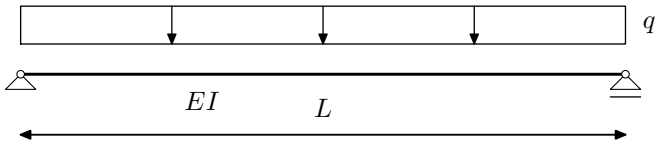
METODA KONAČIH ELEMENATA
Finite Element Method (FEM)

Primjer 1. *Metodom konačnih elemenata odrediti progib $w_{L/2}$ u sredini proste grede raspona L konstantnog poprečnog presjeka I i modula elastičnosti E, opterećene jednoliko distribuiranim opterećenjem q*



Matrica krutosti grednog konačnog elementa (nepoznati pomaci čvorova okomito na os i kutevi zaokreta čvorova, 4 stupnja slobode) duljine L^e glasi

$$\mathbf{k}^e = \begin{bmatrix} \frac{12}{L^{e3}} & -\frac{6}{L^{e2}} & -\frac{12}{L^{e3}} & -\frac{6}{L^{e2}} \\ -\frac{6}{L^{e2}} & \frac{4}{L^e} & \frac{6}{L^{e2}} & \frac{2}{L^e} \\ -\frac{12}{L^{e3}} & \frac{6}{L^{e2}} & \frac{12}{L^{e3}} & \frac{6}{L^{e2}} \\ -\frac{6}{L^{e2}} & \frac{2}{L^e} & \frac{6}{L^{e2}} & \frac{4}{L^e} \end{bmatrix} . \tag{1.1}$$

Vektor sila upetosti takvog grednog konačnog elementa opterećenog jednoliko distribuiranim opterećenjem glasi

$$\mathbf{q}^e = \begin{bmatrix} \frac{qL^e}{2} \\ -\frac{qL^{e2}}{12} \\ \frac{qL^e}{2} \\ \frac{qL^{e2}}{12} \end{bmatrix} . \tag{1.2}$$

Podijelimo područje (gredu) na dva konačna elementa jednake duljine, $L^e = L/2$. Elementarna matrica krutosti svakog konačnog elementa glasi

$$\mathbf{k}^e = \begin{bmatrix} \frac{96}{L^3} & -\frac{24}{L^2} & -\frac{96}{L^3} & -\frac{24}{L^2} \\ -\frac{24}{L^2} & \frac{8}{L} & \frac{24}{L^2} & \frac{4}{L} \\ -\frac{96}{L^3} & \frac{24}{L^2} & \frac{96}{L^3} & \frac{24}{L^2} \\ -\frac{24}{L^2} & \frac{4}{L} & \frac{24}{L^2} & \frac{8}{L} \end{bmatrix} , \tag{1.3}$$

elementarni vektor sila upetosti svakog konačnog elementa glasi

$$\mathbf{q}^e = \begin{bmatrix} \frac{qL}{4} \\ -\frac{qL^2}{48} \\ \frac{qL}{4} \\ \frac{qL^2}{48} \end{bmatrix} . \tag{1.4}$$

Uklapanjem elementarnih matrica krutosti i elementarnih vektora upetosti slijede globalna matrica krutosti

$$\begin{aligned} \mathbf{K} &= \begin{bmatrix} \frac{96}{L^3} & -\frac{24}{L^2} & -\frac{96}{L^3} & -\frac{24}{L^2} & 0 & 0 \\ -\frac{24}{L^2} & \frac{8}{L} & \frac{24}{L^2} & \frac{4}{L} & 0 & 0 \\ -\frac{96}{L^3} & \frac{24}{L^2} & \frac{96}{L^3}+\frac{96}{L^3} & \frac{24}{L^2}-\frac{24}{L^2} & -\frac{96}{L^3} & -\frac{24}{L^2} \\ -\frac{24}{L^2} & \frac{4}{L} & \frac{24}{L^2}-\frac{24}{L^2} & \frac{8}{L}+\frac{8}{L} & \frac{24}{L^2} & \frac{4}{L} \\ 0 & 0 & -\frac{96}{L^3} & \frac{24}{L^2} & \frac{96}{L^3} & \frac{24}{L^2} \\ 0 & 0 & -\frac{24}{L^2} & \frac{4}{L} & \frac{24}{L^2} & \frac{8}{L} \end{bmatrix} \\ &= \begin{bmatrix} \frac{96}{L^3} & -\frac{24}{L^2} & -\frac{96}{L^3} & -\frac{24}{L^2} & 0 & 0 \\ -\frac{24}{L^2} & \frac{8}{L} & \frac{24}{L^2} & \frac{4}{L} & 0 & 0 \\ -\frac{96}{L^3} & \frac{24}{L^2} & \frac{192}{L^3} & 0 & -\frac{96}{L^3} & -\frac{24}{L^2} \\ -\frac{24}{L^2} & \frac{4}{L} & 0 & \frac{16}{L} & \frac{24}{L^2} & \frac{4}{L} \\ 0 & 0 & -\frac{96}{L^3} & \frac{24}{L^2} & \frac{96}{L^3} & \frac{24}{L^2} \\ 0 & 0 & -\frac{24}{L^2} & \frac{4}{L} & \frac{24}{L^2} & \frac{8}{L} \end{bmatrix} , \end{aligned} \tag{1.5}$$

i globalni vektor sila upetosti

$$\mathbf{q} = \begin{bmatrix} \frac{qL}{4} \\ -\frac{qL^2}{48} \\ \frac{qL}{4}+\frac{qL}{4} \\ \frac{qL^2}{48}-\frac{qL^2}{48} \\ \frac{qL}{4} \\ \frac{qL^2}{48} \end{bmatrix} = \begin{bmatrix} \frac{qL}{4} \\ -\frac{qL^2}{48} \\ \frac{qL}{2} \\ 0 \\ \frac{qL}{4} \\ \frac{qL^2}{48} \end{bmatrix} . \tag{1.6}$$

Dobivamo sustav jednadžbi

$$\begin{bmatrix} \frac{96}{L^3} & -\frac{24}{L^2} & -\frac{96}{L^3} & -\frac{24}{L^2} & 0 & 0 \\ -\frac{24}{L^2} & \frac{8}{L} & \frac{24}{L^2} & \frac{4}{L} & 0 & 0 \\ -\frac{96}{L^3} & \frac{24}{L^2} & \frac{192}{L^3} & 0 & -\frac{96}{L^3} & -\frac{24}{L^2} \\ -\frac{24}{L^2} & \frac{4}{L} & 0 & \frac{16}{L} & \frac{24}{L^2} & \frac{4}{L} \\ 0 & 0 - \frac{96}{L^3} & \frac{24}{L^2} & \frac{96}{L^3} & \frac{24}{L^2} & \\ 0 & 0 & -\frac{24}{L^2} & \frac{4}{L} & \frac{24}{L^2} & \frac{8}{L} \end{bmatrix} \begin{bmatrix} w_0 \\ \varphi_0 \\ w_1 \\ \varphi_1 \\ w_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} \frac{qL}{4} \\ -\frac{qL^2}{48} \\ \frac{qL}{2} \\ 0 \\ \frac{qL}{4} \\ \frac{qL^2}{48} \end{bmatrix} . \tag{1.7}$$

Uvrštavanjem rubnih uvjeta, slobodno oslonjeni rubovi $x = 0$ i $x = L$, $w_0 = w_2 = 0$, slijedi manji sustav

$$\begin{bmatrix} \frac{8}{L} & \frac{24}{L^2} & \frac{4}{L} & 0 \\ \frac{24}{L^2} & \frac{192}{L^3} & 0 & -\frac{24}{L^2} \\ \frac{4}{L} & 0 & \frac{16}{L} & \frac{4}{L} \\ 0 & -\frac{24}{L^2} & \frac{4}{L} & \frac{8}{L} \end{bmatrix} \begin{bmatrix} \varphi_0 \\ w_1 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} -\frac{qL^2}{48} \\ \frac{qL}{2} \\ 0 \\ \frac{qL^2}{48} \end{bmatrix} . \tag{1.8}$$

Rješenje sustava daje vrijednosti vektora nepoznatih pomaka

$$\begin{bmatrix} \varphi_0 \\ w_1 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} -\frac{qL^3}{24EI} \\ \frac{5qL^4}{384EI} \\ 0 \\ \frac{qL^3}{24EI} \end{bmatrix} . \tag{1.9}$$