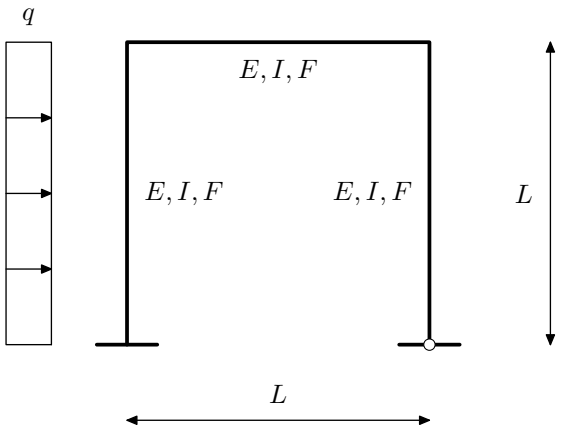
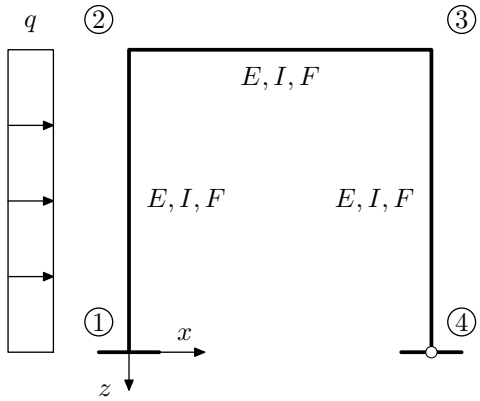
Na zadanom okviru metodom pomaka odredite dijagram momenata



oznake čvorova i globalni koordinatni sustav:



numeriki primjer proračunat je uz vrijednosti
$L = 4,0$ m, $B = 30$ cm, $H = 40$ cm, $E = 3 \cdot 10^7$ kN/m^2, $q = 10\ kN/m$

krutosti štapova:

$$k^a_{ik} = \frac{E_{ik}F_{ik}}{L_{ik}} \quad \Rightarrow \quad k^a_{12} = k^a_{23} = k^a_{34} = \frac{EF}{L}$$
$$k^f_{ik} = \frac{E_{ik}I_{ik}}{L_{ik}} \quad \Rightarrow \quad k^f_{12} = k^f_{23} = k^f_{34} = \frac{EI}{L}$$

vektori sila upetosti u lokalnom koordinatnom sustavu:

$$\overline{\mathbf{f}}_{12} = \begin{bmatrix} 0 \\ -\frac{qL}{2} \\ \frac{qL^2}{12} \\ 0 \\ -\frac{qL}{2} \\ -\frac{qL^2}{12} \end{bmatrix}, \ \overline{\mathbf{f}}_{23} = \overline{\mathbf{f}}_{34} = \mathbf{0}$$

matrica krutosti štapova u lokalnom kordinatnom sustavu (jednaka za sve štapove zbog jednake geometrije):

$$\mathbf{K}_{ik}^{loc} = \begin{bmatrix} k_{ik}^a & 0 & 0 & -k_{ik}^a & 0 & 0 \\ 0 & \frac{12k_{ik}^f}{L_{ik}^2} & -\frac{6k_{ik}^f}{L_{ik}} & 0 & -\frac{12k_{ik}^f}{L_{ik}^2} & -\frac{6k_{ik}^f}{L_{ik}} \\ 0 & -\frac{6k_{ik}^f}{L_{ik}} & 4k_{ik}^f & 0 & \frac{6k_{ik}^f}{L_{ik}} & 2k_{ik}^f \\ -k_{ik}^a & 0 & 0 & k_{ik}^a & 0 & 0 \\ 0 & -\frac{12k_{ik}^f}{L_{ik}^2} & \frac{6k_{ik}^f}{L_{ik}} & 0 & \frac{12k_{ik}^f}{L_{ik}^2} & \frac{6k_{ik}^f}{L_{ik}} \\ 0 & -\frac{6k_{ik}^f}{L_{ik}} & 2k_{ik}^f & 0 & \frac{6k_{ik}^f}{L_{ik}} & 4k_{ik}^f \end{bmatrix}$$

$$= \begin{bmatrix} \frac{EF}{L} & 0 & 0 & -\frac{EF}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & 4\frac{EI}{L} & 0 & \frac{6EI}{L^2} & 2\frac{EI}{L} \\ -\frac{EF}{L} & 0 & 0 & \frac{EF}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & 2\frac{EI}{L} & 0 & \frac{6EI}{L^2} & 4\frac{EI}{L} \end{bmatrix}$$

$$= \begin{bmatrix} 9 \cdot 10^5 & 0 & 0 & -9 \cdot 10^5 & 0 & 0 \\ 0 & 9000 & -18000 & 0 & -9000 & -18000 \\ 0 & -18000 & 48000 & 0 & 18000 & 24000 \\ -9 \cdot 10^5 & 0 & 0 & 9 \cdot 10^5 & 0 & 0 \\ 0 & -9000 & 18000 & 0 & 9000 & 18000 \\ 0 & -18000 & 24000 & 0 & 18000 & 48000 \end{bmatrix}$$

kondenzacija lokalne matrice krutosti štapa (34):

$$
\begin{aligned}
M_{43} = 0 \quad &\Rightarrow \quad k_{62}w_{34} + k_{63}\varphi_{34} + k_{65}w_{43} + k_{66}\varphi_{43} = 0 \\
&\Rightarrow \quad \varphi_{43} = -\frac{1}{k_{66}}\left(k_{62}w_{34} + k_{63}\varphi_{34} + k_{65}w_{43}\right) \\
&\qquad = -\frac{1}{\frac{4EI}{L}}\left(-\frac{6EI}{L^2}w_{34} + \frac{2EI}{L}\varphi_{34} + \frac{6EI}{L^2}w_{43}\right) \\
&\qquad = \frac{3}{2L}w_{34} - \frac{1}{2}\varphi_{34} - \frac{3}{2L}w_{43}
\end{aligned}
$$

lokalna matrica krutosti štapa (34):

$$
\mathbf{K}_{34}^{loc} = \begin{bmatrix}
k_{34}^a & 0 & 0 & -k_{34}^a & 0 & 0 \\
0 & \frac{3k_{ik}^f}{L_{ik}^2} & -\frac{3k_{ik}^f}{L_{ik}} & 0 & -\frac{3k_{ik}^f}{L_{ik}^2} & 0 \\
0 & -\frac{3k_{ik}^f}{L_{ik}} & 3k_{ik}^f & 0 & \frac{3k_{ik}^f}{L_{ik}} & 0 \\
-k_{ik}^a & 0 & 0 & k_{ik}^a & 0 & 0 \\
0 & -\frac{3k_{ik}^f}{L_{ik}^2} & \frac{3k_{ik}^f}{L_{ik}} & 0 & \frac{3k_{ik}^f}{L_{ik}^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{EF}{L} & 0 & 0 & -\frac{EF}{L} & 0 & 0 \\
0 & \frac{3EI}{L^3} & -\frac{3EI}{L^2} & 0 & -\frac{3EI}{L^3} & 0 \\
0 & -\frac{3EI}{L^2} & 3\frac{EI}{L} & 0 & \frac{3EI}{L^2} & 0 \\
-\frac{EF}{L} & 0 & 0 & \frac{EF}{L} & 0 & 0 \\
0 & -\frac{3EI}{L^3} & \frac{3EI}{L^2} & 0 & \frac{3EI}{L^3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
9 \cdot 10^5 & 0 & 0 & -9 \cdot 10^5 & 0 & 0 \\
0 & 2250 & -9000 & 0 & -2250 & 0 \\
0 & -9000 & 36000 & 0 & 9000 & 0 \\
-9 \cdot 10^5 & 0 & 0 & 9 \cdot 10^5 & 0 & 0 \\
0 & -2250 & 9000 & 0 & 2250 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

matrice transformacije

$$\mathbf{r}_{ik}^{loc.\to gl.} = \begin{bmatrix} \cos\alpha_{ik} & \sin\alpha_{ik} & 0 \\ -\sin\alpha_{ik} & \cos\alpha_{ik} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}_{12}^{loc.\to gl.} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}_{23}^{loc.\to gl.} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}_{34}^{loc.\to gl.} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}_{ik}^{gl.\to loc.} = \begin{bmatrix} \cos\alpha_{ik} & -\sin\alpha_{ik} & 0 \\ \sin\alpha_{ik} & \cos\alpha_{ik} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}_{12}^{gl.\to loc.} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}_{23}^{gl.\to loc.} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r}_{34}^{gl.\to loc.} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

globalni vektor upetosti

$$\mathbf{f}_{ik}^{gl} = \begin{bmatrix} \mathbf{r}_{ik}^{loc.\to gl.}\mathbf{f}_{iki} \\ \mathbf{r}_{ik}^{loc.\to gl.}\mathbf{f}_{ikk} \end{bmatrix}$$

$$\mathbf{f}_{12}^{gl} = \begin{bmatrix} -\frac{ql}{2} \\ 0 \\ \frac{ql^2}{12} \\ -\frac{qL}{2} \\ -\frac{qL^2}{12} \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ 13.33 \\ -20 \\ 0 \\ -13.33 \end{bmatrix}$$

globalne matrice krutosti

$$\mathbf{K}_{ik}^{gl} = \begin{bmatrix} \mathbf{r}_{ik}^{loc.\to gl.}\mathbf{k}_{ikii}\mathbf{r}_{ik}^{gl.\to loc.} & \mathbf{r}_{ik}^{loc.\to gl.}\mathbf{k}_{ikik}\mathbf{r}_{ik}^{gl.\to loc.} \\ \mathbf{r}_{ik}^{loc.\to gl.}\mathbf{k}_{ikki}\mathbf{r}_{ik}^{gl.\to loc.} & \mathbf{r}_{ik}^{loc.\to gl.}\mathbf{k}_{ikkk}\mathbf{r}_{ik}^{gl.\to loc.} \end{bmatrix}$$

$$k_{12}^{gl} = \begin{bmatrix} 9000 & 0 & -18000 & -9000 & 0 & -18000 \\ 0 & 9 \cdot 10^5 & 0 & 0 & -9 \cdot 10^5 & 0 \\ -18000 & 0 & 48000 & 18000 & 0 & 24000 \\ -9000 & 0 & 18000 & 9000 & 0 & 18000 \\ 0 & -9 \cdot 10^5 & 0 & 0 & 9 \cdot 10^5 & 0 \\ -18000 & 0 & 24000 & 18000 & 0 & 48000 \end{bmatrix}$$

$$k_{23}^{gl} = \begin{bmatrix} 9 \cdot 10^5 & 0 & 0 & -9 \cdot 10^5 & 0 & 0 \\ 0 & 9000 & -18000 & 0 & -9000 & -18000 \\ 0 & -18000 & 48000 & 0 & 18000 & 24000 \\ -9 \cdot 10^5 & 0 & 0 & 9 \cdot 10^5 & 0 & 0 \\ 0 & -9000 & 18000 & 0 & 9000 & 18000 \\ 0 & -18000 & 24000 & 0 & 18000 & 48000 \end{bmatrix}$$

$$k_{34}^{gl} = \begin{bmatrix} 2250 & 0 & 9000 & -2250 & 0 & 0 \\ 0 & 9 \cdot 10^5 & 0 & 0 & -9 \cdot 10^5 & 0 \\ 9000 & 0 & 36000 & 9000 & 0 & 0 \\ -2250 & 0 & 9000 & -2250 & 0 & 0 \\ 0 & -9 \cdot 10^5 & 0 & 0 & 9 \cdot 10^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

uklapanjem matrica krutosti štapova slijedi matrica krutosti konstrukcije 12×12, uvrštavanjem rubnih uvjeta slijedi globalna matrica krutosti, 6×6:

$$\mathbf{K}^{gl} = \begin{bmatrix} 909000 & 0 & 18000 & -900000 & 0 & 0 \\ 0 & 909000 & -18000 & 0 & -9000 & -18000 \\ 18000 & -18000 & 96000 & 0 & 18000 & 24000 \\ -900000 & 0 & 0 & 902250 & 0 & 9000 \\ 0 & -9000 & 18000 & 0 & 909000 & 18000 \\ 0 & -18000 & 24000 & 9000 & 18000 & 84000 \end{bmatrix}$$

globalni vektor upetosti:

$$\mathbf{f}^{gl} = \begin{bmatrix} -20 \\ 0 \\ -13.33 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sustav jednadžbi sada glasi:

$$\begin{bmatrix} 909000 & 0 & 18000 & -900000 & 0 & 0 \\ 0 & 909000 & -18000 & 0 & -9000 & -18000 \\ 18000 & -18000 & 96000 & 0 & 18000 & 24000 \\ -900000 & 0 & 0 & 902250 & 0 & 9000 \\ 0 & -9000 & 18000 & 0 & 909000 & 18000 \\ 0 & -18000 & 24000 & 9000 & 18000 & 84000 \end{bmatrix} \begin{bmatrix} x_2 \\ z_2 \\ \varphi_2 \\ x_3 \\ z_3 \\ \varphi_3 \end{bmatrix} + \begin{bmatrix} -20 \\ 0 \\ -13.33 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

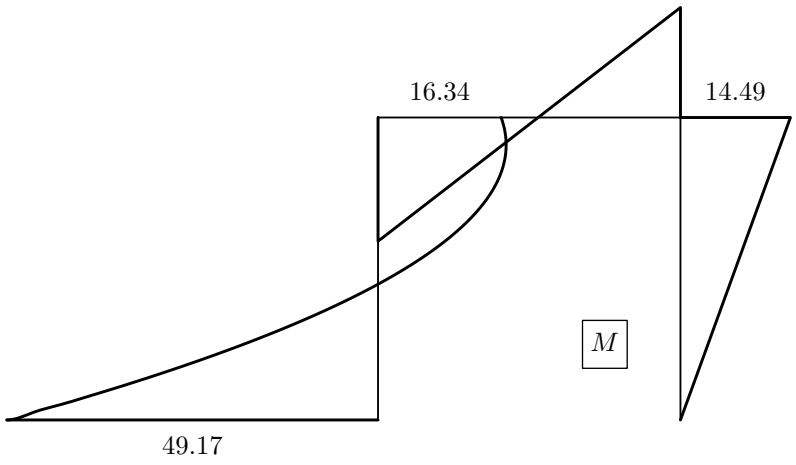
približno rješenje sustava jednadžbi:

$$\begin{aligned} x_2 &= 0.0023335 \text{ m} \\ z_2 &= -0.0000086 \text{ m} \\ \varphi_2 &= -0.0002569 \text{ rad} \\ x_2 &= 0.0023295 \text{ m} \\ z_2 &= 0.0000086 \text{ m} \\ \varphi_3 &= -0.0001799 \text{ rad} \end{aligned}$$

momenti na krajevima štapova (kao funkcije nepoznatih pomaka u globalnom koordinatnom sustavu):

$$
\begin{aligned}
M_{12} &= b_{12}\varphi_2 + c_{12}x_2 + \overline{M_{12}} \\
&= \frac{2EI}{L}\varphi_2 + \frac{6EI}{L^2}x_2 + \frac{qL^2}{12} = 49.171 \text{ kNm} \\
M_{21} &= a_{21}\varphi_2 + c_{21}x_2 + \overline{M_{12}} \\
&= \frac{4EI}{L}\varphi_2 + \frac{6EI}{L^2}x_2 - \frac{qL^2}{12} = 16.339 \text{ kNm} \\
M_{23} &= a_{23}\varphi_2 + b_{23}\varphi_3 - c_{23}z_2 + c_{23}z_3 \\
&= \frac{4EI}{L}\varphi_2 + \frac{2EI}{L}\varphi_3 - \frac{6EI}{L^2}z_2 + \frac{6EI}{L^2}z_3 = -16.339 \text{ kNm} \\
M_{32} &= b_{32}\varphi_2 + a_{32}\varphi_3 - c_{23}z_2 + c_{23}z_3 \\
&= \frac{2EI}{L}\varphi_2 + \frac{4EI}{L}\varphi_3 - \frac{6EI}{L^2}z_2 + \frac{6EI}{L^2}z_3 = -14.490 \text{ kNm} \\
M_{34} &= a_{34}\varphi_3 + c_{34}x_3 \\
&= \frac{4EI}{L}\varphi_3 + \frac{3EI}{L^2}x_3 = 14.490 \text{ kNm}
\end{aligned}
$$

momentni dijagram



ako riješimo zadatak bez kondenzacije lokalne matrice krutosti štapa (34) slijedi:

$$k_{34}^{gl} = \begin{bmatrix} 9000 & 0 & 18000 & -9000 & 0 & -18000 \\ 0 & 9 \cdot 10^5 & 0 & 0 & -9 \cdot 10^5 & 0 \\ 18000 & 0 & 48000 & 18000 & 0 & 24000 \\ -9000 & 0 & 18000 & -9000 & 0 & 18000 \\ 0 & -9 \cdot 10^5 & 0 & 0 & 9 \cdot 10^5 & 0 \\ 18000 & 0 & 24000 & -18000 & 0 & 48000 \end{bmatrix}$$

uklapanjem matrica krutosti štapova slijedi matrica krutosti konstrukcije 12×12, uvrštavanjem rubnih uvjeta slijedi globalna matrica krutosti, 7×7:

$$\mathbf{K}^{gl} = \begin{bmatrix} 909000 & 0 & 18000 & -900000 & 0 & 0 & 0 \\ 0 & 909000 & -18000 & 0 & -9000 & -18000 & 0 \\ 18000 & -18000 & 96000 & 0 & 18000 & 24000 & 0 \\ -900000 & 0 & 0 & 909000 & 0 & 18000 & -18000 \\ 0 & -9000 & 18000 & 0 & 909000 & 18000 & 0 \\ 0 & -18000 & 24000 & 18000 & 18000 & 96000 & 24000 \\ 0 & 0 & 0 & 18000 & 0 & 24000 & 48000 \end{bmatrix}$$

globalni vektor upetosti:

$$\mathbf{f}^{gl} = \begin{bmatrix} -20 \\ 0 \\ -13.33 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sustav jednadžbi sada glasi:

$$\begin{bmatrix} 909000 & 0 & 18000 & -900000 & 0 & 0 & 0 \\ 0 & 909000 & -18000 & 0 & -9000 & -18000 & 0 \\ 18000 & -18000 & 96000 & 0 & 18000 & 24000 & 0 \\ -900000 & 0 & 0 & 909000 & 0 & 18000 & -18000 \\ 0 & -9000 & 18000 & 0 & 909000 & 18000 & 0 \\ 0 & -18000 & 24000 & 18000 & 18000 & 96000 & 24000 \\ 0 & 0 & 0 & 18000 & 0 & 24000 & 48000 \end{bmatrix} \begin{bmatrix} x_2 \\ z_2 \\ \varphi_2 \\ x_3 \\ z_3 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} + \begin{bmatrix} -20 \\ 0 \\ -13.33 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

približno rješenje sustava jednadžbi:

$$\begin{aligned} x_2 &= 0.0023335 \text{ m} \\ z_2 &= -0.0000086 \text{ m} \\ \varphi_2 &= -0.0002569 \text{ rad} \\ x_2 &= 0.0023295 \text{ m} \\ z_2 &= 0.0000086 \text{ m} \\ \varphi_3 &= -0.0001799 \text{ rad} \\ \varphi_4 &= -0.0007836 \text{ rad} \end{aligned}$$

uvrštavanjem dobivenih vrijednosti u izraz kojim smo izrazili kut φ_4 kao funkciju ostalih nepoznanica slijedi identična vrijednsot za kut kao u rješenju sustava, na taj način jasno je da su i momenti na krajevima štapova jednaki kao i slučju rješavanja s prethodnom kondenzacijom