka stupa u obliku prvog vlastitog vektora pri čemu je pomak vrha stupa 10 cm.

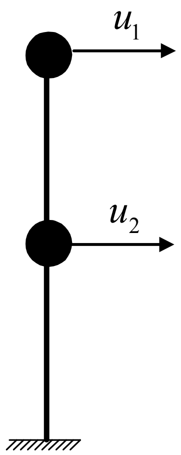
Konstrukcija je prije djelovanja pobude mirovala.

I. Određivanje dinamičkih stupnjeva slobode sustava



$H = 4,0$ m
$m_1 = 2$ t
$m_2 = 4$ t
$E = 3,0 \times 10^7$ kN/m^2
$b/h = 40/40$ cm

$$I = \frac{0,4^4}{12} = 2,133 \cdot 10^{-3} \text{m}^4$$
$$EI = 64000 \text{ kNm}^2$$



Homogena diferencijalna jednadžba gibanja bez prigušenja u općem obliku

$$[m]\{\ddot{u}(t)\} + [k]\{u(t)\} = \{0\}$$

Za konstrukciju s 2 stupnja slobode sustav se sastoji od 2 međusobno zavisne diferencijalne jednadžbe:

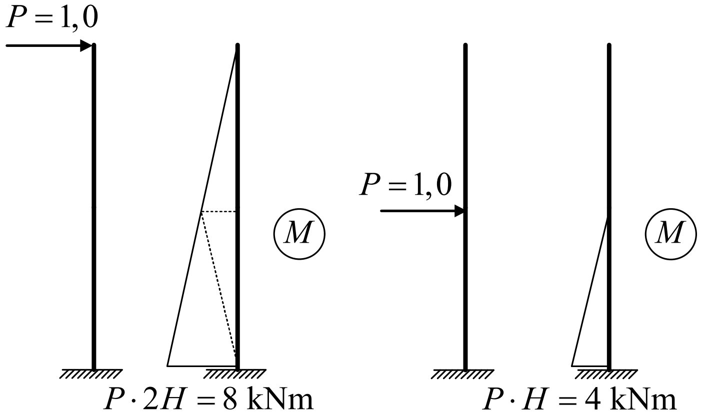
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

II. a) Određivanje koeficijenata matrice masa sustava

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

b) Određivanje koeficijenata krutosti sustava

- budući da se radi o statički određenom sustavu, lakše je odrediti matricu fleksibilnosti pa njenim invertiranjem dobiti matricu krutosti; uzimaju se u obzir samo deformacije od savijanja



$$\delta_{11} = \frac{1}{EI}\left(\frac{1}{2}\cdot 8\cdot 8\cdot\frac{2}{3}\cdot 8\right) = \frac{1}{375}$$

$$\delta_{12} = \frac{1}{EI}\left(\frac{1}{2}\cdot 4\cdot 4\cdot\frac{1}{3}\cdot 4 + \frac{1}{2}\cdot 8\cdot 4\cdot\frac{2}{3}\cdot 4\right) = \frac{1}{1200}$$

$$\delta_{22} = \frac{1}{EI}\left(\frac{1}{2}\cdot 4\cdot 4\cdot\frac{2}{3}\cdot 4\right) = \frac{1}{3000}$$

$$\delta_{21} = \delta_{12}$$

Globalna matrica krutosti sustava:

$$[k] = [F]^{-1} = \frac{1}{\det[F]}\begin{bmatrix} \delta_{22} & -\delta_{12} \\ -\delta_{21} & \delta_{11} \end{bmatrix} = \frac{1}{\frac{1}{375}\cdot\frac{1}{3000} - \left(\frac{1}{1200}\right)^2}\begin{bmatrix} \frac{1}{3000} & -\frac{1}{1200} \\ -\frac{1}{1200} & \frac{1}{375} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{12000}{7} & -\frac{30000}{7} \\ -\frac{30000}{7} & \frac{96000}{7} \end{bmatrix}$$

c) Određivanje vlastitih frekvencija i oblika osciliranje

Pretpostavljeno rješenje za funkcije pomaka

$$\{u(t)\} = [\phi]\{q(t)\} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11}q_1(t) + \phi_{12}q_2(t) \\ \phi_{21}q_1(t) + \phi_{22}q_2(t) \end{Bmatrix}$$

$$\{u_n(t)\} = \{\phi_n\}q_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\left[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right]$$

$$\{\dot{u}_n(t)\} = \{\phi_n\}\dot{q}_n(t) = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\left[-\omega_n A_n\sin(\omega_n t) + \omega_n B_n\cos(\omega_n t)\right]$$

$$\{\ddot{u}_n(t)\} = \{\phi_n\}\ddot{q}_n(t) = \begin{Bmatrix}\phi_{1n}\\ \phi_{2n}\end{Bmatrix}\left[-\omega_n{}^2 A_n\cos(\omega_n t) - \omega_n{}^2 B_n \sin(\omega_n t)\right]$$

uvrštavanjem slijede vlastite frekvencije i oblici titranja.

$$\left(-[m]\{\phi_n\}\omega_n{}^2 + [k]\{\phi_n\}\right)\cdot\left[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right] = \{0\}$$

Slijedi da je:

$$[k]\{\phi_n\} = \omega_n{}^2[m]\{\phi_n\}$$ - problem vlastitih vrijednosti

$$\left([k] - \omega_n{}^2[m]\right)\{\phi_n\} = \{0\}$$

$$\det\left\|[k] - \omega_n{}^2[m]\right\| = 0$$

Raspisivanjem determinante imamo:

$$\left(k_{11} - \omega_n{}^2\cdot m_1\right)\cdot\left(k_{22} - \omega_n{}^2\cdot m_2\right) - k_{12}\cdot k_{21} = 0$$

$$k_{11}\cdot k_{22} - k_{12}{}^2 - \omega_n{}^2\left(k_{11}\cdot m_2 + k_{22}\cdot m_1\right) + \omega_n{}^4 m_1 m_2 = 0 \qquad \text{uz} \qquad \lambda = \omega_n{}^2$$

$$m_1\cdot m_2\cdot\lambda^2 - \left(k_{11}\cdot m_2 + k_{22}\cdot m_1\right)\cdot\lambda + k_{11}\cdot k_{22} - k_{12}{}^2 = 0$$

$$\lambda_{1,2} = \frac{k_{11}\cdot m_2 + k_{22}\cdot m_1 \pm \sqrt{\left(k_{11}\cdot m_2 + k_{22}\cdot m_1\right)^2 - 4\cdot m_1\cdot m_2\cdot\left(k_{11}\cdot k_{22} - k_{12}{}^2\right)}}{2\cdot m_1\cdot m_2}$$

Za zadane vrijednosti imamo vrijednosti:

$$\lambda_1 = \omega_1^2 = 155{,}653 \;\rightarrow\; \omega_1 = 12{,}476\; s^{-1} \;\rightarrow\; \mathrm{T}_1 = 0{,}504\; s$$

$$\lambda_2 = \omega_2^2 = 4130{,}06 \;\rightarrow\; \omega_2 = 64{,}266\; s^{-1} \;\rightarrow\; \mathrm{T}_2 = 0{,}098\; s$$

Vratimo se na početnu jednadžbu problema vlastitih vrijednosti da bismo odredili vlastite oblike

$$\left([k] - \omega_n{}^2[m]\right)\cdot\{\phi_n\} = \{0\} \qquad \left(\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix} - \omega_n{}^2\cdot\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\right)\cdot\begin{Bmatrix}\phi_{1n}\\ \phi_{2n}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$k_{11}\phi_{1n} + k_{12}\phi_{2n} - \omega_n{}^2\cdot m_1\phi_{1n} = 0$$

$$k_{21}\phi_{1n} + k_{22}\phi_{2n} - \omega_n{}^2\cdot m_2\phi_{2n} = 0$$

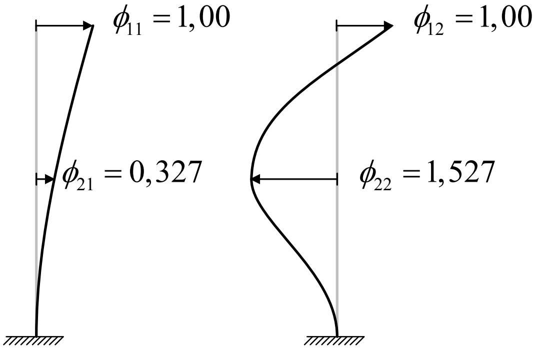
Vlastiti oblici su bezdimenzionalne veličine i određene su do na konstantu. Poznati su nam samo omjeri amplituda pa je potrebno pretpostaviti pojedine veličine.

$$\phi_{1n} = 1{,}0 \qquad \phi_{2n} = -\frac{k_{11} - \omega_n^2 m_1}{k_{12}}$$

za n=1 $\quad \phi_{11} = 1{,}0 \quad \phi_{21} = -\dfrac{12000/7 - 155{,}653\cdot 2}{-30000/7} = 0{,}327$

za n=2 $\quad \phi_{12} = 1{,}0 \quad \phi_{22} = -\dfrac{12000/7 - 4130{,}06\cdot 2}{-30000/7} = -1{,}527$

$$\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1,000 \\ 0,327 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1,000 \\ -1,527 \end{Bmatrix}$$



III. Određivanje odziva konstrukcije na zadani početni pomak u obliku prvog vlastitog vektora s pomakom vrha od 10 cm

$$\{u(t)\} = \{\phi_1\} q_1(t) + ... + \{\phi_N\} q_N(t) = \sum_{n=1}^{N} \{\phi_n\} q_n(t) = \sum_{n=1}^{N} \{\phi_n\} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right]$$

$$\{\dot{u}(t)\} = \sum_{n=1}^{N} \{\phi_n\} \dot{q}_n(t) = \sum_{n=1}^{N} \{\phi_n\} \omega_n \left[-A_n \sin(\omega_n t) + B_n \cos(\omega_n t) \right]$$

Za $\{u(0)\} = \begin{Bmatrix} \phi_{11} * \\ \phi_{21} * \end{Bmatrix} = \begin{Bmatrix} 1 \cdot 0,1 \\ 0,327 \cdot 0,1 \end{Bmatrix} = \begin{Bmatrix} 0,1 \\ 0,0327 \end{Bmatrix}$:

$$A_1 = \frac{\{\phi_1\}^T [m] \{u(0)\}}{\{\phi_1\}^T [m] \{\phi_1\}} = \frac{\{1,0\ 0,327\} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} 0,1 \\ 0,0327 \end{Bmatrix}}{\{1,0\ 0,327\} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} 1,0 \\ 0,327 \end{Bmatrix}} = \frac{0,2428}{2,428} = 0,1$$

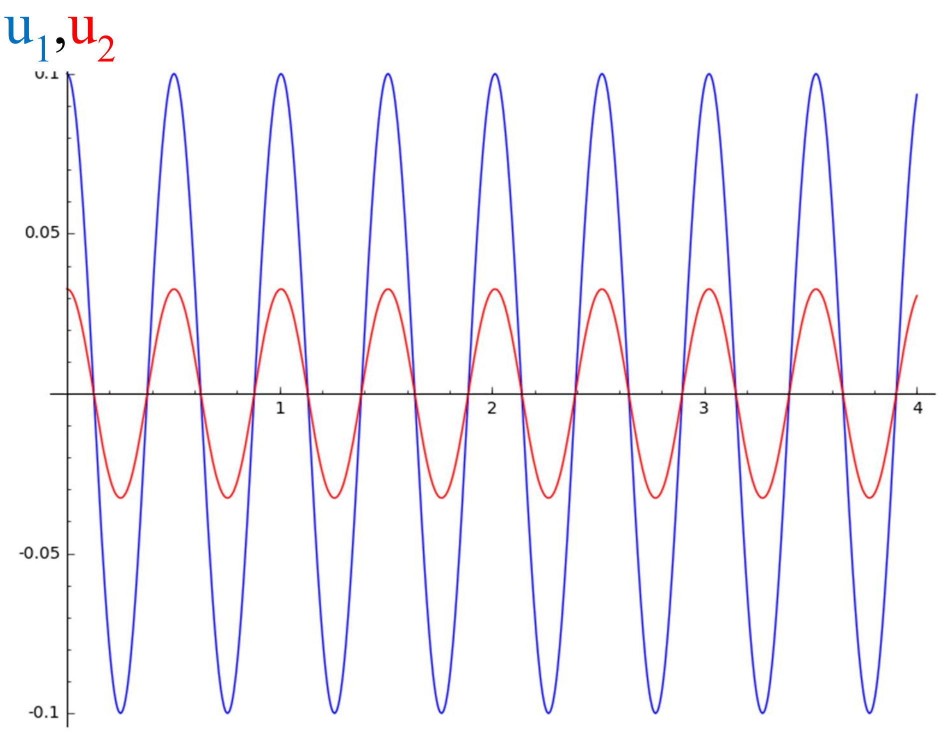
$$A_2 = \frac{\{\phi_2\}^T [m] \{u(0)\}}{\{\phi_2\}^T [m] \{\phi_2\}} == \frac{\{1,0\ -1,527\} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} 0,1 \\ 0,0327 \end{Bmatrix}}{\{1,0\ -1,527\} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} 1,0 \\ -1,527 \end{Bmatrix}} = \frac{0,000268}{11,327} = 2,37 \cdot 10^{-5}$$

Za $\{\dot{u}(0)\} = \{0\}$: $B_1 = \dfrac{1}{\omega_1} \dfrac{\{\phi_1\}^T [m] \{\dot{u}(0)\}}{\{\phi_1\}^T [m] \{\phi_1\}} = 0 \quad i \quad B_2 = \dfrac{1}{\omega_2} \dfrac{\{\phi_2\}^T [m] \{\dot{u}(0)\}}{\{\phi_2\}^T [m] \{\phi_2\}} = 0$

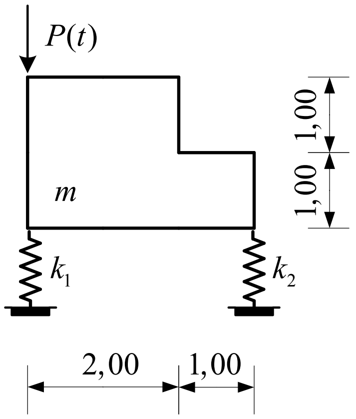
Ukupni odziv:

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \cdot \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \left[A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \right] + \phi_{12} \left[A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \right] \\ \phi_{21} \left[A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \right] + \phi_{22} \left[A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \right] \end{Bmatrix} =$$

$$= \begin{Bmatrix} 1,000 \cdot 0,1 \cos(12,476t) + 1,000 \cdot 2,37 \cdot 10^{-5} \cos(64,266t) \\ 0,327 \cdot 0,1 \cos(12,476t) \underbrace{- 1,527 \cdot 2,37 \cdot 10^{-5} \cos(64,266t)}_{\approx 0!!!\ \text{nema utjecaja 2. oblika titranja}} \end{Bmatrix}$$


u1,u2
0.1
0.05
-0.05
-0.1
1
2
3
4

Zadatak 3. Temelj



$m = 2\ \text{t/m}^2$

$k_1 = 20000\ \text{kN/m}$

$k_2 = 10000\ \text{kN/m}$

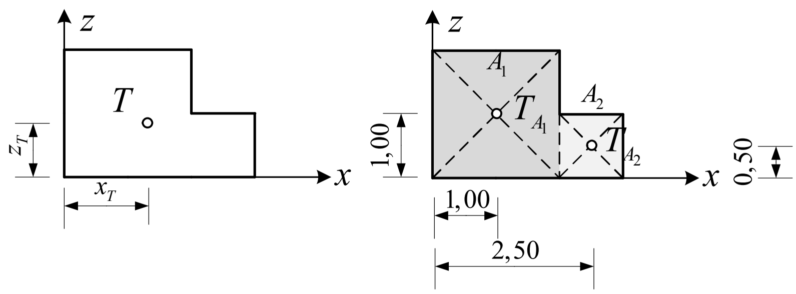
$P(t) = 50\sin(10t)\ [\text{kN}]$

Za zadani temelj u ravnini, oslonjen na elastičnu podlogu, odrediti:

- dinamičke stupnjeve slobode,
- vlastite frekvencije i forme (forme prikazati grafički)
- te oscilacije prikazane konstrukcije nastale uslijed djelovanja pobude.

Konstrukcija je prije djelovanja pobude mirovala.

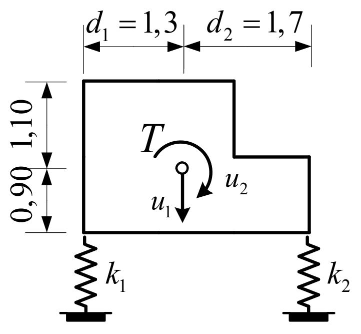
I. Određivanje centra masa (težišta)



$$x_T = \frac{\sum A_i x_i}{\sum A_i} = \frac{4\cdot 1 + 1\cdot 2{,}5}{5} = 1{,}3\ \text{m}$$

$$z_T = \frac{\sum A_i z_i}{\sum A_i} = \frac{4\cdot 1 + 1\cdot 0{,}5}{5} = 0{,}9\ \text{m}$$

II. Određivanje dinamičkih stupnjeva slobode sustava (u težištu!)



Homogena diferencijalna jednadžba gibanja bez prigušenja u općem obliku

$$[m]\{\ddot{u}(t)\} + [k]\{u(t)\} = \{0\}$$

Za konstrukciju s 2 stupnja slobode sustav se sastoji od 2 međusobno zavisne diferencijalne jednadžbe:

$$\begin{bmatrix} m_1 & 0 \\ 0 & I_t \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

III. **a) Određivanje matrice masa sustava**

$$m_1 = m_{A_1} + m_{A_2} = 2 \cdot (2 \cdot 2) + 2 \cdot (1 \cdot 1) = 10 \text{ t}$$

$$I_t = m_{A_1} \cdot \left(\frac{a_{A_1}^{\ 2} + b_{A_1}^{\ 2}}{12} + d_{A_1}^{\ 2} \right) + m_{A_2} \cdot \left(\frac{a_{A_2}^{\ 2} + b_{A_2}^{\ 2}}{12} + d_{A_2}^{\ 2} \right) =$$

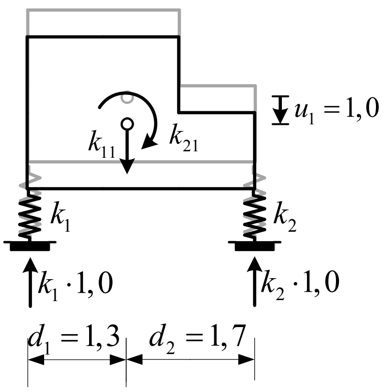
$$= 8 \cdot \left(\frac{2^2 + 2^2}{12} + 0,3^2 + 0,1^2 \right) + 2 \cdot \left(\frac{1^2 + 1^2}{12} + 1,2^2 + 0,4^2 \right) =$$

$$= 9,667 \text{ tm}^2$$

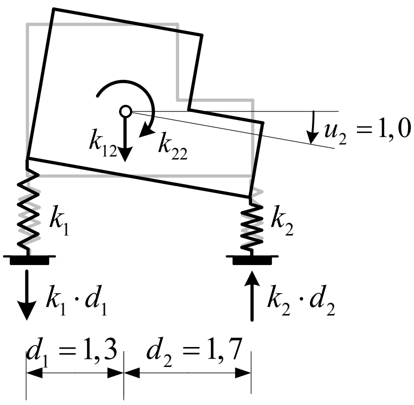
$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & I_t \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 9,667 \end{bmatrix}$$

b) Određivanje matrice krutosti sustava

Slučaj 1: u_1=1,0; u_2=0



Slučaj 1: u_2=1,0; u_1=0



$$\sum F_z = 0$$
$$k_{11} - k_1 - k_2 = 0$$
$$k_{11} = k_1 + k_2 = 30000 \text{ kN/m}$$

$$\sum M_t = 0$$
$$k_{21} + k_1 \cdot 1,0 \cdot d_1 - k_2 \cdot 1,0 \cdot d_2 = 0$$
$$k_{21} = k_2 \cdot 1,0 \cdot d_2 - k_1 \cdot 1,0 \cdot d_1$$
$$= 10000 \cdot 1,7 - 20000 \cdot 1,3$$
$$= -9000 \text{ kN}$$

$$\sum F_z = 0$$
$$k_{12} + k_1 \cdot d_1 - k_2 \cdot d_2 = 0$$
$$k_{12} = k_2 \cdot d_2 - k_1 \cdot d_1 = -9000 \text{ kN}$$

$$\sum M_t = 0$$
$$k_{22} - k_1 \cdot d_1 \cdot d_1 - k_2 \cdot d_2 \cdot d_2 = 0$$
$$k_{22} = k_1 \cdot d_1^{\ 2} + k_2 \cdot d_2^{\ 2} =$$
$$= 20000 \cdot 1,7^2 + 10000 \cdot 1,3^2$$
$$= 62700 \text{ kNm}$$

Globalna matrica krutosti sustava:

$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} 30000 & -9000 \\ -9000 & 62700 \end{bmatrix}$$

c) Određivanje vlastitih frekvencija i oblika osciliranje

Pretpostavljeno rješenje za funkcije pomaka

$$\{u(t)\}=[\phi]\{q(t)\}=\begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}\begin{Bmatrix}q_1(t)\\ q_2(t)\end{Bmatrix}=\begin{Bmatrix}\phi_{11}q_1(t)+\phi_{12}q_2(t)\\ \phi_{21}q_1(t)+\phi_{22}q_2(t)\end{Bmatrix}=\begin{Bmatrix}u_{11}(t)+u_{12}(t)\\ u_{21}(t)+u_{22}(t)\end{Bmatrix}$$

$$\{u(t)\}=\begin{Bmatrix}u_1(t)\\ u_2(t)\end{Bmatrix},\quad \{q(t)\}=\begin{Bmatrix}q_1(t)\\ q_2(t)\end{Bmatrix}=\begin{Bmatrix}A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t)\\ A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t)\end{Bmatrix}=A_n\cos(\omega_n t)+B_n\sin(\omega_n t)$$

$$\{u_n(t)\}=\{\phi_n\}q_n(t)=\begin{Bmatrix}\phi_{1n}\\ \phi_{2n}\end{Bmatrix}\left[A_n\cos(\omega_n t)+B_n\sin(\omega_n t)\right]$$

$$\{\dot{u}_n(t)\}=\{\phi_n\}\dot{q}_n(t)=\begin{Bmatrix}\phi_{1n}\\ \phi_{2n}\end{Bmatrix}\left[-\omega_n A_n\sin(\omega_n t)+\omega_n B_n\cos(\omega_n t)\right]$$

$$\{\ddot{u}_n(t)\}=\{\phi_n\}\ddot{q}_n(t)=\begin{Bmatrix}\phi_{1n}\\ \phi_{2n}\end{Bmatrix}\left[-\omega_n^2 A_n\cos(\omega_n t)-\omega_n^2 B_n\sin(\omega_n t)\right]$$

uvrštavanjem slijede vlastite frekvencije i oblici titranja.

$$\left(-[m]\{\phi_n\}\omega_n^2+[k]\{\phi_n\}\right)\cdot\left[A_n\cos(\omega_n t)+B_n\sin(\omega_n t)\right]=\{0\}$$

Slijedi da je:

$[k]\{\phi_n\}=\omega_n^2[m]\{\phi_n\}$ - problem vlastitih vrijednosti

$$\left([k]-\omega_n^2[m]\right)\{\phi_n\}=\{0\}$$

$$\det\left|[k]-\omega_n^2[m]\right|=0$$

Raspisivanjem determinante imamo:

$$\left(k_{11}-\omega_n^2\cdot m_1\right)\cdot\left(k_{22}-\omega_n^2\cdot I_t\right)-k_{12}\cdot k_{21}=0$$

$$k_{11}\cdot k_{22}-k_{12}^2-\omega_n^2\left(k_{11}\cdot I_t+k_{22}\cdot m_1\right)+\omega_n^4 m_1 I_t=0 \qquad \text{uz} \qquad \lambda=\omega_n^2$$

$$m_1\cdot I_t\cdot\lambda^2-\left(k_{11}\cdot I_t+k_{22}\cdot m_1\right)\cdot\lambda+k_{11}\cdot k_{22}-k_{12}^2=0$$

$$\lambda_{1,2}=\frac{k_{11}\cdot I_t+k_{22}\cdot m_1\pm\sqrt{\left(k_{11}\cdot I_t+k_{22}\cdot m_1\right)^2-4\cdot m_1\cdot I_t\cdot\left(k_{11}\cdot k_{22}-k_{12}^2\right)}}{2\cdot m_1\cdot I_t}$$

Za zadane vrijednosti imamo vrijednosti:

$$\lambda_{1,2}=\frac{917010\pm 380635{,}44}{193{,}34}$$

$$\lambda_1 = \omega_1^2 = 2774,26 \;\rightarrow\; \omega_1 = 52,671\, s^{-1} \;\rightarrow\; T_1 = 0,119\, s$$

$$\lambda_2 = \omega_2^2 = 6711,73 \;\rightarrow\; \omega_2 = 81,925\, s^{-1} \;\rightarrow\; T_2 = 0,077\, s$$

Vratimo se na početnu jednadžbu problema vlastitih vrijednosti da bismo odredili vlastite oblike

$$\left([k] - \omega_n^2 [m]\right) \cdot \{\phi_n\} = \{0\} \qquad \left(\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_n^2 \cdot \begin{bmatrix} m_1 & 0 \\ 0 & I_t \end{bmatrix}\right) \cdot \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_{11}\phi_{1n} + k_{12}\phi_{2n} - \omega_n^2 \cdot m_1 \phi_{1n} = 0$$

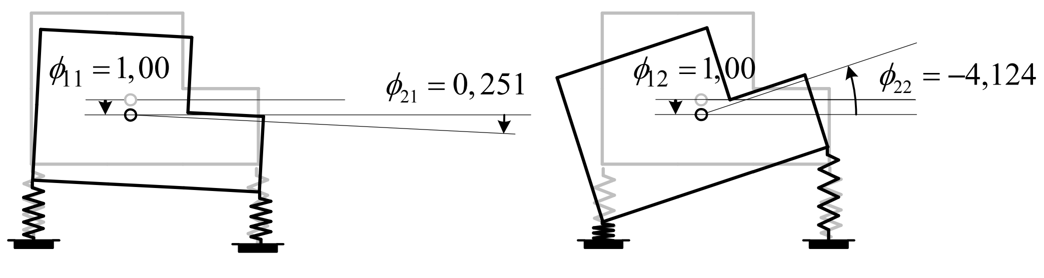
$$k_{21}\phi_{1n} + k_{22}\phi_{2n} - \omega_n^2 \cdot I_t \phi_{2n} = 0$$

Vlastiti oblici su bezdimenzionalne veličine i određene su do na konstantu. Poznati su nam samo omjeri amplituda pa je potrebno pretpostaviti pojedine veličine.

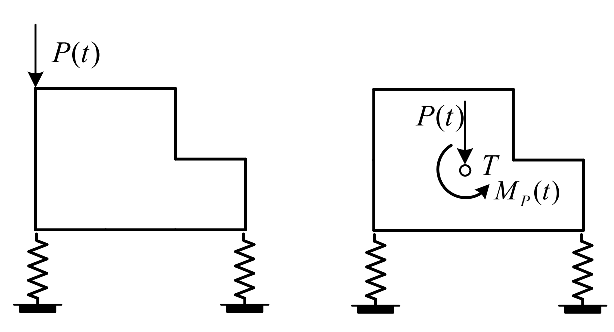
$$\phi_{1n} = 1,0 \qquad \phi_{2n} = -\frac{k_{11} - \omega_n^2 m_1}{k_{12}}$$

$$\phi_{21} = -\frac{30000 - 2774,26 \cdot 10}{-9000} = 0,251 \qquad \phi_{22} = -\frac{30000 - 6711,73 \cdot 10}{-9000} = -4,124$$

$$\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1,000 \\ 0,251 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1,000 \\ -4,124 \end{Bmatrix}$$



IV. Određivanje odziva konstrukcije na zadanu pobudu



$$\{p(t)\} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} = \begin{Bmatrix} P(t) \\ M_P(t) \end{Bmatrix} = \begin{Bmatrix} 50\sin(10t) \\ -50\sin(10t) \cdot 1,3 \end{Bmatrix} = \begin{Bmatrix} 50\sin(10t) \\ -65\sin(10t) \end{Bmatrix}$$

Modalna analiza

Diferencijalna jednadžba uz pretpostavljeno rješenje

$$[m]\cdot[\phi]\cdot\{\ddot{q}(t)\}+[k]\cdot[\phi]\cdot\{q(t)\}=\{p(t)\} \qquad /\cdot\{\phi_n\}^T$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & I_t \end{bmatrix}\cdot\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}\cdot\begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix}+\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\cdot\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}\cdot\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}=\begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \qquad /\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T$$

Uvjeti ortogonalnosti: $\{\phi_i\}^T\cdot[m]\cdot\{\phi_j\}=0, \quad \{\phi_i\}^T\cdot[k]\cdot\{\phi_j\}=0 \quad \forall\ i\neq j$

Dobivamo sustav neovisnih diferencijalnih jednadžbi

$$\{\phi_n\}^T\cdot[m]\cdot\{\phi_n\}\cdot\ddot{q}_n(t)+\{\phi_n\}^T\cdot[k]\cdot\{\phi_n\}\cdot q_n(t)=\{\phi_n\}^T\cdot\{p(t)\}$$

$$\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{bmatrix} m_1 & 0 \\ 0 & I_t \end{bmatrix}\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\cdot\ddot{q}_n(t)+\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}\cdot q_n(t)=\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

$M_n\cdot\ddot{q}_n(t)+K_n\cdot q_n(t)=P_n(t)$ gdje su modalni koeficijenti:

$$M_n=\{\phi_n\}^T\cdot[m]\cdot\{\phi_n\},\quad K_n=\{\phi_n\}^T\cdot[k]\cdot\{\phi_n\},\quad P_n(t)=\{\phi_n\}^T\cdot\{p(t)\}$$

Kod dva stupnja slobode nam ostaju dvije neovisne diferencijalne jednadžbe

$$M_1\cdot\ddot{q}_1(t)+K_1\cdot q_1(t)=P_1(t)$$
$$M_2\cdot\ddot{q}_2(t)+K_2\cdot q_2(t)=P_2(t)$$

$$M_n=\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot[m]\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix},\quad K_n=\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot[k]\cdot\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix},\quad P_n(t)=\begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \end{Bmatrix}^T\cdot\begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

Za zadane vrijednosti imamo:

$$M_1=\{\phi_1\}^T[m]\{\phi_1\}=\{1\quad 0,251\}\begin{bmatrix} 10 & 0 \\ 0 & 9,667 \end{bmatrix}\begin{Bmatrix} 1 \\ 0,251 \end{Bmatrix}=10,609$$

$$M_2=\{\phi_2\}^T[m]\{\phi_2\}=\{1\quad -4,124\}\begin{bmatrix} 10 & 0 \\ 0 & 9,667 \end{bmatrix}\begin{Bmatrix} 1 \\ -4,124 \end{Bmatrix}=174,41$$

$$K_1=\omega_1^2M_1=29432,124$$

$$K_2=\omega_2^2M_2=1170592,829$$

$$P_1=\{\phi_1\}^T\{p(t)\}=\{1\quad 0,251\}\begin{Bmatrix} 50\sin(10t) \\ -65\sin(10t) \end{Bmatrix}=33,685\sin(10t)$$

$$P_2 = \{\phi_2\}^T \{p(t)\} = \{1 \quad -4{,}124\} \begin{Bmatrix} 50\sin(10t) \\ -65\sin(10t) \end{Bmatrix} = 318{,}06\sin(10t)$$

Potrebno je riješiti dvije neovisne diferencijalne jednadžbe sa konstantnim koeficijentima (pogledati prethodno izvedena rješenja za traženi oblik pobude kod jednog stupnja slobode).

1. Diferencijalna jednadžba

$$10{,}609 \cdot \ddot{q}_1(t) + 29432{,}124 \cdot q_1(t) = 33{,}685\sin(10t)$$

...izvesti odziv!!! *(pogledati u nastavku)

Rješenje:

$$q_1(t) = \frac{33{,}685}{29432{,}124 \cdot \left(1 - \frac{10^2}{52{,}671^2}\right)} \cdot \left[\sin(10 \cdot t) - \frac{10}{52{,}671} \cdot \sin(52{,}671 \cdot t)\right]$$
$$= 1{,}187 \cdot 10^{-3} \cdot \left[\sin(10 \cdot t) - 0{,}19 \cdot \sin(52{,}671 \cdot t)\right]$$

2. Diferencijalna jednadžba

$$174{,}41 \cdot \ddot{q}_2(t) + 1170592{,}83 \cdot q_2(t) = 318{,}06 \cdot \sin(10 \cdot t)$$

...izvesti odziv!!! *(pogledati u nastavku)

Rješenje:

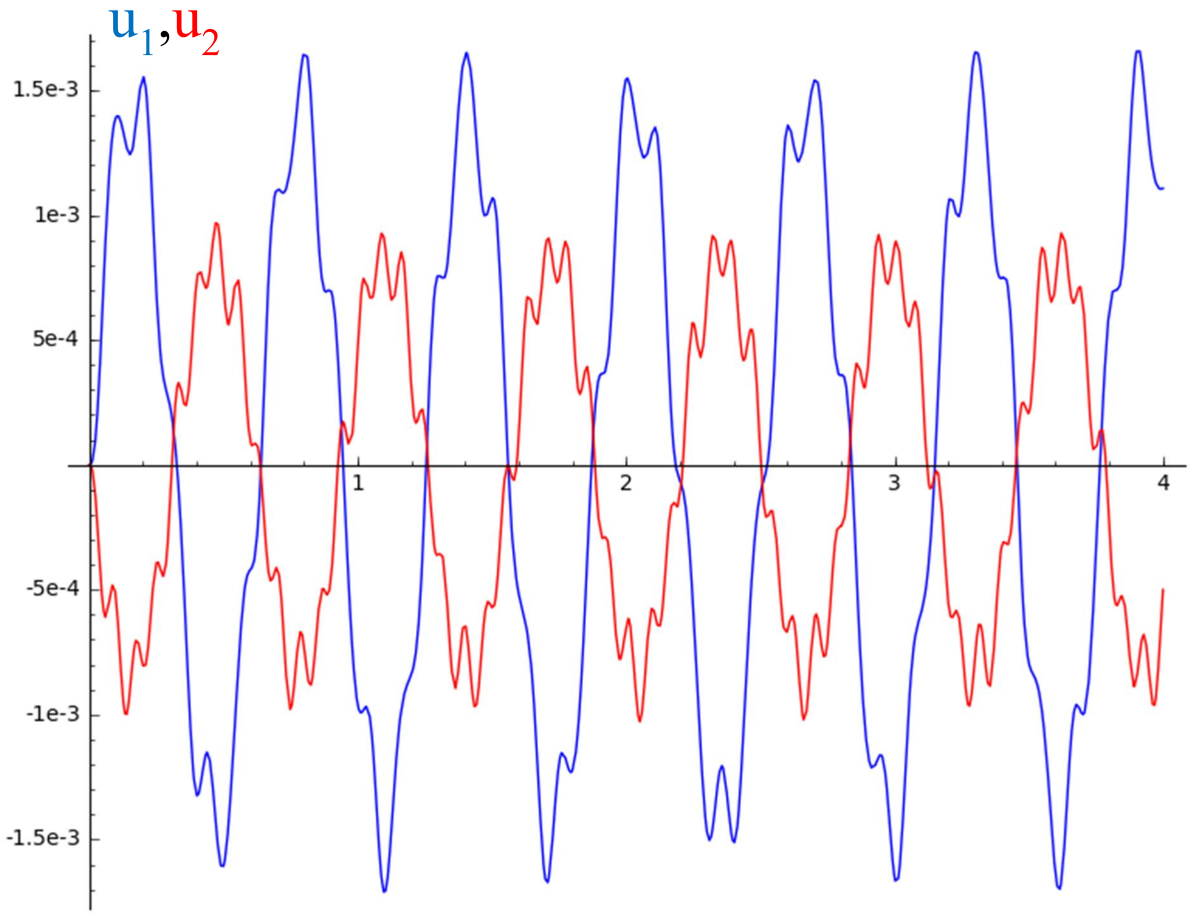
$$q_2(t) = \frac{318{,}06}{1170592{,}83 \cdot \left(1 - \frac{10^2}{81{,}925^2}\right)} \cdot \left[\sin(10 \cdot t) - \frac{10}{81{,}925} \cdot \sin(81{,}925 \cdot t)\right]$$
$$= 2{,}758 \cdot 10^{-4} \cdot \left[\sin(10 \cdot t) - 0{,}12 \cdot \sin(81{,}925 \cdot t)\right]$$

Ukupni odziv se dobije linearnom kombinacijom dva dobivena rješenja.

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0{,}251 & -4{,}124 \end{bmatrix} \begin{Bmatrix} 1{,}187 \cdot 10^{-3} \cdot \left[\sin(10 \cdot t) - 0{,}19 \cdot \sin(52{,}671 \cdot t)\right] \\ 2{,}758 \cdot 10^{-4} \cdot \left[\sin(10 \cdot t) - 0{,}12 \cdot \sin(81{,}925 \cdot t)\right] \end{Bmatrix}$$

$$u_1(t) = \phi_{11} \cdot q_1(t) + \phi_{12} \cdot q_2(t) = 1{,}187 \cdot 10^{-3} \cdot \left[\sin(10 \cdot t) - 0{,}19 \cdot \sin(52{,}671 \cdot t)\right] -$$
$$+ 1 \cdot 2{,}758 \cdot 10^{-4} \cdot \left[\sin(10 \cdot t) - 0{,}12 \cdot \sin(81{,}925 \cdot t)\right]$$

$$u_2(t) = \phi_{21} \cdot q_1(t) + \phi_{22} \cdot q_2(t) = 0{,}251 \cdot 1{,}187 \cdot 10^{-3} \cdot \left[\sin(10 \cdot t) - 0{,}19 \cdot \sin(52{,}671 \cdot t)\right] -$$
$$- 4{,}124 \cdot 2{,}758 \cdot 10^{-4} \cdot \left[\sin(10 \cdot t) - 0{,}12 \cdot \sin(81{,}925 \cdot t)\right]$$



Ako se traži pomak u nekom trenutku *t*, paziti da se vrijednosti uvrštavaju u radijanima, a ne stupnjevima!!!

Npr. vertikalni pomak težišta temelja: $u_1\left(t=3\ \mathrm{s}\right)=-0{,}00165\ \mathrm{m}$

* izvod odziva na sinusnu pobudu

$$M \cdot \ddot{q}(t) + K \cdot q(t) = P_0 \cdot \sin(\omega \cdot t)$$
$$q(t) = q_{\text{hom}}(t) + q_{\text{part}}(t)$$

$$q_{\text{hom}}(t) = A \cdot \cos(\omega_n \cdot t) + B \cdot \sin(\omega_n \cdot t)$$
$$q_{\text{part}}(t) \quad \rightarrow \quad \text{iz} \quad m \cdot \ddot{q}_{\text{part}}(t) + k \cdot q_{\text{part}}(t) = P_0 \cdot \sin(\omega \cdot t)$$

pretpostavka: $q_{\text{part}}(t) = C \cdot \sin(\omega \cdot t)$

$$\dot{q}_{\text{part}}(t) = C \cdot \omega \cdot \cos(\omega \cdot t)$$
$$\ddot{q}_{\text{part}}(t) = -C \cdot \omega^2 \cdot \sin(\omega \cdot t)$$

uvrštavanje u : $m \cdot \ddot{q}_{\text{part}}(t) + k \cdot q_{\text{part}}(t) = P_0 \cdot \sin(\omega \cdot t)$

$$-m \cdot C \cdot \omega^2 \cdot \cancel{\sin(\omega \cdot t)} + k \cdot C \cdot \cancel{\sin(\omega \cdot t)} = P_0 \cdot \cancel{\sin(\omega \cdot t)}$$
$$-m \cdot C \cdot \omega^2 + k \cdot C = P_0$$
$$C = \frac{P_0}{k - m \cdot \omega^2} = \frac{P_0}{k - \frac{k}{\omega_n^2} \cdot \omega^2} = \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \rightarrow q_{\text{part}}(t) = \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \sin(\omega \cdot t)$$

$$q(t) = q_{\text{hom}}(t) + q_{\text{part}}(t)$$
$$= A \cdot \cos(\omega_n \cdot t) + B \cdot \sin(\omega_n \cdot t) + \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \sin(\omega \cdot t)$$
$$\dot{q}(t) = -A \cdot \omega_n \cdot \sin(\omega_n \cdot t) + B \cdot \omega_n \cdot \cos(\omega_n \cdot t) + \frac{P_0 \cdot \omega}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \cos(\omega \cdot t)$$

početni uvjeti: $q(t=0) = 0$ i $\dot{q}(t=0) = 0$

$$q(t=0) = A \cdot \cos(\omega_n \cdot 0) + B \cdot \sin(\omega_n \cdot 0) + \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \sin(\omega \cdot 0) = 0$$
$$= A \cdot 1{,}0 = 0 \rightarrow A = 0$$
$$\dot{q}(t=0) = -A \cdot \omega_n \cdot \sin(\omega_n \cdot 0) + B \cdot \omega_n \cdot \cos(\omega_n \cdot 0) + \frac{P_0 \cdot \omega}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \cos(\omega \cdot 0) = 0$$
$$= B \cdot \omega_n \cdot 1{,}0 + \frac{P_0 \cdot \omega}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} = 0 \rightarrow B = -\frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \frac{\omega}{\omega_n}$$

$$q(t) = A \cdot \cos(\omega_n \cdot t) + B \cdot \sin(\omega_n \cdot t) + \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \sin(\omega \cdot t)$$
$$= -\frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right) \cdot} \cdot \frac{\omega}{\omega_n} \cdot \sin(\omega_n \cdot t) + \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \sin(\omega \cdot t)$$
$$= \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \cdot \left[\sin(\omega \cdot t) - \frac{\omega}{\omega_n} \cdot \sin(\omega_n \cdot t)\right]$$