MATEMATIKA 2, 7.7.2021.

1. Riješite sljedeće diferencijalne jednadžbe:

   a) (8 bodova) $(4+x^2)y' = y\cancel{\ln}y$,

   b) (12 bodova) $y'' + 4y = \sin(2x)$.

2. **a)** (8 bodova) Neka je $z = \frac{x^2}{y} + \varphi(x^2+y^2)$. Odredite $\frac{1}{x}\frac{\partial z}{\partial x} - \frac{1}{y}\frac{\partial z}{\partial y}$.

   b) (12 bodova) Ispitajte ekstreme funkcije $f(x,y) = e^x - x + y^3 - 27y + 1$.

3. (20 bodova) Prelaskom na sferne koordinate izračunajte

$$\int_0^{\pi} \sin^2\varphi \, \mathrm{d}\varphi \int_0^1 \rho^3 \, \mathrm{d}\rho \int_{-\sqrt{1-\rho^2}}^{\sqrt{1-\rho^2}} \mathrm{d}z.$$

   Skicirajte područje integracije.

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4. **a)** (8 bodova) Izračunajte grad$u$ u točki $T(0,1,1)$, ako je

$$u(x,y,z) = e^x y + x\cos(\pi y) + \ln(2z).$$

   Je li polje 'grad u' potencijalno?

   b) (14 bodova) Izračunajte

$$\int_{\vec{\Gamma}} x^2 \, \mathrm{d}x + y^2 \, \mathrm{d}y + xy \, \mathrm{d}z,$$

   ako je krivulja $\vec{\Gamma}$ zadana parametrizacijom $x(t) = e^t$, $y(t) = \cos t$ i $z(t) = t$ za $t \in [0, \frac{\pi}{2}]$.

5. (18 bodova) Izračunajte $\int\int_{\vec{\Sigma}} \vec{a} \, \mathrm{d}\vec{S}$ ako je $\vec{a} = y^2\vec{i} - x\vec{j} + yz\vec{k}$, a $\vec{\Sigma}$ je dio ravnine $x+y+z=2$ u 1. oktantu orijentirane normalom koja zatvara ~~oštar~~ kut s vektorom $\vec{k}$. Skicirajte plohu.

   ORIJENTIRANE   ŠILJASTI

Za polaganje ispita treba sakupiti 50 bodova (od toga barem 24 boda iz prvog dijela i barem 16 bodova iz drugog dijela).

1. a) (8b) $(4+x^2)y' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{4+x^2}$$

$$\int \frac{dx}{4+x^2} \begin{matrix} :4 \\ :4 \end{matrix} = \frac{1}{4}\int \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \left| \begin{matrix} t = \frac{x}{2} \\ dt = \frac{dx}{2} \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\operatorname{arctg} t + C = \underbrace{\frac{1}{2}\operatorname{arctg}\frac{x}{2} + C}$$

$$\boxed{\ln y = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + C}$$

1. b) (12b) $y'' + 4y = \sin(2x)$

$\lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i \quad , \beta = 2$

$\underbrace{y_H = C_1\cos(2x) + C_2\sin(2x)}$

$f(x) = \sin(2x) \quad , \quad \lambda_{1,2} = \pm \beta i$

$\Rightarrow \quad y_P = x\,(A\cos(2x) + B\sin(2x))$

$y_P' = A\cos(2x) + B\sin(2x) + x(-2A\sin(2x) + 2B\cos(2x))$

$y_P'' = -2A\sin(2x) + 2B\cos(2x) - 2A\sin(2x) + 2B\cos(2x)$

$\quad + x(-4A\cos(2x) - 4B\sin(2x))$

$\quad = -4A\sin(2x) + 4B\cos(2x) - x(4A\cos(2x) + 4B\sin(2x))$

$-4A\sin(2x) + 4B\cos(2x) - \cancel{4x(A\cos(2x) + B\sin(2x))}$

$+ \cancel{4x(A\cos(2x) + B\sin(2x))} = \sin(2x)$

$\Rightarrow -4A = 1 \Rightarrow \underbrace{A = -\frac{1}{4}} \quad i \quad \underbrace{B = 0}$

$\Rightarrow \underbrace{y_P = -\frac{x}{4}\cos(2x)}$

$$\boxed{y = y_H + y_P = C_1\cos(2x) + C_2\sin(2x) - \frac{x}{4}\cos(2x)}$$

2.a) (8b) $z = \frac{x^2}{y} + \varphi(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{y} + 2x\,\varphi'(x^2+y^2)$$

$$\frac{\partial z}{\partial y} = -\frac{x^2}{y^2} + 2y\,\varphi'(x^2+y^2)$$

$$\boxed{\frac{1}{x}\frac{\partial z}{\partial x} - \frac{1}{y}\frac{\partial z}{\partial y}} = \frac{2}{y} + \cancel{2\varphi'(x^2+y^2)} + \frac{x^2}{y^3} - \cancel{2\varphi'(x^2+y^2)}$$

$$= \boxed{\frac{2}{y} + \frac{x^2}{y^3}}$$

2.b) (12b) $f(x,y) = e^x - x + y^3 - 27y + 1$ Ekstremi = ?

$$\frac{\partial f}{\partial x} = e^x - 1 = 0 \quad \Rightarrow \quad e^x = 1 \Rightarrow \boxed{x = 0}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 27 = 0 \quad \Rightarrow \quad y^2 = 9 \Rightarrow \boxed{y_{1,2} = \pm 3}$$

$\underline{T_1(0,-3),\ T_2(0,3)}$, su stacionarne točke

$$\frac{\partial^2 f}{\partial x^2} = e^x, \quad \frac{\partial^2 f}{\partial y \partial x} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 6y$$

$\underline{T_1(0,-3)}$, $A = 1$, $B = 0$, $C = -18$, $AC - B^2 = -18 < 0$

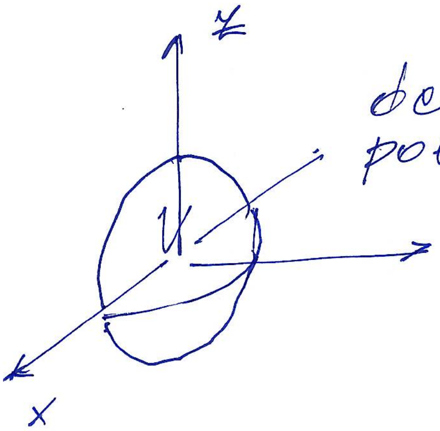
$\Rightarrow$ $\underline{T_1(0,-3) \text{ je sedlasta točka}}$

$\underline{T_2(0,3)}$, $A = 1$, $B = 0$, $C = 18$, $AC - B^2 = 18 > 0$, $A > 0$

$\Rightarrow$ $\underline{T_2(0,3) \text{ je lokalni minimum}}$

3. (20b)

$$I = \int_0^{\pi} \sin^2\varphi \, d\varphi \int_0^1 \rho^3 \, d\rho \int_{-\sqrt{1-\rho^2}}^{\sqrt{1-\rho^2}} dz \quad , \quad \text{Sferne k.}$$



desna polukugla

$$V \ldots \quad 0 \le \varphi \le \pi$$
$$0 \le \vartheta \le \pi$$
$$0 \le r \le 1$$

Podint. fja: $\rho^2 \sin^2\varphi = y^2 = r^2 \sin^2\vartheta \sin^2\varphi$

$$I = \int_0^{\pi} d\varphi \int_0^{\pi} d\vartheta \int_0^1 r^2 \sin^2\vartheta \sin^2\varphi \cdot r^2 \sin\vartheta \, dr$$

$$= \int_0^{\pi} \underbrace{\sin^3\vartheta \, d\vartheta}_{\substack{(1-\cos^2\vartheta)\sin\vartheta \\ t=\cos\vartheta}} \int_0^{\pi} \underbrace{\sin^2\varphi}_{\frac{1-\cos(2\varphi)}{2}} d\varphi \int_0^1 r^4 dr$$

$$= \int_{-1}^{1} (1-t^2)\,dt \cdot \frac{1}{2}\int_0^{\pi} (1-\cos(2\varphi))\,d\varphi \cdot \frac{r^5}{5}\Big|_0^1$$

$$= 2\left(t - \frac{t^3}{3}\right)\Big|_0^1 \cdot \frac{1}{2}\left(\varphi - \frac{1}{2}\sin(2\varphi)\right)\Big|_0^{\pi} \cdot \frac{1}{5}$$

$$= \frac{\cancel{4}^{\,2}}{3} \cdot \frac{\pi}{\cancel{2}} \cdot \frac{1}{5} = \boxed{\frac{2\pi}{15}}$$

4. a) (8b) $u(x,y,z) = e^x y + x\cos(\pi y) + \ln(2z)$

$$\operatorname{grad} u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}$$

$$= \underbrace{(e^x y + \cos(\pi y))\vec{i} + (e^x - \pi x \sin(\pi y))\vec{j} + \frac{1}{z}\vec{k}}$$

$$\operatorname{grad} u \Big|_{T(0,1,1)} = (1-1)\vec{i} + (1-0)\vec{j} + \vec{k} = \boxed{\vec{j} + \vec{k}}$$

Polje „grad u" je potencijalno po definiciji potencijala. Potencijal toga polja je skalarno polje „u".

4. b) (14b) $\vec{r} \ldots \; x(t) = e^t$, $y(t) = \cos t$, $z(t) = t$, $t \in [0, \frac{\pi}{2}]$

$x'(t) = e^t$, $y'(t) = -\sin t$, $z'(t) = 1$

$$\int_{\Gamma} x^2 dx + y^2 dy + xy\,dz = \int_0^{\frac{\pi}{2}} \big(\underbrace{e^{2t}\cdot e^t}_{e^{3t}} - \underbrace{\cos^2 t \sin t}_{\substack{u=\cos t \\ du=-\sin t\,dt}} + e^t \cos t\big)\,dt$$

$$= \left(\frac{1}{3}e^{3t} + \frac{1}{3}\cos^3 t\right)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^t \cos t\,dt \; \cancel{+} \; \frac{\cos^3 t}{3}\Big|_0^{\frac{\pi}{2}} = -\frac{1}{3}$$

$$= \left| \underline{\int_0^{\frac{\pi}{2}} e^t \cos t\,dt = I} = \left| \begin{matrix} u = e^t & du = e^t dt \\ dv = \cos t\,dt & v = \sin t \end{matrix} \right| = e^t \sin t \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^t \sin t\,dt \right.$$

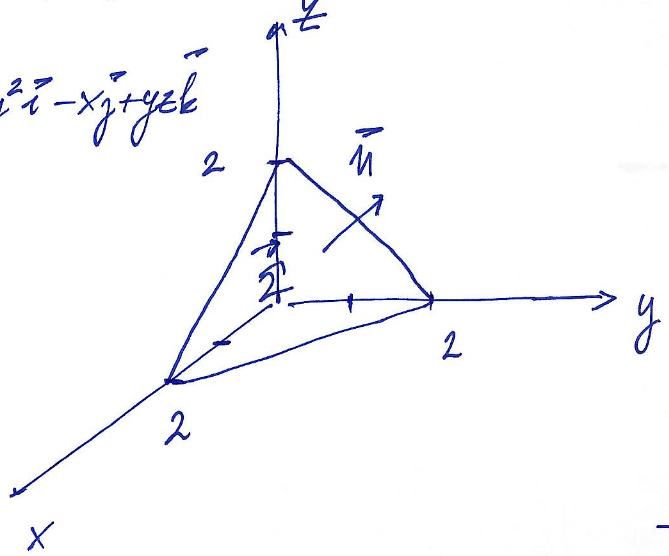
$$= \left| \begin{matrix} u = e^t \\ dv = \sin t\,dt \end{matrix} \right| = e^{\frac{\pi}{2}} + e^t \cos t \Big|_0^{\frac{\pi}{2}} - I = e^{\frac{\pi}{2}} - 1 - I \Rightarrow I = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

$$= \boxed{\frac{1}{3}\left(e^{\frac{3\pi}{2}} - 1\right) + \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)} -\frac{1}{3} = \boxed{\frac{1}{3}e^{\frac{3\pi}{2}} + \frac{1}{2}e^{\frac{\pi}{2}} - \frac{5}{6} - \frac{1}{3}} = -\frac{7}{6}$$

$$= \boxed{\frac{1}{6}\left(2e^{\frac{3\pi}{2}} + 3e^{\frac{\pi}{2}} - \cancel{7}\right)}$$

5. (18b)

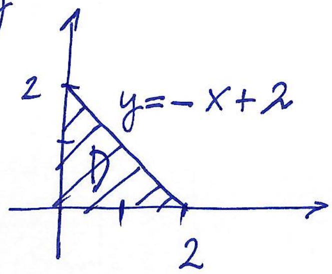
$\bar{a} = y^2\bar{i} - x\bar{j} + yz\bar{k}$



$\bar{\Sigma}_1 \ldots \underbrace{z}_{} = f(x,y) = \underline{2-x-y},$

$n = -\frac{\partial f}{\partial x}\bar{i} - \frac{\partial f}{\partial y}\bar{j} + \bar{k}$

$= \underline{+\bar{i} + \bar{j} + \bar{k}}$



$$\boxed{D.. \; 0 \le x \le 2 \qquad 0 \le y \le -x+2}$$

$$\bar{a}\cdot\bar{n} = y^2 - x + yz = y^2 - x + y(2-x-y)$$

$$= \cancel{y^2} - x + 2y - xy - \cancel{y^2} = \underline{-x + 2y - xy},$$

$$\iint_{\bar{\Sigma}_1} \bar{a}\,d\bar{S} = \iint_D \bar{a}\cdot\bar{n}\,dx\,dy = \int_0^2 dx \int_0^{-x+2} (-x + 2y - xy)\,dy$$

$$= \int_0^2 \left(-xy + y^2 - x\cdot\frac{y^2}{2}\right)\Big|_0^{-x+2} dx$$

$$= \int_0^2 \left(-x(-x+2) + (-x+2)^2 - \frac{x}{2}\cdot(-x+2)^2\right)dx$$

$$= \int_0^2 \left(x^2 - 2x + x^2 - 4x + 4 - \frac{x^3}{2} + 2x^2 - 2x\right)dx$$

$$= \int_0^2 \left(-\frac{x^3}{2} + 4x^2 - 8x + 4\right)dx$$

$$= \left(-\frac{x^4}{8} + 4\frac{x^3}{3} - 4x^2 + 4x\right)\Big|_0^2$$

$$= -2 + \frac{32}{3} - 16 + 8 = \frac{32}{3} - 10 = \boxed{\frac{2}{3}}$$