Prisilne oscilacije sustava s jednim stupnjem slobode
(bez prigušenja)

SADRŽAJ

Prisilne oscilacije sustava s jednim stupnjem slobode (bez prigušenja)

Diferencijalna jednadžba gibanja u općem obliku

- s prigušenjem: $$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) + k \cdot u(t) = p(t) \quad (1)$$
- bez prigušenja: $$m \cdot \ddot{u}(t) + k \cdot u(t) = p(t) \quad (2)$$

a) Analitičko rješenje

Za pobude koje mogu biti opisane glatkom (neprekidno derivabilnom) funkcijom $P(t)$ moguće je jednostavno naći analitičko rješenje diferencijalne jednadžbe gibanja izražene jednadžbom (2).

Ukupno traženo rješenje može se prikazati kao zbroj komplementarnog rješenja $u_c(t)$ i partikularnog rješenja $u_p(t)$:

$$u(t) = u_c(t) + u_p(t) \quad (3)$$

Komplementarno rješenje $u_c(t)$ je rješenje homogene jednadžbe (4) kojom su opisane slobodne oscilacije sustava s jednim stupnjem slobode bez prigušenja i može se pretpostaviti u obliku danom jednadžbom (5).

$$m \cdot \ddot{u}_c(t) + k \cdot u_c(t) = 0 \quad (4)$$

$$u_c(t) = A \cdot \cos(\omega_n \cdot t) + B \cdot \sin(\omega_n \cdot t) \quad (5)$$

Kružna frekvencija gibanja ω_n dana je izrazom (6), te povezuje krutost k i masu m sustava. Odnos kružne frekvencije ω_n i perioda osciliranja T_n izražen je jednadžbom (7).

$$\omega_n^2 = \frac{k}{m} \quad (6)$$

$$\omega_n = \frac{2\pi}{T_n} \quad (7)$$

Partikularno rješenje $u_p(t)$ uvijek ima oblik funkcije pobude $P(t)$ i mora zadovoljiti opći oblik diferencijalne jednadžbe gibanja (2), stoga vrijedi izraz (8).

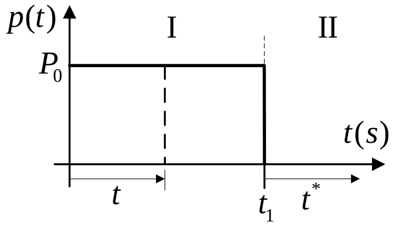
$$m \cdot \ddot{u}_p(t) + k \cdot u_p(t) = p(t) \quad (8)$$

Konstante A i B mogu se odrediti iz početnih uvjeta, danih izrazima (9) i (10), koji određuju pomak i brzinu na početku gibanja, odnosno u trenutku $t = t_0$. U dinamici konstrukcija početni uvjeti su često homogeni, odnosno početni pomak i brzina su tada jednaki nula.

$$u(t = t_0) = u_0 \quad (9)$$

$$\dot{u}(t = t_0) = \dot{u}_0 \quad (10)$$

1. Konstantna pobuda



$$P(t)=\begin{cases} P_0,\ t\in\langle 0,t_1\rangle \\ 0,\ t\in\langle t_1,\infty\rangle \end{cases}$$

I. INTERVAL: $t\in\langle 0,t_1\rangle$

Diferencijalna jednadžba

- bez prigušenja: $$m\cdot\ddot{u}(t)+k\cdot u(t)=P_0 \tag{11}$$

Pretpostavljeno ukupno rješenje:

$$u(t)=A\cdot\cos(\omega_n\cdot t)+B\cdot\sin(\omega_n\cdot t)+u_p(t) \tag{12}$$

Pretpostavljeno partikularno rješenje mora imati oblik funkcije pobude *p*(*t*) (konstantne funkcije):

$$u_p(t)=C \tag{13}$$

$$\ddot{u}_p(t)=0 \tag{14}$$

Partikularno rješenje može se odrediti iz uvjeta da zadovoljava opći oblik diferencijalne jednadžbe (15):

$$m\cdot\ddot{u}_p+k\cdot u_p=P_0 \tag{15}$$

$$m\cdot 0+k\cdot C=P_0 \quad\Rightarrow\quad C=\frac{P_0}{k} \tag{16}$$

Zatim traženo rješenje (pomak) i odgovarajuća derivacija (brzina) primaju oblik prikazan izrazima (17) i (18):

$$u(t)=A\cdot\cos(\omega_n\cdot t)+B\cdot\sin(\omega_n\cdot t)+\frac{P_0}{k} \tag{17}$$

$$\dot{u}(t)=-A\cdot\omega_n\sin(\omega_n\cdot t)+B\cdot\omega_n\cdot\cos(\omega_n\cdot t) \tag{18}$$

Za početne uvjete u trenutku $t_0 = 0$ konstante A i B mogu se odrediti iz jednadžbe (17) odnosno (18):

$$u(t=0)=u_0 \quad\Rightarrow\quad u_0=A\cdot 1+B\cdot 0+\frac{P_0}{k} \quad\Rightarrow\quad A=u_0-\frac{P_0}{k} \tag{19}$$

$$\dot{u}(t=0)=\dot{u}_0 \quad\Rightarrow\quad \dot{u}_0=-A\cdot\omega_n\cdot 0+B\cdot\omega_n\cdot 1 \quad\Rightarrow\quad B=\frac{\dot{u}_0}{\omega_n} \tag{20}$$

Isto tako, za homogene početne uvjete vrijedi:

$$u(t=0)=0 \quad\Rightarrow\quad 0=A\cdot 1+B\cdot 0+\frac{P_0}{k} \quad\Rightarrow\quad A=-\frac{P_0}{k} \tag{21}$$

$$\dot{u}(t=0)=0 \quad\Rightarrow\quad 0=-A\cdot\omega_n\cdot 0+B\cdot\omega_n\cdot 1 \quad\Rightarrow\quad B=0 \tag{22}$$

Konačno rješenje u obliku funkcije pomaka u vremenu za I. interval dano je izrazom (23), odnosno izrazom (24) za homogene početne uvjete.

$$u(t) = u_0 \cdot \cos(\omega_n \cdot t) + \frac{\dot{u}_0}{\omega_n} \cdot \sin(\omega_n \cdot t) + \frac{P_0}{k} \cdot \left[1 - \cos(\omega_n \cdot t)\right] \tag{23}$$

$$u(t) = \frac{P_0}{k} \cdot \left[1 - \cos(\omega_n \cdot t)\right] \tag{24}$$

Odgovarajuće derivacije, odnosno funkcije brzine u vremenu, dane su izrazima (25) i (26).

$$\dot{u}(t) = -u_0 \cdot \omega_n \cdot \sin(\omega_n \cdot t) + \dot{u}_0 \cdot \cos(\omega_n \cdot t) + \frac{P_0 \cdot \omega_n}{k} \cdot \sin(\omega_n \cdot t) \tag{25}$$

$$\dot{u}(t) = \frac{P_0 \cdot \omega_n}{k} \cdot \sin(\omega_n \cdot t) \tag{26}$$

II. INTERVAL: $t \in \langle t_1, \infty \rangle$, $t^* \in \langle 0, \infty \rangle$

Na početku II. intervala, prestankom djelovanja pobude, sustav počinje slobodno oscilirati, uz nehomogene početne uvjete koji odgovaraju pomaku i brzini na kraju I. intervala. Zbog jednostavnosti, gibanje se može promatrati u pomaknutom koordinatnom sustavu $t^* = t - t_1$. Slobodne oscilacije sustava s jednim stupnjem slobode bez prigušenja opisane su jednadžbom (27).

$$m \cdot \ddot{u}(t) + k \cdot u(t) = 0 \tag{27}$$

Stoga je pretpostavljeno rješenje (pomak) određeno izrazom (28), odnosno (29), a njegova derivacija (brzina) izrazom (30).

$$u(t^*) = A_0 \cdot \cos(\omega_n \cdot t^*) + B_0 \cdot \sin(\omega_n \cdot t^*) \tag{28}$$

$$u(t) = A_0 \cdot \cos\left[\omega_n (t - t_1)\right] + B_0 \sin\left[\omega_n (t - t_1)\right] \tag{29}$$

$$\dot{u}(t) = -A_0 \cdot \omega_n \cdot \sin\left[\omega_n \cdot (t - t_1)\right] + B_0 \cdot \omega_n \cdot \cos\left[\omega_n \cdot (t - t_1)\right] \tag{30}$$

Za početne uvjete u trenutku $t = t_0 = t_1$, odnosno $t^* = 0$, uz homogene početne uvjete u trenutku $t_0 = 0$, prema izrazima (24) i (26) vrijedi:

$$u(t = t_1) = u(t^* = 0) = \frac{P_0}{k} \cdot \left[1 - \cos(\omega_n \cdot t_1)\right] \tag{31}$$

$$\dot{u}(t = t_1) = u(t^* = 0) = \frac{P_0 \cdot \omega_n}{k} \cdot \sin(\omega_n \cdot t_1) \tag{32}$$

Iz izraza (29) i (30) za $t = t_0 = t_1$, odnosno $t^* = 0$, uz početne uvjete (31) i (32) mogu se odrediti konstante A_0 i B_0:

$$\frac{P_0}{k} \cdot \left[1 - \cos(\omega_n \cdot t_1)\right] = A_0 \cdot 1 + B_0 \cdot 0 \quad \Rightarrow \quad A_0 = \frac{P_0}{k} \cdot \left[1 - \cos(\omega_n \cdot t_1)\right] \tag{33}$$

$$\frac{P_0 \cdot \omega_n}{k} \cdot \sin(\omega_n \cdot t_1) = -A_0 \cdot \omega_n \cdot 0 + B_0 \cdot \omega_n \cdot 1 \quad \Rightarrow \quad B_0 = \frac{P_0}{k} \cdot \sin(\omega_n \cdot t_1) \tag{34}$$

Konačno rješenje u obliku funkcije pomaka u vremenu za II. interval dano je izrazom (35), odnosno izrazom (38), primjenom osnovnih trigonometrijskih jednakosti (36a,b) nakon pojednostavljenja (37).

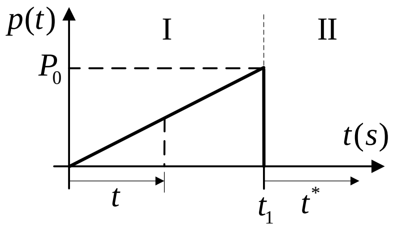
$$u(t)=\frac{P_0}{k}\cdot\left[1-\cos\left(\omega_n\cdot t_1\right)\right]\cdot\cos\left[\omega_n\cdot\left(t-t_1\right)\right]+\frac{P_0}{k}\cdot\sin\left(\omega_n\cdot t_1\right)\cdot\sin\left[\omega_n\cdot\left(t-t_1\right)\right] \tag{35}$$

$$\begin{aligned}\sin x\sin y&=\frac{1}{2}\left[\cos\left(x-y\right)-\cos\left(x+y\right)\right]\\ \cos x\cos y&=\frac{1}{2}\left[\cos\left(x-y\right)+\cos\left(x+y\right)\right]\end{aligned} \tag{36a,b}$$

$$\begin{aligned}u(t)=&\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot\left(t-t_1\right)\right]\\ &-\frac{P_0}{k}\cdot\cos\left(\omega_n\cdot t_1\right)\cos\left[\omega_n\cdot\left(t-t_1\right)\right]+\frac{P_0}{k}\cdot\sin\left(\omega_n\cdot t_1\right)\sin\left[\omega_n\cdot\left(t-t_1\right)\right]\\ =&\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot\left(t-t_1\right)\right]\\ &-\frac{P_0}{k}\cdot\frac{1}{2}\cdot\left[\cos\left(\omega_n\cdot t_1-\omega_n\cdot t+\omega_n\cdot t_1\right)+\cos\left(\cancel{\omega_n\cdot t_1}-\omega_n\cdot t-\cancel{\omega_n\cdot t_1}\right)\right]\\ &+\frac{P_0}{k}\cdot\frac{1}{2}\cdot\left[\cos\left(\omega_n\cdot t_1-\omega_n\cdot t+\omega_n\cdot t_1\right)-\cos\left(\cancel{\omega_n\cdot t_1}-\omega_n\cdot t-\cancel{\omega_n\cdot t_1}\right)\right]\\ =&\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot\left(t-t_1\right)\right]\\ &-\frac{P_0}{k}\cdot\frac{1}{2}\cdot\left[\cancel{\cos\left[\omega_n\cdot\left(2\cdot t_1-t\right)\right]}+\cos\left(\omega_n\cdot t\right)-\cancel{\cos\left[\omega_n\cdot\left(2\cdot t_1-t\right)\right]}+\cos\left(\omega_n\cdot t\right)\right]\\ =&\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot\left(t-t_1\right)\right]-\frac{P_0}{k}\cdot\cos\left(\omega_n\cdot t\right)\end{aligned} \tag{37}$$

$$u(t)=\frac{P_0}{k}\cdot\left\{\cos\left[\omega_n\cdot\left(t-t_1\right)\right]-\cos\left(\omega_n\cdot t\right)\right\} \tag{38}$$

2. Linearno rastuća pobuda



$$p(t)=\begin{cases}\dfrac{P_0}{t_1}\cdot t,\ t\in\langle 0,t_1\rangle\\ 0,\ t\in\langle t_1,\infty\rangle\end{cases}$$

I. INTERVAL: $t\in\langle 0,t_1\rangle$

Diferencijalna jednadžba

- bez prigušenja: $$m\cdot\ddot{u}(t)+k\cdot u(t)=\frac{P_0}{t_1}\cdot t \qquad (39)$$

Pretpostavljeno ukupno rješenje:

$$u(t)=A\cdot\cos(\omega_n\cdot t)+B\cdot\sin(\omega_n\cdot t)+u_p(t) \qquad (40)$$

Pretpostavljeno partikularno rješenje mora imati oblik funkcije pobude $p(t)$ (linearne funkcije):

$$u_p(t)=C\cdot t \qquad (41)$$

$$\ddot{u}_p(t)=0 \qquad (42)$$

Partikularno rješenje može se odrediti iz uvjeta da zadovoljava opći oblik diferencijalne jednadžbe (43):

$$m\cdot\ddot{u}_p(t)+k\cdot u_p(t)=\frac{P_0}{t_1}\cdot t \qquad (43)$$

$$m\cdot 0+k\cdot C\cdot t=\frac{P_0}{t_1}\cdot t \quad\Rightarrow\quad C=\frac{P_0}{k\cdot t_1} \qquad (46)$$

Zatim traženo rješenje (pomak) i odgovarajuća derivacija (brzina) primaju oblik prikazan izrazima (47) i (48):

$$u(t)=A\cdot\cos(\omega_n\cdot t)+B\cdot\sin(\omega_n\cdot t)+\frac{P_0}{k\cdot t_1}\cdot t \qquad (47)$$

$$\dot{u}(t)=-A\cdot\omega_n\cdot\sin(\omega_n\cdot t)+B\cdot\omega_n\cdot\cos(\omega_n\cdot t)+\frac{P_0}{k\cdot t_1} \qquad (48)$$

Za početne uvjete u trenutku $t_0=0$ konstante A i B mogu se odrediti iz jednadžbe (47) odnosno (48):

$$u(t=0)=u_0 \quad\Rightarrow\quad u_0=A\cdot 1+B\cdot 0+\frac{P_0}{k\cdot t_1}\cdot 0 \quad\Rightarrow\quad A=u_0 \qquad (49)$$

$$\dot{u}(t=0)=\dot{u}_0 \quad\Rightarrow\quad \dot{u}_0=-A\cdot\omega_n\cdot 0+B\cdot\omega_n\cdot 1+\frac{P_0}{k\cdot t_1} \quad\Rightarrow\quad B=\frac{\dot{u}_0}{\omega_n}-\frac{P_0}{k\cdot t_1\cdot\omega_n} \qquad (50)$$

Isto tako, za homogene početne uvjete vrijedi:

$$u(t=0)=0 \quad\Rightarrow\quad 0=A\cdot 1+B\cdot 0+\frac{P_0}{k\cdot t_1}\cdot 0 \quad\Rightarrow\quad A=0 \qquad (51)$$

$$\dot{u}(t=0)=0 \quad \Rightarrow \quad 0=-A\cdot\omega_n\cdot 0+B\cdot\omega_n\cdot 1+\frac{P_0}{k\cdot t_1} \quad \Rightarrow \quad B=-\frac{P_0}{k\cdot t_1\cdot\omega_n} \tag{52}$$

Konačno rješenje u obliku funkcije pomaka u vremenu za I. interval dano je izrazom (53), odnosno izrazom (54) za homogene početne uvjete.

$$u(t)=u_0\cdot\cos(\omega_n\cdot t)+\frac{\dot{u}_0}{\omega_n}\cdot\sin(\omega_n\cdot t)+\frac{P_0}{k\cdot t_1}\left[t-\frac{1}{\omega_n}\cdot\sin(\omega_n\cdot t)\right] \tag{53}$$

$$u(t)=\frac{P_0}{k\cdot t_1}\left[t-\frac{1}{\omega_n}\cdot\sin(\omega_n\cdot t)\right] \tag{54}$$

Odgovarajuće derivacije, odnosno funkcije brzine u vremenu, dane su izrazima (55) i (56).

$$\dot{u}(t)=-u_0\cdot\omega_n\cdot\sin(\omega_n\cdot t)+\dot{u}_0\cdot\cos(\omega_n\cdot t)+\frac{P_0}{k\cdot t_1}\cdot\left[1-\cos(\omega_n\cdot t)\right] \tag{55}$$

$$\dot{u}(t)=\frac{P_0}{k\cdot t_1}\cdot\left[1-\cos(\omega_n\cdot t)\right] \tag{56}$$

II. INTERVAL: $t\in\langle t_1,\infty\rangle$, $t^*\in\langle 0,\infty\rangle$

Na početku II. intervala, prestankom djelovanja pobude, sustav počinje slobodno oscilirati, uz nehomogene početne uvjete koji odgovaraju pomaku i brzini na kraju I. intervala. Zbog jednostavnosti, gibanje se može promatrati u pomaknutom koordinatnom sustavu $t^* = t - t_1$. Slobodne oscilacije sustava s jednim stupnjem slobode bez prigušenja opisane su jednadžbom (57).

$$m\cdot\ddot{u}(t)+k\cdot u(t)=0 \tag{57}$$

Stoga je pretpostavljeno rješenje (pomak) određeno izrazom (58), odnosno (59), a njegova derivacija (brzina) izrazom (60).

$$u(t^*)=A_0\cdot\cos(\omega_n\cdot t^*)+B_0\cdot\sin(\omega_n\cdot t^*) \tag{58}$$

$$u(t)=A_0\cdot\cos\left[\omega_n\cdot(t-t_1)\right]+B_0\cdot\sin\left[\omega_n\cdot(t-t_1)\right] \tag{59}$$

$$\dot{u}(t)=-A_0\cdot\omega_n\sin\left[\omega_n\cdot(t-t_1)\right]+B_0\cdot\omega_n\cdot\cos\left[\omega_n\cdot(t-t_1)\right] \tag{60}$$

Za početne uvjete u trenutku $t = t_0 = t_1$, odnosno $t^* = 0$, uz homogene početne uvjete u trenutku $t_0 = 0$, prema izrazima (54) i (56) vrijedi:

$$u(t=t_1)=u(t^*=0)=\frac{P_0}{k\cdot t_1}\cdot\left[t_1-\frac{1}{\omega_n}\cdot\sin(\omega_n\cdot t_1)\right] \tag{61}$$

$$\dot{u}(t=t_1)=u(t^*=0)=\frac{P_0}{k\cdot t_1}\cdot\left[1-\cos(\omega_n\cdot t_1)\right] \tag{62}$$

Iz izraza (59) i (60) za $t = t_0 = t_1$, odnosno $t^* = 0$, uz početne uvjete (61) i (62) mogu se odrediti konstante A_0 i B_0:

$$\frac{P_0}{k \cdot t_1} \cdot \left[t_1 - \frac{1}{\omega_n} \sin\left(\omega_n \cdot t_1\right) \right] = A_0 \cdot 1 + B_0 \cdot 0 \quad \Rightarrow \quad A_0 = \frac{P_0}{k \cdot t_1} \cdot \left[t_1 - \frac{1}{\omega_n} \sin\left(\omega_n \cdot t_1\right) \right] \tag{63}$$

$$\frac{P_0}{k \cdot t_1} \cdot \left[1 - \cos\left(\omega_n \cdot t_1\right)\right] = -A_0 \cdot \omega_n \cdot 0 + B_0 \cdot \omega_n \cdot 1 \quad \Rightarrow \quad B_0 = \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \left[1 - \cos\left(\omega_n \cdot t_1\right)\right] \tag{64}$$

Konačno rješenje u obliku funkcije pomaka u vremenu za II. interval dano je izrazom (65), odnosno izrazom (68), primjenom osnovnih trigonometrijskih jednakosti (66) nakon pojednostavljenja (67).

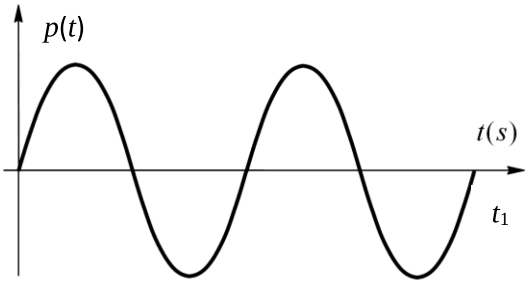
$$\begin{aligned} u(t) = {} & \frac{P_0}{k \cdot t_1} \cdot \left[t_1 - \frac{1}{\omega_n} \cdot \sin\left(\omega_n \cdot t_1\right) \right] \cdot \cos\left[\omega_n \cdot \left(t - t_1\right)\right] \\ & + \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \left[1 - \cos\left(\omega_n \cdot t_1\right)\right] \cdot \sin\left[\omega_n \cdot \left(t - t_1\right)\right] \end{aligned} \tag{65}$$

$$\sin x \cos y + \cos x \sin y = \sin\left(x + y\right) \tag{66}$$

$$\begin{aligned} u(t) = {} & \frac{P_0}{k} \cdot \cos\left[\omega_n \cdot \left(t - t_1\right)\right] - \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \sin\left(\omega_n \cdot t_1\right) \cdot \cos\left[\omega_n \cdot \left(t - t_1\right)\right] \\ & + \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \sin\left[\omega_n \cdot \left(t - t_1\right)\right] - \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \cos\left(\omega_n \cdot t_1\right) \cdot \sin\left[\omega_n \cdot \left(t - t_1\right)\right] \\ = {} & \frac{P_0}{k} \cdot \cos\left[\omega_n \cdot \left(t - t_1\right)\right] + \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \sin\left[\omega_n \cdot \left(t - t_1\right)\right] \\ & - \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \sin\left(\omega_n \cdot t_1 + \omega_n \cdot t - \omega_n \cdot t_1\right) \\ = {} & \frac{P_0}{k} \cdot \cos\left[\omega_n \cdot \left(t - t_1\right)\right] + \frac{P_0}{k \cdot t_1 \cdot \omega_n} \sin\left[\omega_n \cdot \left(t - t_1\right)\right] - \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \sin\left(\omega_n \cdot t\right) \end{aligned} \tag{67}$$

$$u(t) = \frac{P_0}{k} \cdot \cos\left[\omega_n \cdot \left(t - t_1\right)\right] + \frac{P_0}{k \cdot t_1 \cdot \omega_n} \cdot \left\{ \sin\left[\omega_n \cdot \left(t - t_1\right)\right] - \sin\left(\omega_n \cdot t\right) \right\} \tag{68}$$

3. Harmonijska pobuda



$$p(t) = P_0 \cdot \sin(\omega \cdot t)$$

I. INTERVAL: $t \in \langle 0, t_1 \rangle$

Diferencijalna jednadžba

- bez prigušenja: $m \cdot \ddot{u}(t) + k \cdot u(t) = P_0 \cdot \sin(\omega \cdot t)$ (69)

Pretpostavljeno ukupno rješenje:

$$u(t) = A \cdot \cos(\omega_n \cdot t) + B \cdot \sin(\omega_n \cdot t) + u_p(t) \tag{70}$$

Pretpostavljeno partikularno rješenje mora imati oblik funkcije pobude $p(t)$ (sinusne funkcije):

$$\begin{aligned} u_p(t) &= C \cdot \sin(\omega \cdot t) \\ \dot{u}_p(t) &= C \cdot \omega \cdot \cos(\omega \cdot t) \\ \ddot{u}_p(t) &= -C \cdot \omega^2 \cdot \sin(\omega \cdot t) \end{aligned} \tag{71}$$

Partikularno rješenje može se odrediti iz uvjeta da zadovoljava opći oblik diferencijalne jednadžbe (69):

$$\begin{aligned} & m \cdot \ddot{u}_p(t) + k \cdot u_p(t) = P_0 \cdot \sin(\omega \cdot t) \\ & -m \cdot C \cdot \omega^2 \cdot \cancel{\sin(\omega \cdot t)} + k \cdot C \cdot \cancel{\sin(\omega \cdot t)} = P_0 \cdot \cancel{\sin(\omega \cdot t)} \\ & -m \cdot C \cdot \omega^2 + k \cdot C = P_0 \end{aligned} \tag{72}$$

$$\begin{aligned} C &= \frac{P_0}{k - m \cdot \omega^2} = \frac{P_0}{k - \frac{k}{\omega_n^{\,2}} \cdot \omega^2} = \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^{\,2}}\right)} \\ u_p(t) &= \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^{\,2}}\right)} \cdot \sin(\omega \cdot t) \end{aligned} \tag{73}$$

Zatim traženo rješenje (pomak) i odgovarajuća derivacija (brzina) primaju oblik prikazan izrazima (74) i (75):

$$u(t) = A \cdot \cos(\omega_n \cdot t) + B \cdot \sin(\omega_n \cdot t) + \frac{P_0}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^{\,2}}\right)} \cdot \sin(\omega \cdot t) \tag{74}$$

$$\dot{u}(t) = -A \cdot \omega_n \cdot \sin(\omega_n \cdot t) + B \cdot \omega_n \cdot \cos(\omega_n \cdot t) + \frac{P_0 \cdot \omega}{k \cdot \left(1 - \frac{\omega^2}{\omega_n^{\,2}}\right)} \cdot \cos(\omega \cdot t) \tag{75}$$

Za homogene početne uvjete u trenutku $t_0 = 0$ konstante A i B mogu se odrediti iz jednadžbe (74) odnosno (75):

$$u(t=0) = A\cdot\cos(\omega_n\cdot 0) + B\cdot\sin(\omega_n\cdot 0) + \frac{P_0}{k\cdot\left(1-\frac{\omega^2}{\omega_n{}^2}\right)}\cdot\sin(\omega\cdot 0) = 0$$

$$= A\cdot 1{,}0 = 0 \rightarrow A = 0 \tag{76}$$

$$\dot{u}(t=0) = -A\cdot\omega_n\cdot\sin(\omega_n\cdot 0) + B\cdot\omega_n\cdot\cos(\omega_n\cdot 0) + \frac{P_0\cdot\omega}{k\cdot\left(1-\frac{\omega^2}{\omega_n{}^2}\right)}\cdot\cos(\omega\cdot 0) = 0$$

$$= B\cdot\omega_n\cdot 1{,}0 + \frac{P_0\cdot\omega}{k\cdot\left(1-\frac{\omega^2}{\omega_n{}^2}\right)} = 0$$

$$B = -\frac{P_0}{k\cdot\left(1-\frac{\omega^2}{\omega_n{}^2}\right)}\cdot\frac{\omega}{\omega_n} \tag{77}$$

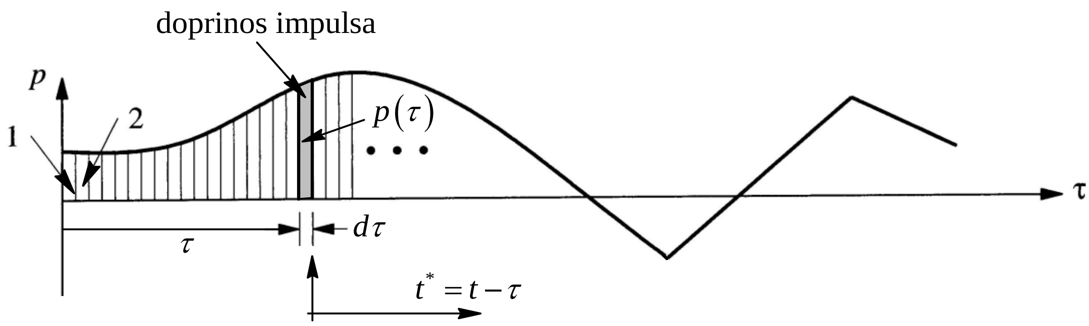
Konačno rješenje u obliku funkcije pomaka u vremenu za I. interval dano je izrazom (78):

$$u(t) = A\cdot\cos(\omega_n\cdot t) + B\cdot\sin(\omega_n\cdot t) + \frac{P_0}{k\cdot\left(1-\frac{\omega^2}{\omega_n{}^2}\right)}\cdot\sin(\omega\cdot t)$$

$$= -\frac{P_0}{k\cdot\left(1-\frac{\omega^2}{\omega_n{}^2}\right)\cdot}\cdot\frac{\omega}{\omega_n}\cdot\sin(\omega_n\cdot t) + \frac{P_0}{k\cdot\left(1-\frac{\omega^2}{\omega_n{}^2}\right)}\cdot\sin(\omega\cdot t) \tag{78}$$

b) Duhamelov integral

ako je $p(t)$ integrabilna funkcija tražimo analitičko rješenje, ako nije potrebno ju je numerički riješiti funkcija opterećenja prikazana je nizom kratkih impulsa



$$\frac{d(m\cdot\dot{u})}{d\tau} = p(\tau)$$ - promjena količine gibanja

$$d(m\cdot\dot{u}) = p(\tau)d\tau \qquad d\dot{u} = \frac{p(\tau)}{m}d\tau = \frac{S}{m} \tag{79}$$

$$du(t) = A\cdot\cos(\omega_n\cdot t^*) + B\cdot\sin(\omega_n\cdot t^*) = du_0\cdot\cos(\omega_n\cdot t^*) + \frac{d\dot{u}_0}{\omega_n}\cdot\sin(\omega_n\cdot t^*) \tag{80}$$

za djelovanje impulsa je $du_0 = 0$ pa imamo $du(t) = \frac{d\dot{u}_0}{\omega_n}\cdot\sin(\omega_n\cdot t^*)$ (81)

$$du(t) = \frac{p(\tau)\cdot d\tau}{m\cdot\omega_n}\cdot\sin(\omega_n\cdot t^*) \tag{82}$$

uz $t^* = t - \tau$ imamo:

$$u(t) = \frac{1}{m\cdot\omega_n}\int_{\tau=0}^{\tau=t} p(\tau)\cdot\sin[\omega_n\cdot(t-\tau)]d\tau$$ ili

$$u(t) = \frac{\omega_n}{k}\int_{\tau=0}^{\tau=t} p(\tau)\cdot\sin[\omega_n\cdot(t-\tau)]d\tau \tag{83}$$

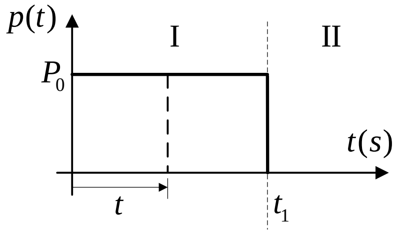
ako uvedemo prigušenje uz pretpostavku $u_0 = 0$ i $\dot{u}_0 = 0$ imamo

$$u(t) = \frac{1}{m\cdot\omega_D}\int_{\tau=0}^{\tau=t} p(\tau)\cdot e^{-\zeta\cdot\omega_n\cdot(t-\tau)}\cdot\sin[\omega_D\cdot(t-\tau)]d\tau \tag{84}$$

Rješenje za slučaj bez prigušenja te uz pretpostavku $u_0 \neq 0$ i $\dot{u}_0 \neq 0$

$$u(t) = u_0\cdot\cos(\omega_n\cdot t) + \frac{\dot{u}_0}{\omega_n}\cdot\sin(\omega_n\cdot t) + \frac{1}{m\cdot\omega_n}\int_{\tau=0}^{\tau=t} p(\tau)\cdot\sin[\omega_n\cdot(t-\tau)]d\tau \tag{85}$$

1. Konstantna pobuda



$$p(t)=\begin{cases}P_0,\ t\in\langle 0,t_1\rangle\\ 0,\ t\in\langle t_1,\infty\rangle\end{cases}$$

I. INTERVAL: $t\in<0,t_1>$ $\qquad p(\tau)=P_0=\text{konstanta}$

$$u(t)=\frac{\omega_n}{k}\int_{\tau=0}^{\tau=t}P_0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau \tag{86}$$

$$u(t)=\frac{\omega_n\cdot P_0}{k}\int_{\tau=0}^{\tau=t}\sin\left[\omega_n\cdot(t-\tau)\right]d\tau \tag{87}$$

$$u(t)=\frac{\omega_n\cdot P_0}{k}\cdot\frac{1}{\omega_n}\cdot\cos\left[\omega_n\cdot(t-\tau)\right]/_{\tau=0}^{\tau=t} \tag{88}$$

$$u(t)=\frac{P_0}{k}\cdot\left[1-\cos(\omega_n\cdot t)\right] \tag{89}$$

$$u(t)=u_{stat}(t)\cdot DF \tag{90}$$

gdje je $\frac{P_0}{k}$ progib od statičke sile

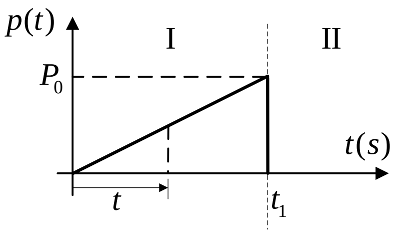
II. INTERVAL: $t\in<t_1,\infty>$ $\qquad p(\tau)=0$

$$u(t)=\frac{\omega_n}{k}\int_{\tau=0}^{\tau=t_1}P_0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau+\frac{\omega_n}{k}\int_{\tau=t_1}^{\tau=t}0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau \tag{91}$$

$$u(t)=\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-\tau)\right]/_{\tau=0}^{\tau=t_1} \tag{92}$$

$$u(t)=\frac{P_0}{k}\cdot\left[\cos\left[\omega_n\cdot(t-t_1)\right]-\cos(\omega_n\cdot t)\right] \tag{93}$$

2. Linearno rastuća pobuda



$$p(t)=\begin{cases}\frac{P_0}{t_1}\cdot t,\; t\in\langle 0,t_1\rangle\\ 0,\; t\in\langle t_1,\infty\rangle\end{cases}$$

I. INTERVAL: $t\in<0,t_1>$ $\qquad p(\tau)=\frac{P_0}{t_1}\cdot\tau$

$$u(t)=\frac{\omega_n}{k}\int_{\tau=0}^{\tau=t}\frac{P_0}{t_1}\cdot\tau\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau \tag{94}$$

$$u(t)=\frac{\omega_n\cdot P_0}{k\cdot t_1}\int_{\tau=0}^{\tau=t}\tau\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau \tag{95}$$

parcijalna integracija: $\int uv'=uv-\int vu'$, $u=\tau$, $v=\frac{1}{\omega_n}\cdot\cos\left[\omega_n\cdot(t-\tau)\right]$ (96)

$$u(t)=\frac{\omega_n\cdot P_0}{k\cdot t_1}\cdot\left(\tau\cdot\frac{1}{\omega_n}\cdot\cos\left[\omega_n\cdot(t-\tau)\right]/_{\tau=0}^{\tau=t}-\int_{\tau=0}^{\tau=t}\frac{1}{\omega_n}\cdot\cos\left[\omega_n\cdot(t-\tau)\right]\cdot 1\cdot d\tau\right) \tag{97}$$

$$u(t)=\frac{\omega_n\cdot P_0}{k\cdot t_1}\cdot\left(t\cdot\frac{1}{\omega_n}\cdot\cos\left[\omega_n\cdot(t-t)\right]-0+\frac{1}{\omega_n}\cdot\frac{1}{\omega_n}\cdot\sin\left[\omega_n\cdot(t-\tau)\right]/_{\tau=0}^{\tau=t}\right) \tag{98}$$

$$u(t)=\frac{P_0}{k\cdot t_1}\cdot\left(t+\frac{1}{\omega_n}\cdot\left(0-\sin[\omega_n\cdot t]\right)\right) \tag{99}$$

$$u(t)=\frac{P_0}{k\cdot t_1}\cdot\left(t-\frac{1}{\omega_n}\cdot\sin[\omega_n\cdot t]\right) \tag{100}$$

II. INTERVAL: $t\in<t_1,\infty>$ $\qquad p(\tau)=0$

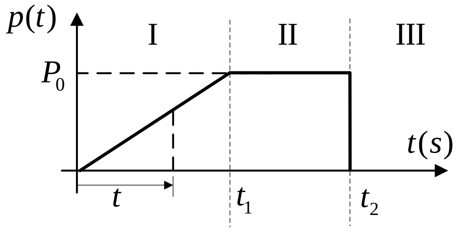
$$u(t)=\frac{\omega_n}{k}\int_{\tau=0}^{\tau=t_1}\frac{P_0}{t_1}\cdot\tau\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau+\frac{\omega_n}{k}\int_{\tau=t_1}^{\tau=t}0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau \tag{101}$$

$$u(t)=\frac{\omega_n\cdot P_0}{k\cdot t_1}\cdot\left(\tau\cdot\frac{1}{\omega_n}\cdot\cos\left[\omega_n\cdot(t-\tau)\right]/_{\tau=0}^{\tau=t_1}+\frac{1}{\omega_n^2}\cdot\sin\left[\omega_n\cdot(t-\tau)\right]/_{\tau=0}^{\tau=t_1}\right) \tag{102}$$

$$u(t)=\frac{\omega_n\cdot P_0}{k\cdot t_1}\cdot\left(t_1\cdot\frac{1}{\omega_n}\cdot\cos\left[\omega_n\cdot(t-t_1)\right]-0+\frac{1}{\omega_n^2}\cdot\sin\left[\omega_n\cdot(t-t_1)\right]-\frac{1}{\omega_n^2}\cdot\sin[\omega_n\cdot t]\right) \tag{103}$$

$$u(t)=\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_1)\right]+\frac{P_0}{k\cdot t_1\cdot\omega_n}\left(\sin\left[\omega_n\cdot(t-t_1)\right]-\sin[\omega_n\cdot t]\right) \tag{104}$$

3. Linearno rastuća + konstantna pobuda



$$p(t)=\begin{cases}\frac{P_0}{t_1}\cdot t,\ t\in\langle 0,t_1\rangle\\ P_0,\ t\in\langle t_1,t_2\rangle\\ 0,\ t\in\langle t_2,\infty\rangle\end{cases}$$

I. INTERVAL: $t\in<0,t_1>$ $\qquad p(\tau)=\frac{P_0}{t_1}\cdot\tau$

$$u(t)=\frac{P_0}{k\cdot t_1}\cdot\left(t-\frac{1}{\omega_n}\cdot\sin\left[\omega_n\cdot t\right]\right) \tag{105}$$

II. INTERVAL: $t\in<t_1,t_2>$ $\qquad p(\tau)=P_0$

$$u(t)=\frac{\omega_n}{k}\int_{\tau=0}^{\tau=t_1}\frac{P_0}{t_1}\cdot\tau\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau+\frac{\omega_n}{k}\int_{\tau=t_1}^{\tau=t}P_0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau \tag{106}$$

$$u(t)=\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_1)\right]+\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot(t-t_1)\right]-\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot t\right]+ \\ +\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-\tau)\right]/_{\tau=t_1}^{\tau=t} \tag{107}$$

$$u(t)=\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_1)\right]+\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot(t-t_1)\right]-\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot t\right]+ \\ +\frac{P_0}{k}\cdot\left(\cos\left[\omega_n\cdot(t-t)\right]-\cos\left[\omega_n\cdot(t-t_1)\right]\right) \tag{108}$$

$$u(t)=\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_1)\right]+\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot(t-t_1)\right]-\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot t\right]+ \\ +\frac{P_0}{k}-\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_1)\right] \tag{109}$$

$$u(t)=\frac{P_0}{k}+\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot(t-t_1)\right]-\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot t\right] \tag{110}$$

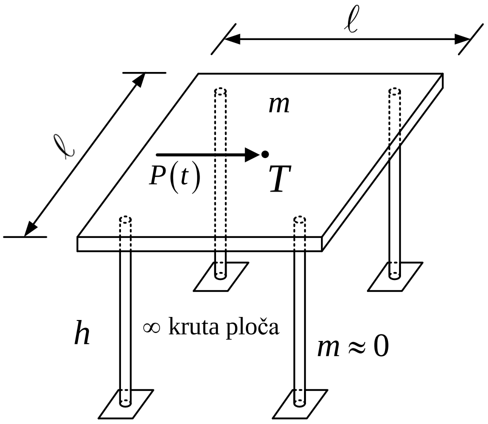
III. INTERVAL: $t\in<t_2,\infty>$ $\qquad p(\tau)=0$

$$u(t)=\frac{\omega_n}{k}\int_{\tau=0}^{\tau=t_1}\frac{P_0}{t_1}\cdot\tau\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau+\frac{\omega_n}{k}\int_{\tau=t_1}^{\tau=t_2}P_0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau+\frac{\omega_n}{k}\int_{\tau=t_2}^{\tau=t}0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]d\tau$$

$$u(t)=\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_1)\right]+\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot(t-t_1)\right]-\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot t\right]+ \\ +\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_2)\right]-\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_1)\right] \tag{111}$$

$$u(t)=\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot(t-t_1)\right]-\frac{P_0}{k\cdot t_1\cdot\omega_n}\cdot\sin\left[\omega_n\cdot t\right]+\frac{P_0}{k}\cdot\cos\left[\omega_n\cdot(t-t_2)\right] \tag{112}$$

c) Primjer



$h = 6$ m, $\ell = 8$ m, $t_p = 30$ cm

$m = \ell^2 \cdot t_p \cdot \gamma = 8,0^2 \cdot 0,3 \cdot 2,5 = 48$ t

$k = 523,6$ kN/m

$P_0 = 10$ kN

stupovi - $\phi 20$ cm

$I = d^4 \cdot \pi / 64 = 7854 \text{ cm}^4$

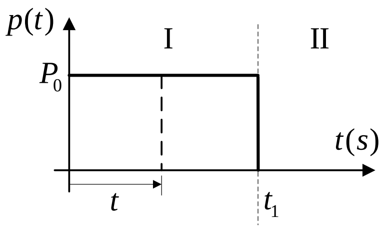
$E = 30000000 \text{ kN/m}^2$

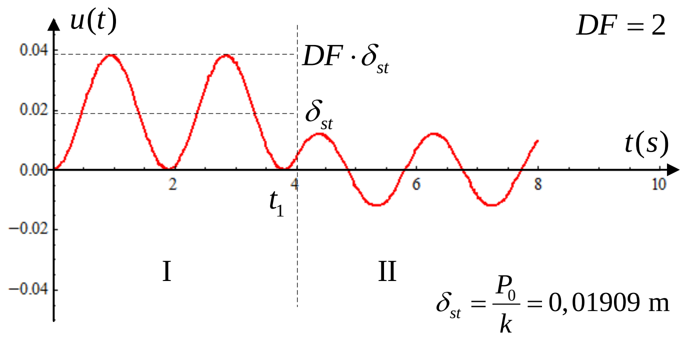
$EI = 2356,2 \text{ kNm}^2$

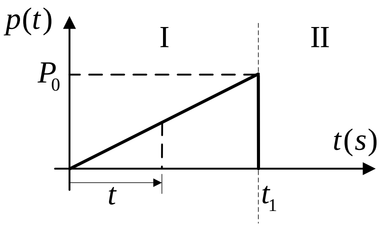
$k = 4 \cdot 12 EI / h^3 = 523,6$ kN/m

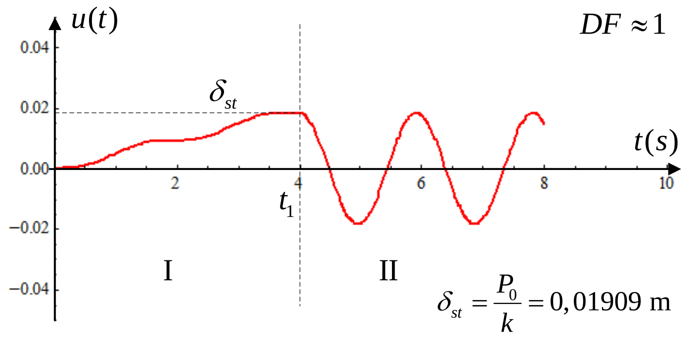
$\omega_n = \sqrt{\frac{k}{m}} = 3,302$ r/s

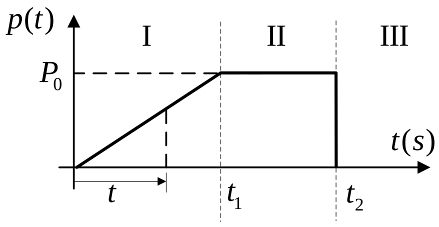
$\mathrm{T_n} = \frac{2\pi}{\omega_n} = 1,9$ s

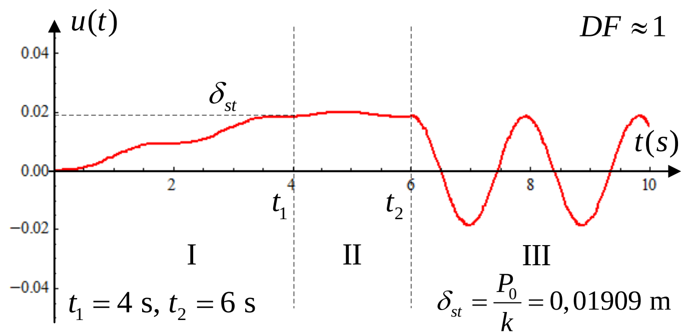

$p(t)$
I
II
P_0
$t(s)$
t
t_1

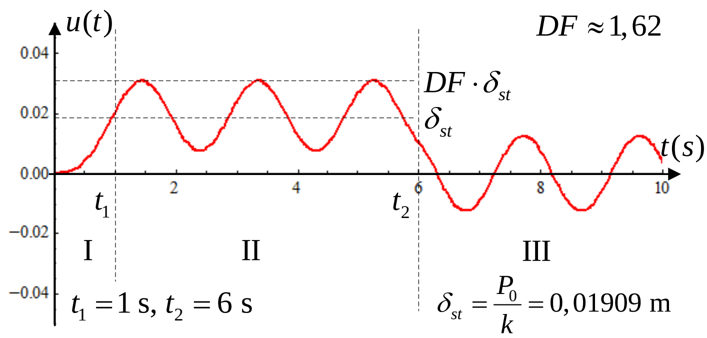

$u(t)$
$DF = 2$
$DF \cdot \delta_{st}$
δ_{st}
$t(s)$
t_1
I
II
$\delta_{st} = \frac{P_0}{k} = 0,01909$ m

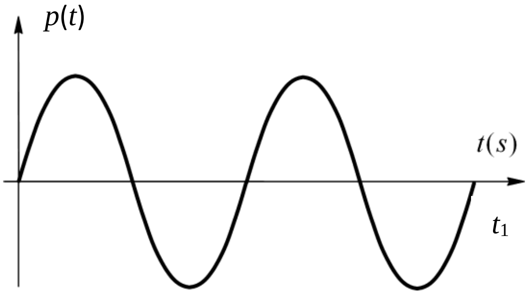

$p(t)$
I
II
P_0
$t(s)$
t
t_1


$u(t)$
$DF \approx 1$
δ_{st}
$t(s)$
t_1
I
II
$\delta_{st} = \frac{P_0}{k} = 0,01909$ m


$p(t)$
I
II
III
P_0
$t(s)$
t
t_1
t_2


$u(t)$
$DF \approx 1$
δ_{st}
$t(s)$
t_1
t_2
I
II
III
$t_1 = 4$ s, $t_2 = 6$ s
$\delta_{st} = \frac{P_0}{k} = 0,01909$ m


$u(t)$
$DF \approx 1,62$
$DF \cdot \delta_{st}$
δ_{st}
$t(s)$
t_1
t_2
I
II
III
$t_1 = 1$ s, $t_2 = 6$ s
$\delta_{st} = \frac{P_0}{k} = 0,01909$ m



$$p(t) = P_0 \cdot \sin(\omega \cdot t)$$