2.3 Parcijalne derivacije

Definicija. Neka je $\Omega \subseteq \mathbb{R}^2$ otvoren podskup, $f\colon \Omega \to \mathbb{R}$ i $(x_0, y_0) \in \Omega$. *Parcijalna derivacija od f po varijabli x u točki* (x_0, y_0) je sljedeći limes (ako postoji):

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}.$$

Parcijalna derivacija od f po varijabli y u točki (x_0, y_0) je sljedeći limes (ako postoji):

$$\frac{\partial f}{\partial y}(x_0, y_0) := \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.$$

Ako funkcija f ima parcijalne derivacije po varijabli x (ili po varijabli y) u svim točkama skupa Ω, onda je dobro definirana funkcija $\frac{\partial f}{\partial x}\colon \Omega \to \mathbb{R}$ (ili $\frac{\partial f}{\partial y}\colon \Omega \to \mathbb{R}$), koju zovemo *parcijalna derivacija funkcije f po varijabli x* (ili *po varijabli y*).

Ukoliko postoji $\frac{\partial f}{\partial x}$, tada se radi također o funkciji dvije varijable, koja može imati svoje pripadne parcijalne derivacije po varijabli x, odnosno y; ako postoje, označavamo ih redom $\frac{\partial^2 f}{\partial x^2}$ i $\frac{\partial^2 f}{\partial y \partial x}$. Analogno, koristimo oznake $\frac{\partial^2 f}{\partial x \partial y}$ i $\frac{\partial^2 f}{\partial y^2}$ za parcijalne derivacije funkcije $\frac{\partial f}{\partial y}$ po varijablama x i y redom.

Kažemo da su funkcije $\frac{\partial f}{\partial x}$ i $\frac{\partial f}{\partial y}$ parcijalne derivacije *prvog reda* od f, te funkcije $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$ parcijalne derivacije *drugog reda* od f. Analogno definiramo parcijalne derivacije od f viših redova.

Kažemo da je funkcija *klase C^k*, ako ima parcijalne derivacije svih redova do uključivo k i one su neprekidne. Kažemo da je funkcija *glatka* ili *klase C^∞*, ako ima neprekidne parcijalne derivacije svih redova. Termin "*dovoljno glatka*" podrazumijeva da postoje i neprekidne su sve parcijalne derivacije koje se javljaju u računu.

Potpuno analogno definiramo parcijalne derivacije funkcija u tri ili više varijabli, te klase C^k.

Teorem (Schwarz)**.** *Neka funkcija f ima neprekidne parcijalne derivacije* *prvog i drugog reda. Tada vrijedi*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Analogna tvrdnja vrijedi za funkcije u više od dvije varijable. Dakle, parcijalne derivacije funkcije više varijabli možemo uzimati u bilo kojem poretku.

Stoga za funkciju f klase C^2 ne razlikujemo $\dfrac{\partial^2 f}{\partial y \partial x}$ i $\dfrac{\partial^2 f}{\partial x \partial y}$, te oboje zovemo *mješovitom parcijalnom derivacijom drugog reda.*

Parcijalne derivacije glatke funkcije u pravilu ne računamo po definiciji, nego ju deriviramo kao funkciju jedne varijable (one po kojoj uzimamo parcijalnu derivaciju), a prema svim ostalim varijablama se ponašamo kao da su konstante.

Zadatak 2.6. Izračunajte parcijalne derivacije prvog i drugog reda sljedećih funkcija:

(a) $f(x, y) = x^2 y$

(b) $f(x, y) = \cos \dfrac{x}{y}$

(c) $f(x, y, z) = \dfrac{x + y}{z}$

(d) $f(x, y, z) = z^{xy}$. Odredite i $\left.\dfrac{\partial^2 f}{\partial x \partial y}\right|_{(1,2,3)}$.

Rješenje:

(a)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 y\right) = y \cdot \frac{\partial}{\partial x}\left(x^2\right) = y \cdot 2x = 2xy$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y\right) = x^2 \cdot \frac{\partial}{\partial y}(y) = x^2 \cdot 1 = x^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy) = 2y \cdot \frac{\partial}{\partial x}(x) = 2y \cdot 1 = 2y$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy) = 2x \cdot \frac{\partial}{\partial y}(y) = 2x \cdot 1 = 2x$$

$$\frac{\partial^2 f}{\partial x \partial y} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial y \partial x} = 2x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2\right) = 0.$$

Kad deriviramo po varijabli y, prema x^2 se ponašamo kao da je konstanta i zato je $\frac{\partial}{\partial y}\left(x^2\right) = 0$ (jer je derivacija konstante 0).

(b)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos\frac{x}{y}\right) = -\sin\left(\frac{x}{y}\right) \cdot \frac{1}{y} = -\frac{1}{y}\sin\frac{x}{y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\cos\frac{x}{y}\right) = -\sin\left(\frac{x}{y}\right) \cdot \frac{-x}{y^2} = \frac{x}{y^2}\sin\frac{x}{y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{y}\sin\frac{x}{y}\right) = -\frac{1}{y}\cos\left(\frac{x}{y}\right) \cdot \frac{1}{y} = -\frac{1}{y^2}\cos\frac{x}{y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{1}{y}\sin\frac{x}{y}\right)$$

$$= -\frac{-1}{y^2} \cdot \sin\frac{x}{y} - \frac{1}{y} \cdot \cos\left(\frac{x}{y}\right) \cdot \frac{-x}{y^2} = \frac{1}{y^2}\sin\frac{x}{y} + \frac{x}{y^3}\cos\frac{x}{y}$$

$$\frac{\partial^2 f}{\partial x \partial y} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{y^2}\sin\frac{x}{y} + \frac{x}{y^3}\cos\frac{x}{y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{y^2}\sin\frac{x}{y}\right)$$

$$= \frac{-2x}{y^3} \cdot \sin\frac{x}{y} + \frac{x}{y^2} \cdot \cos\left(\frac{x}{y}\right) \cdot \frac{-x}{y^2} = -\frac{2x}{y^3}\sin\frac{x}{y} - \frac{x^2}{y^4}\cos\frac{x}{y}.$$

(c) Vrijedi $f(x, y, z) = \dfrac{x+y}{z} = \dfrac{x}{z} + \dfrac{y}{z}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{z} + \frac{y}{z}\right) = \frac{1}{z}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{z} + \frac{y}{z}\right) = \frac{1}{z}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x+y}{z}\right) = -\frac{x+y}{z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{z}\right) = 0$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{z}\right) = 0$$
$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial y \partial x} = 0$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{z}\right) = 0$$
$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$$

$$\frac{\partial^2 f}{\partial x \partial z} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial z \partial x} = -\frac{1}{z^2}$$
$$\frac{\partial^2 f}{\partial y \partial z} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial z \partial y} = -\frac{1}{z^2}$$
$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(-\frac{x+y}{z^2}\right) = \frac{2x+2y}{z^3}.$$

(d)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(z^{xy}\right) = z^{xy} \cdot \ln z \cdot y$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(z^{xy}\right) = z^{xy} \cdot \ln z \cdot x$$
$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(z^{xy}\right) = xy \cdot z^{xy-1}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(z^{xy} \cdot \ln z \cdot y\right) = z^{xy} \cdot \ln^2 z \cdot y^2$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(z^{xy} \cdot \ln z \cdot y\right)$$
$$= z^{xy} \cdot \ln z \cdot x \cdot \ln z \cdot y + z^{xy} \cdot \ln z = z^{xy} \cdot \ln^2 z \cdot xy + z^{xy} \cdot \ln z$$
$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial z}\left(z^{xy} \cdot \ln z \cdot y\right)$$
$$= xy \cdot z^{xy-1} \cdot \ln z \cdot y + z^{xy} \cdot \frac{1}{z} \cdot y = xy^2 \cdot z^{xy-1} \cdot \ln z + y \cdot z^{xy-1}$$

$$\frac{\partial^2 f}{\partial x \partial y} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial y \partial x} = z^{xy} \cdot \ln^2 z \cdot xy + z^{xy} \cdot \ln z$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(z^{xy} \cdot \ln z \cdot x\right) = z^{xy} \cdot \ln^2 z \cdot x^2$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\left(z^{xy} \cdot \ln z \cdot x\right) =$$

$$= xy \cdot z^{xy-1} \cdot \ln z \cdot x + z^{xy} \cdot \frac{1}{z} \cdot x = x^2 y \cdot z^{xy-1} \cdot \ln z + x \cdot z^{xy-1}$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,2,e)} = \frac{\partial^2 f}{\partial x \partial y}(1,2,e) = e^2 \cdot \ln^2 e \cdot 2 + e^2 \cdot \ln e = 3e^2$$

$$\frac{\partial^2 f}{\partial x \partial z} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial z \partial x} = xy^2 \cdot z^{xy-1} \cdot \ln z + y \cdot z^{xy-1}$$

$$\frac{\partial^2 f}{\partial y \partial z} \overset{\text{Schwarzov teorem}}{=} \frac{\partial^2 f}{\partial z \partial y} = x^2 y \cdot z^{xy-1} \cdot \ln z + x \cdot z^{xy-1}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(xy \cdot z^{xy-1}\right) = xy(xy-1)z^{xy-2}.$$

□

Zadatak 2.7. Neka je $\varphi\colon I \to \mathbb{R}$ glatka funkcija definirana na otvorenom intervalu $I \subseteq \mathbb{R}$. Odredite $\frac{\partial f}{\partial x}$ i $\frac{\partial f}{\partial y}$ za

$$f(x,y) = y\varphi(\sqrt{x^2+y^2}).$$

Rješenje: Računamo prvo prvu parcijalnu derivaciju po x:

$$\begin{aligned}\frac{\partial f}{\partial x} = y \cdot \frac{\partial}{\partial x}\left(\varphi(\sqrt{x^2+y^2})\right) &= y \cdot \varphi'(\sqrt{x^2+y^2}) \cdot \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right) \\ &= y \cdot \varphi'(\sqrt{x^2+y^2}) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x \\ &= \frac{xy\varphi'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}.\end{aligned}$$

Sad računamo prvu parcijalnu derivaciju po y

$$\begin{aligned}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y\varphi(\sqrt{x^2+y^2})\right) &= \varphi(\sqrt{x^2+y^2}) + y\cdot\varphi'(\sqrt{x^2+y^2})\cdot\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right)\\ &= \varphi(\sqrt{x^2+y^2}) + y\cdot\varphi'(\sqrt{x^2+y^2})\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y\\ &= \varphi(\sqrt{x^2+y^2}) + \frac{y^2\varphi'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}.\end{aligned}$$

□

Zadatak 2.8. Neka je $\varphi\colon I\to\mathbb{R}$ glatka funkcija definirana na otvorenom intervalu $I\subseteq\mathbb{R}$, i stavimo

$$z(x,y) = \frac{4x}{\varphi(x-y^2)-x-y^2}.$$

Pokažite da vrijedi jednakost:

$$2x\cdot\frac{\partial z}{\partial x} + \frac{x}{y}\cdot\frac{\partial z}{\partial y} = 2z + z^2.$$

Rješenje: Izračunat ćemo posebno lijevu i posebno desnu stranu gornje jednakosti i pokazati da su one jednake.

Računamo prvo parcijalne derivacije prvog reda funkcije z:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{4\cdot(\varphi(x-y^2)-x-y^2) - 4x\cdot(\varphi'(x-y^2)\cdot 1 - 1 + 0)}{\left(\varphi(x-y^2)-x-y^2\right)^2}\\ &= \frac{4\cdot\varphi(x-y^2) - 4x\cdot\varphi'(x-y^2) - 4y^2}{\left(\varphi(x-y^2)-x-y^2\right)^2}\\ \frac{\partial z}{\partial y} &= \frac{-4x\cdot(\varphi'(x-y^2)\cdot(-2y) - 0 - 2y)}{\left(\varphi(x-y^2)-x-y^2\right)^2}\\ &= \frac{8xy\cdot\varphi'(x-y^2) + 8xy}{\left(\varphi(x-y^2)-x-y^2\right)^2}.\end{aligned}$$

Zatim ih uvrstimo u lijevu stranu tražene jednakosti da bi dobili:

$$\begin{aligned}2x\cdot\frac{\partial z}{\partial x} + \frac{x}{y}\cdot\frac{\partial z}{\partial y} &= 2x\cdot\frac{4\cdot\varphi(x-y^2) - 4x\cdot\varphi'(x-y^2) - 4y^2}{\left(\varphi(x-y^2)-x-y^2\right)^2}\\ &\quad + \frac{x}{y}\cdot\frac{8xy\cdot\varphi'(x-y^2) + 8xy}{\left(\varphi(x-y^2)-x-y^2\right)^2}\\ &= \frac{8x\cdot\varphi(x-2y) - 8xy^2 + 8x^2}{\left(\varphi(x-y^2)-x-y^2\right)^2}\end{aligned}$$

Konačno, desnu stranu tražene jednakosti možemo raspisati kao

$$\begin{aligned}
2z + z^2 &= \frac{8x}{\varphi(x-y^2) - x - y^2} + \frac{16x^2}{\left(\varphi(x-y^2) - x - y^2\right)^2} \\
&= \frac{8x \cdot \varphi(x-y^2) - 8x^2 - 8xy^2 + 16x^2}{\left(\varphi(x-y^2) - x - y^2\right)^2} \\
&= \frac{8x \cdot \varphi(x-y^2) - 8xy^2 + 8x^2}{\left(\varphi(x-y^2) - x - y^2\right)^2}.
\end{aligned}$$

Dakle, vrijedi tražena jednakost. □

Zadatak 2.9. Neka je $\varphi\colon I \to \mathbb{R}$ glatka funkcija definirana na otvorenom intervalu $I \subseteq \mathbb{R}$, i stavimo

$$z(x,y) = xy + \frac{y}{\varphi\left(\frac{y}{x}\right) - \frac{x}{y}}.$$

Pokažite da je funkcija z rješenje sljedeće parcijalne diferencijalne jednadžbe:

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = xy + z.$$

Rješenje: Opet prvo računamo parcijalne derivacije prvog reda funkcije z.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y + \frac{-y \cdot \left(\varphi'\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2} - \frac{1}{y}\right)}{\left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right)^2} = y + \frac{\frac{y^2}{x^2} \cdot \varphi'\left(\frac{y}{x}\right) + 1}{\left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right)^2} \\
\frac{\partial z}{\partial y} &= x + \frac{1 \cdot \left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right) - y \cdot \left(\varphi'\left(\frac{y}{x}\right) \cdot \frac{1}{x} - \frac{-x}{y^2}\right)}{\left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right)^2} \\
&= x + \frac{\varphi\left(\frac{y}{x}\right) - \frac{x}{y} - \frac{y}{x} \cdot \varphi'\left(\frac{y}{x}\right) - \frac{x}{y}}{\left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right)^2} = x + \frac{1}{\varphi\left(\frac{y}{x}\right) - \frac{x}{y}} - \frac{\frac{y}{x} \cdot \varphi'\left(\frac{y}{x}\right) + \frac{x}{y}}{\left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right)^2}
\end{aligned}$$

Uvrštavanjem u lijevu stranu jednakosti dobivamo

$$\begin{aligned}
x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} &= xy + \frac{\frac{y^2}{x} \cdot \varphi'\left(\frac{y}{x}\right) + x}{\left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right)^2} + \underbrace{xy + \frac{y}{\varphi\left(\frac{y}{x}\right) - \frac{x}{y}}}_{z} - \frac{\frac{y^2}{x} \cdot \varphi'\left(\frac{y}{x}\right) + x}{\left(\varphi\left(\frac{y}{x}\right) - \frac{x}{y}\right)^2} \\
&= xy + z.
\end{aligned}$$

Dakle, vrijedi tražena jednakost. □

Zadatak 2.10. Neka je $\varphi\colon I \to \mathbb{R}$ glatka funkcija definirana na otvorenom intervalu $I \subseteq \mathbb{R}$, i stavimo

$$z(x,y) = \frac{y}{1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)}.$$

Pokažite da vrijedi jednakost:

$$(x^2-y^2)\cdot\frac{\partial z}{\partial x} + xy\cdot\frac{\partial z}{\partial y} = xz.$$

Rješenje: Računamo parcijalne derivacije prvog reda funkcije z:

$$\frac{\partial z}{\partial x} = \frac{-y\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right)\cdot\left(\frac{2x}{2y^2}\right)}{\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right)^2} = \frac{-\frac{x}{y}\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right)}{\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right)^2}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{1\cdot\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right) - y\cdot\left(\varphi'\left(\ln y + \frac{x^2}{2y^2}\right)\cdot\left(\frac{1}{y}+\frac{-2x^2}{2y^3}\right)\right)}{\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right)^2} \\
&= \frac{1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right) - \varphi'\left(\ln y + \frac{x^2}{2y^2}\right) + \frac{x^2}{y^2}\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right)}{\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right)^2} \\
&= \frac{-\varphi'\left(\ln y + \frac{x^2}{2y^2}\right) + \frac{x^2}{y^2}\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right)}{\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right)^2} + \frac{1}{1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)}
\end{aligned}$$

Uvrštavanjem u lijevu stranu jednakosti slijedi

$$\begin{aligned}
(x^2-y^2)\cdot\frac{\partial z}{\partial x} + xy\cdot\frac{\partial z}{\partial y} &= \frac{-\frac{x^3}{y}\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right) + xy\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right)}{\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right)^2} \\
&\quad + \frac{-xy\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right) + \frac{x^3}{y}\cdot\varphi'\left(\ln y + \frac{x^2}{2y^2}\right)}{\left(1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)\right)^2} \\
&\quad + \frac{xy}{1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)} \\
&= x\cdot\frac{y}{1+\varphi\left(\ln y + \frac{x^2}{2y^2}\right)} = xz.
\end{aligned}$$

Dakle, vrijedi tražena jednakost. □

2.3.1 Zadaci za vježbu

Zadatak 2.11. Dokažite da je funkcija $f(x, y) = (x-y) \cdot \varphi((x-y)^2)$ rješenje parcijalne diferencijalne jednadžbe $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$.

Rješenje: Kako i u prethodnim zadacima potrebno je prvo izračunati parcijalne derivacije prvog reda funkcije f. Zatim je te parcijalne derivacije potrebno zbrojiti i uvjeriti se da je njihov zbroj 0. □

Zadatak 2.12. Odredite parcijalne derivacije prvog i drugog reda sljedećih funkcija:

(a) $f(x, y) = x\sqrt{y-x}$

(b) $f(x, y) = \ln \dfrac{x-y}{x+y}$

(c) $f(x, y) = \operatorname{arctg}(y^2 - x^2)$.

Rješenje: Zadaća. □

Zadatak 2.13. Odredite parcijalne derivacije prvog reda sljedećih funkcija:

(a) $f(x, y) = \ln(x \cdot \ln(x-y))$

(b) $f(x, y) = \cos^2(y^2 - x^2)$

Rješenje: Zadaća. □

Zadatak 2.14. Neka je $\varphi\colon I \to \mathbb{R}$ glatka funkcija definirana na otvorenom intervalu $I \subseteq \mathbb{R}$. Odredite parcijalne derivacije prvog reda sljedećih funkcija:

(a) $f(x, y) = \varphi(y^2 - x^2)$

(b) $f(x, y) = x + \varphi(x^2 + y^2)$

(c) $f(x, y) = y + \varphi\left(\dfrac{x}{y}\right)$

(d) $f(x, y) = xy \cdot \varphi\left(\dfrac{x}{y}\right)$.

Rješenje: Zadaća. □