MATEMATIKA 2, 26. 6. 2024.

Ime i prezime: ______________________________

1.	2.	3.	4.	5.

1. dio	2. dio	$\sum$

1. Riješite diferencijalne jednadžbe:

   (a) (10 bodova) $y' + xye^x = y$,

   (b) (10 bodova) $y'' + 4y = x^2$.

2. (a) (13 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \arcsin\left(\frac{y}{x^2+1}\right) + \ln(y - x + 1).$$

   (b) (12 bodova) Odredite lokalne ekstreme funkcije

$$f(x,y) = \frac{x^2}{2} + xy^2 + 2x + 2y^3.$$

3. (15 bodova) Odredite površinu skupa

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 - 2x + y^2 \geq 0,\ x^2 - 4x + y^2 \leq 0,\ y \geq x\}.$$

   Skicirajte D.

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4. (a) (13 bodova) Zadano je vektorsko polje

$$\vec{a} = z\cos(xz)\vec{i} + (ze^{yz} + 2)\vec{j} + (x\cos(xz) + ye^{yz})\vec{k}.$$

   Je li polje $\vec{a}$ potencijalno? Ako jest, odredite mu potencijal.

   (b) (12 bodova) Izračunajte

$$\int_\Gamma xy\,ds,$$

   ako je Γ presječnica ploha $y = 2 - x$ i $z = 1$ u prvom oktantu. Skicirajte krivulju.

5. (15 bodova) Izračunajte tok vektorskog polja $\vec{a} = \vec{j} + z^3\vec{k}$ kroz sferu $x^2 + y^2 + z^2 = 1$. Skicirajte sferu.

Prvi dio čine prva tri zadatka. **Drugi dio** čine 4. i 5. zadatak.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
C	0	$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$
x^α	$\alpha x^{\alpha-1}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$	$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{sh} x$	$\operatorname{ch} x$
$\sin x$	$\cos x$	$\operatorname{ch} x$	$\operatorname{sh} x$
$\cos x$	$-\sin x$	$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	$x+C$	$\cos x$	$\sin x + C$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1$	$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x + C$
e^x	$e^x + C$	$\frac{1}{\cos^2 x}$	$\operatorname{tg} x + C$
a^x	$\frac{a^x}{\ln a} + C$	$\frac{1}{1+x^2}$	$\operatorname{arctg} x + C$
$\frac{1}{x}$	$\ln\|x\| + C$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{\sqrt{x^2 \pm 1}}$	$\ln\left\|x + \sqrt{x^2 \pm 1}\right\| + C$

MAT2 , 26.6.'24.

1. a) (10b) $y' + xye^x = y \Rightarrow \frac{dy}{dx} = y - xye^x$

$\Rightarrow \frac{dy}{dx} = y(1-xe^x) \Rightarrow \int \frac{dy}{y} = \int (1-xe^x)dx$

$\int (1-xe^x)dx = x - \int xe^x dx = \left| \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix} \right|$

$= x - xe^x + \int e^x dx = x - xe^x + e^x$

$$\boxed{\ln y = x - xe^x + e^x + C}$$

$$\boxed{y = Ce^{x - xe^x + e^x}}$$

b) (10b) $y'' + 4y = x^2 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i$

$y_H = C_1 \sin 2x + C_2 \cos 2x$

$y_P = Ax^2 + Bx + C, \; y_P' = 2Ax + B, \; y_P'' = 2A$

$2A + 4Ax^2 + 4Bx + 4C = x^2 \Rightarrow$

$4A = 1, \quad 4B = 0, \quad 2A + 4C = 0 \Rightarrow$

$A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{8} \Rightarrow y_P = \frac{x^2}{4} - \frac{1}{8}$

$$\boxed{y = y_H + y_P = C_1 \sin 2x + C_2 \cos 2x + \frac{x^2}{4} - \frac{1}{8}}$$

2. a) (13b) $f(x,y) = \arcsin\left(\frac{y}{x^2+1}\right) + \ln(y-x+1)$

1° $-1 \le \frac{y}{x^2+1} \le 1 \quad / \cdot (x^2+1) > 0$

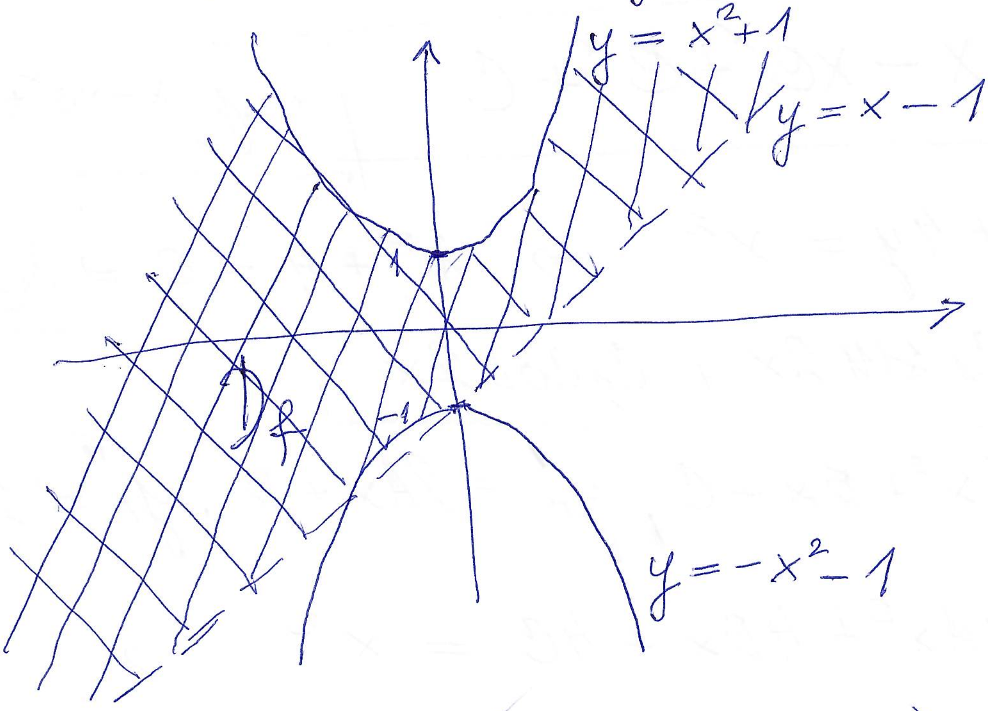
$-x^2-1 \le y$ $\qquad$ $\underline{y \le x^2+1}$ ispod parab.

$\underline{y \ge -x^2-1}$ iznad parab.

2° $y-x+1>0$

$\Rightarrow \underline{y > x-1}$ iznad pravca

$D_f = \{(x,y) \in \mathbb{R}^2 : -x^2-1 \le y \le x^2+1,\ y > x-1\}$



b) (12b) $f(x,y) = \frac{x^2}{2} + xy^2 + 2x + 2y^3$

$\frac{\partial f}{\partial x} = x + y^2 + 2 = 0 \Rightarrow y^2 + 3y + 2 = 0 \Rightarrow y_{1,2} = \frac{+3 \pm \sqrt{9-}}{2}$

$y_1 = -2,\ y_2 =$

$\frac{\partial f}{\partial y} = 2xy + 6y^2 = 0 \Rightarrow y = 0$ ili $x = -3y \Rightarrow x_1 = -6,\ x_2 = -$

$T_1(-6,-2),\ T_2(-3,-1)$ stac. točke, $T_3(-2,0$

$\frac{\partial^2 f}{\partial x^2} = 1,\ \frac{\partial^2 f}{\partial x \partial y} = 2y,\ \frac{\partial^2 f}{\partial y^2} = 2x + 12y$

$\boxed{T_3 \Rightarrow 1 \cdot (-4) - 0 < 0}$

$T_1(-6,-2) \Rightarrow A \cdot C - B^2 = 1 \cdot (-36) - 16 < 0 \Rightarrow T_1$ je sedl. točka

$T_2(-3,-1) \Rightarrow A \cdot C - B^2 = 1 \cdot (-18) - 4 < 0 \Rightarrow T_2$ je —"— —"—

2. b) (12b) Ekstremi od $f(x,y)=\frac{x^2}{2}+xy^2+2x+2y^3$.

$\frac{\partial f}{\partial x}=x+y^2+2=0$

$\frac{\partial f}{\partial y}=2xy+6y^2=0 \Rightarrow \underline{y=0}$ ili $\underline{x=-3y}$

- za $y_1=0$ imamo $x_1=-2$
- za $x=-3y$ rješavamo $y^2-3y+2=0 \rightarrow y_2=1$, $\rightarrow \underline{y_3=2}$

$\Rightarrow x_2=-3$

$x_3=-6$

- stacionarne točke: $T_1(-2,0)$, $T_2(-3,1)$, $T_3(-6,2)$

$\frac{\partial^2 f}{\partial x^2}=1$ $\qquad \frac{\partial^2 f}{\partial x \partial y}=2y$ $\qquad \frac{\partial^2 f}{\partial y^2}=2x+12y$

- $T_1(-2,0) \Rightarrow A\cdot C-B^2=1\cdot(-4)-0^2=-4<0$

$\Rightarrow T_1$ je sedlasta točka

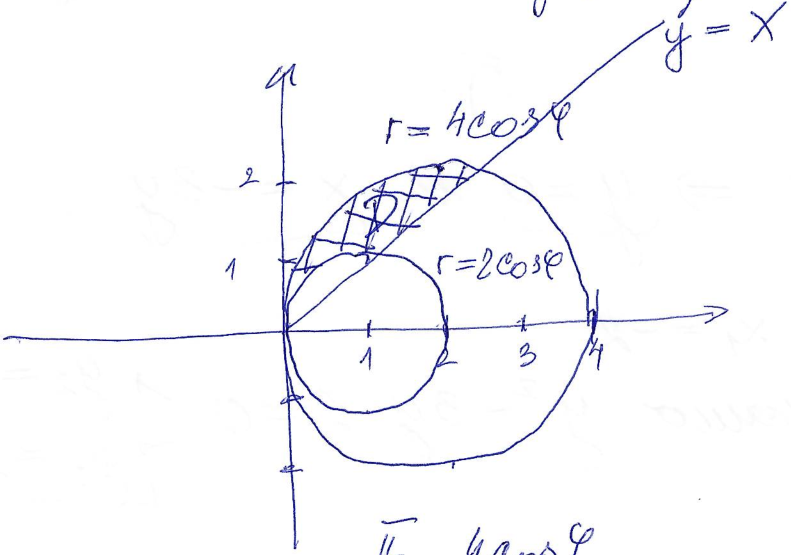
- $T_2(-3,1) \Rightarrow A\cdot C-B^2=1\cdot 6-2^2=2>0$, $A>0$

$\Rightarrow T_2$ je lokalni minimum

- $T_3(-6,2) \Rightarrow A\cdot C-B^2=1\cdot 12-16=-4<0$

$\Rightarrow T_3$ je sedlasta točka

3. Površina od $D=\{(x,y)\in\mathbb{R}^2 : x^2-2x+y^2\geq 0,\ x^2-4x+y^2\leq 0,\ y\geq x\}$



$D\ldots\ \frac{\pi}{4}\leq\varphi\leq\frac{\pi}{2}$

$2\cos\varphi\leq r\leq 4\cos\varphi$

$$Pov(D)=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\varphi\int_{2\cos\varphi}^{4\cos\varphi}r\,dr=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{r^2}{2}\Big|_{2\cos\varphi}^{4\cos\varphi}d\varphi$$

$$=6\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\varphi\,d\varphi=3\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1+\cos 2\varphi)\,d\varphi$$

$$=3\left(\varphi+\frac{1}{2}\sin 2\varphi\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=3\left(\frac{\pi}{2}-\frac{\pi}{4}-\frac{1}{2}\right)$$

$$=3\left(\frac{\pi}{4}-\frac{1}{2}\right)=\boxed{\frac{3}{4}(\pi-2)}$$

[1. dio — 30 min]

10,0
9,4

4. a) (13b)

$$\operatorname{rot}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos(xz) & ze^{yz}+2 & x\cos(xz)+ye^{yz} \end{vmatrix}$$

$$= \vec{i}\left(\cancel{e^{yz}}+\cancel{yze^{yz}}-\cancel{e^{yz}}-\cancel{yze^{yz}}\right)-\vec{j}\left(\cancel{\cos(xz)}-\cancel{xz\sin(xz)}-\cancel{\cos(xz)}+\cancel{xz\sin(xz)}\right)+\vec{k}(0-0)=\vec{0}$$

$\Rightarrow$ polje $\vec{a}$ je potencijalno

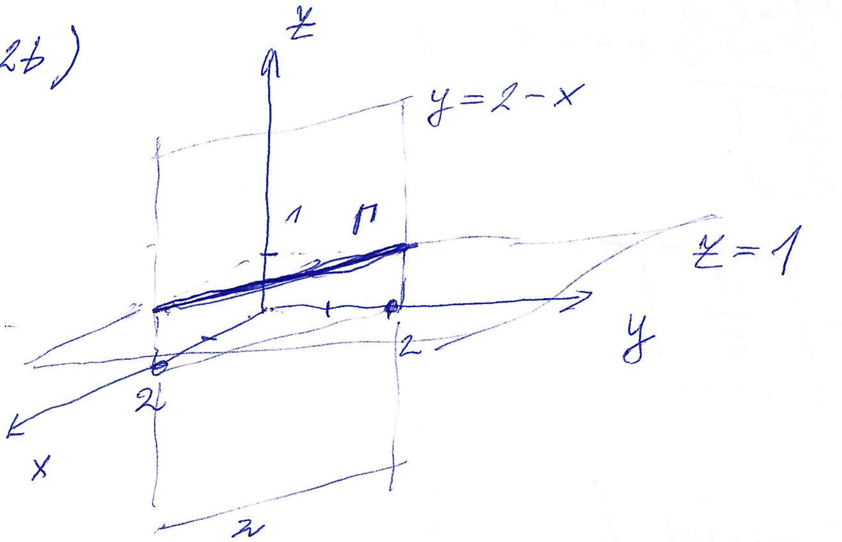
Potencijal :

$$\varphi(x,y,z) = \int_0^x z\cos(tz)\,dt + \int_0^y (ze^{tz}+2)\,dt + \cancel{\int_0^z 0\,dt}$$

$$= \sin(tz)\Big|_0^x + e^{tz}\Big|_0^y + 2t\Big|_0^y + C$$

$$= \boxed{\sin(xz)+e^{yz}+2y+C}$$

b) (12b)

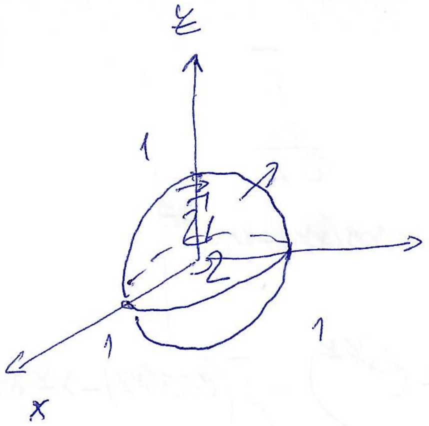


$\Gamma \ldots$ $x(t)=t$, $\quad x'(t)=1$
$y(t)=2-t$, $\quad y'(t)=-1$
$z(t)=1$, $\quad z'(t)=0$
$t\in[0,2]$

$$\int_\Gamma xy\,ds = \int_0^2 t(2-t)\sqrt{1^2+(-1)^2+0^2}\,dt = \sqrt{2}\int_0^2 (2t-t^2)\,dt$$

$$= \sqrt{2}\left(t^2-\frac{t^3}{3}\right)\Big|_0^2 = \sqrt{2}\left(4-\frac{8}{3}\right) = \boxed{\frac{4\sqrt{2}}{3}}$$

5. (15b)



$\Omega \ldots \begin{array}{l} 0 \le \varphi \le 2\pi \\ 0 \le \vartheta \le \pi \\ 0 \le r \le 1 \end{array}$

$\operatorname{div} \bar{a} = 3z^2 = 3r^2\cos^2\vartheta$

$$\iint\limits_{\Pi} \bar{a}\,d\bar{S} = \iiint\limits_{\Omega} \operatorname{div}\bar{a}\,dx\,dy\,dz = 3\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \int_0^1 r^2\cos^2\vartheta\, r^2\sin\vartheta\, dr$$

$$= 6\pi \int_0^{\pi} \cos^2\vartheta \sin\vartheta\, d\vartheta \int_0^1 r^4\,dr =$$

$$= \cancel{6}^{2}\pi \cdot \frac{2}{\cancel{3}} \cdot \frac{1}{5} = \boxed{\frac{4\pi}{5}}$$

$$\int_0^{\pi} \cos^2\vartheta \sin\vartheta\, d\vartheta = \left| \begin{array}{l} t = \cos\vartheta \\ dt = -\sin\vartheta\, d\vartheta \\ \vartheta = 0 \Rightarrow t = 1 \\ \vartheta = \pi \Rightarrow t = -1 \end{array} \right| = \int_{-1}^{1} t^2\,dt = 2\int_0^1 t^2\,dt$$

$$= 2\frac{t^3}{3}\Big|_0^1 = \boxed{\frac{2}{3}}$$

$$\int_0^1 r^4\,dr = \frac{r^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$$

2. dio -- 22 min