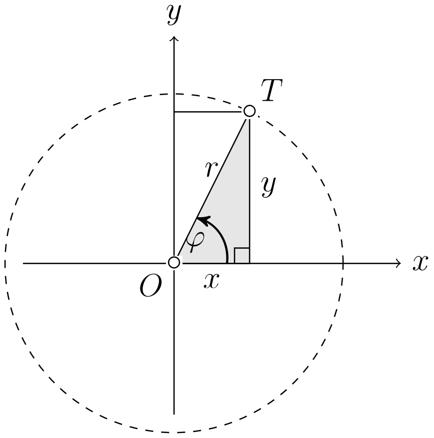
3.1.1 Polarni koordinatni sustav

Osim u Kartezijevim koordinatama (x, y), točku T neke ravnine možemo zadati i u *polarnim koordinatama* (r, φ). Koordinata r je definirana kao udaljenost $r = |\vec{OT}| \geq 0$, a koordinata $\varphi = \measuredangle(\vec{i}, \vec{OT})$ kao usmjeren kut (dakle, gledamo od osi x do spojnice $\vec{OT}$ u smjeru suprotnom od kazaljke na satu). Kut φ možemo ograničiti na npr. $[0, 2\pi\rangle$, ili $\langle -\pi, \pi]$.



Promatrajući pravokutni trokut istaknut na skici, uočavamo relacije za prijelaz iz jednog koordinatnog sustava u drugi:

$$x = r\cos\varphi \qquad\qquad r^2 = x^2 + y^2$$
$$y = r\sin\varphi, \qquad\qquad \operatorname{tg}\varphi = \frac{y}{x}.$$

Prilikom uzimanja arctg radi računanja kuta φ, treba voditi računa o kvadrantu u kojem se točka nalazi, te o slici funkcije arctg. U I. i IV. kvadrantu, kad je $x > 0$, vrijedi $\varphi = \operatorname{arctg}\frac{y}{x}$. Ali npr. točka s Kartezijevim koordinatama $(-1, 1)$ nalazi se u II. kvadrantu i ima $\varphi = \frac{3\pi}{4}$; međutim $\operatorname{arctg}\frac{1}{-1} = -\frac{\pi}{4}$. Zaključno, u II. i III. kvadrantu, kad je $x < 0$, vrijedi $\varphi = \pi + \operatorname{arctg}\frac{y}{x}$.

Istaknimo *koordinatne krivulje* polarnog koordinatnog sustava. Njih dobivamo tako da interpretiramo jednadžbu u kojoj izjednačimo koordinatu s konstantom. Neka je c konstanta.

- Za $r = c$ dobivamo kružnice sa središtem u ishodištu (radijusa c), a
- za $\varphi = c$ dobivamo polupravce koji kreću iz ishodišta.

Izračunajmo *Jacobijan* za prijelaz iz Kartezijevog u polarni sustav:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r.$$

Dakle, u ovom slučaju *teorem o zamjeni varijabli* glasi:

$$\iint_{D(x,y)} f(x,y)\,dx\,dy = \iint_{D(r,\varphi)} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi, \qquad (3.3)$$

gdje je $D(x,y)$ lik u ravnini opisan u Kartezijevim koordinatama (x,y), a $D(r,\varphi)$ isti taj lik opisan u polarnim koordinatama (r,φ). Istaknimo:

$$dx\,dy = r\,dr\,d\varphi.$$

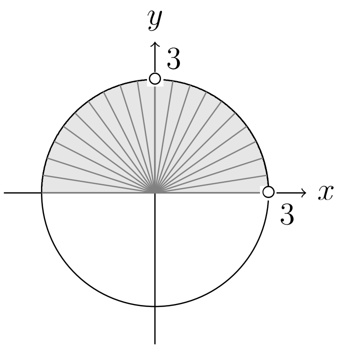
Računanje integrala u polarnom sustavu često se isplati ukoliko je područje integracije "okruglo", tj. ima oblik kruga ili kružnog isječka.

Zadatak 3.8. Izračunajte

$$\iint_D \frac{xy}{x^4 + y^4 + 2x^2y^2 + 1}\,dx\,dy,$$

gdje je $D = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leqslant 9, \ y > 0\}$. Skicirajte D.

Rješenje: Iz opisa vidimo da je D dio kruga iznad x-osi.



$0 \leqslant \varphi \leqslant \pi$

$0 \leqslant r \leqslant 3$

Budući da se D nalazi u I. i II. kvadrantu, vrijedi $0 \leqslant \varphi \leqslant \pi$. Za svaki kut φ vidimo da se udaljenost točaka od ishodišta kreće od 0 do 3 pa je $0 \leqslant r \leqslant 3$.

$$\begin{aligned}
\iint_D \frac{xy}{x^4+y^4+2x^2y^2+1}\,dx\,dy &= \iint_D \frac{xy}{(x^2+y^2)^2+1}\,dx\,dy \\
\text{(prema (3.3))} \quad &= \int_0^{\pi}\int_0^{3} \frac{r\cos\varphi\cdot r\sin\varphi}{(r^2)^2+1}\cdot r\,dr\,d\varphi \\
&= \int_0^{\pi}\int_0^{3} \underbrace{\frac{r^3\,dr}{r^4+1}}_{t=r^4+1}\cdot\cos\varphi\cdot\sin\varphi\,d\varphi \\
&= \frac{1}{4}\int_0^{\pi}\int_1^{82}\frac{dt}{t}\cdot\underbrace{\cos\varphi\cdot\sin\varphi\,d\varphi}_{s=\sin\varphi} \\
&= \frac{\ln 82}{4}\cdot\int_0^0 s\,ds = 0.
\end{aligned}$$

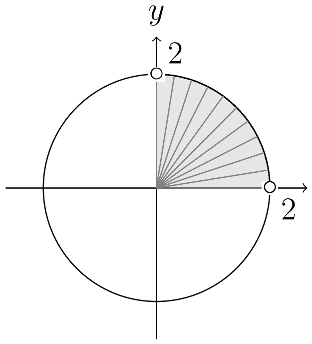
□

Zadatak 3.9. Izračunajte

$$\iint_D \frac{xy}{\sqrt{x^4+y^4+2x^2y^2+1}}\,dx\,dy,$$

gdje je $D = \left\{(x,y)\in\mathbb{R}^2 \;:\; x^2+y^2 \leqslant 4,\; x,y>0\right\}$. Skicirajte D.

Rješenje:



$$0 \leqslant \varphi \leqslant \tfrac{\pi}{2}$$
$$0 \leqslant r \leqslant 2$$

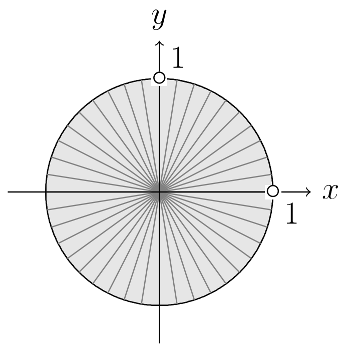
$$
\begin{aligned}
\iint_D \frac{xy}{\sqrt{x^4+y^4+2x^2y^2+1}}\,dx\,dy &= \int_0^{\frac{\pi}{2}}\int_0^2 \underbrace{\frac{r\cos\varphi\cdot r\sin\varphi}{\sqrt{r^4+1}}}_{t=r^4+1}\cdot r\,dr\,d\varphi \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\int_1^{17}\frac{dt}{\sqrt{t}}\cdot\underbrace{\cos\varphi\cdot\sin\varphi\,d\varphi}_{s=\sin\varphi} \\
&= \frac{\sqrt{17}-1}{2}\cdot\int_0^1 s\,ds = \frac{\sqrt{17}-1}{4}. \qquad \square
\end{aligned}
$$

Zadatak 3.10. Izračunajte

$$
\iint_D x^2y^2\sqrt{(x^2+y^2)^3+1}\,dx\,dy,
$$

gdje je $D = \left\{(x,y)\in\mathbb{R}^2 \;:\; x^2+y^2 \leqslant 1\right\}$. Skicirajte D.

Rješenje:



$0 \leqslant \varphi \leqslant 2\pi$

$0 \leqslant r \leqslant 1$

$$
\begin{aligned}
\iint_D x^2y^2\sqrt{(x^2+y^2)^3+1}\,dx\,dy &= \int_0^{2\pi}\int_0^1 r^2\sin^2\varphi\cdot r^2\cos^2\varphi\cdot\underbrace{\sqrt{r^6+1}}_{t=r^6+1}\cdot r\,dr\,d\varphi \\
&= \frac{1}{6}\int_0^{2\pi}\int_1^2\sqrt{t}\,dt\cdot\sin^2\varphi\cos^2\varphi\,d\varphi \\
&= \frac{2\sqrt{2}-1}{9}\cdot\int_0^{2\pi}\frac{1-\cos(4\varphi)}{8}\,d\varphi \\
&= \frac{2\sqrt{2}-1}{36}\pi.
\end{aligned}
$$

U predzadnjem redu iskoristili smo jednakost

$$
\sin^2\varphi\cos^2\varphi = (\sin\varphi\cos\varphi)^2 = \frac{\sin^2(2\varphi)}{4} = \frac{1-\cos(4\varphi)}{8}.
$$

$\square$

Zadatak 3.11. Skicirajte područje integracije sljedećih integrala:

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{6\cos\varphi} f(r,\varphi)\,dr\,d\varphi,$

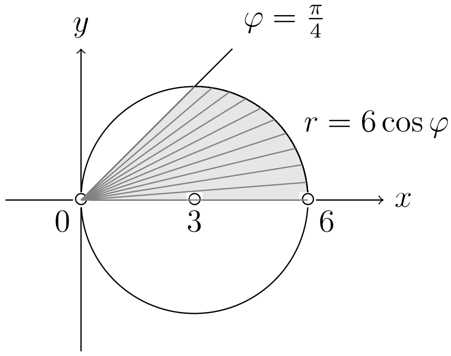
(b) $\displaystyle\int_0^{\pi} \int_0^{1+\cos\varphi} f(r,\varphi)\,dr\,d\varphi.$

Rješenje:

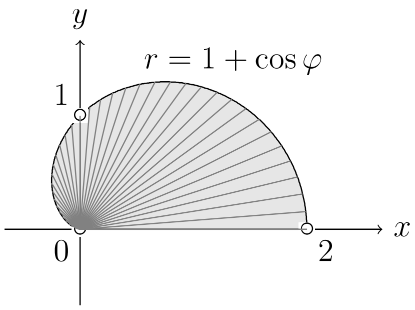
(a) $0 \leqslant \varphi \leqslant \frac{\pi}{4}$ i $0 \leqslant r \leqslant 6\cos\varphi$. Gornja granica za r dana je krivuljom

$$r = 6\cos\varphi \;/\cdot r \;\Rightarrow\; r^2 = 6r\cos\varphi \;\Rightarrow\; x^2+y^2 = 6x \Rightarrow\; x^2-6x+y^2 = 0$$
$$\text{(nadopunjavanje do potpunog kvadrata)} \;\Rightarrow\; (x-3)^2+y^2 = 9,$$

dakle, kružnicom radijusa 3 sa središtem u $(3,0)$.

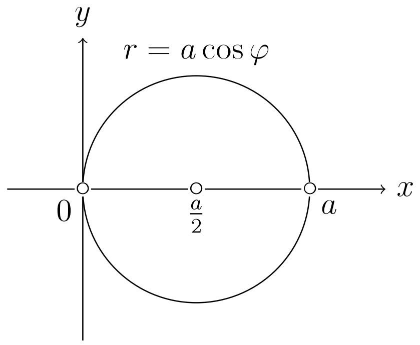


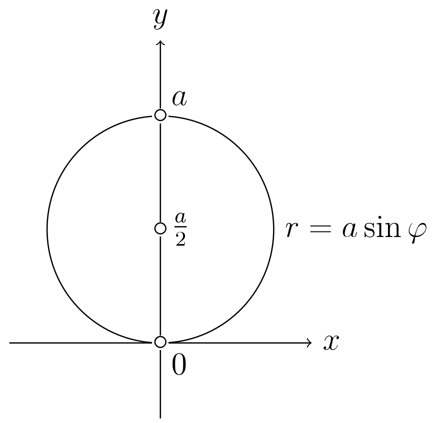
(b) $0 \leqslant \varphi \leqslant \pi$ daje I. i II. kvadrant, tj. iznad x-osi, a $0 \leqslant r \leqslant 1+\cos\varphi$ daje unutrašnjost tzv. kardioide.



□

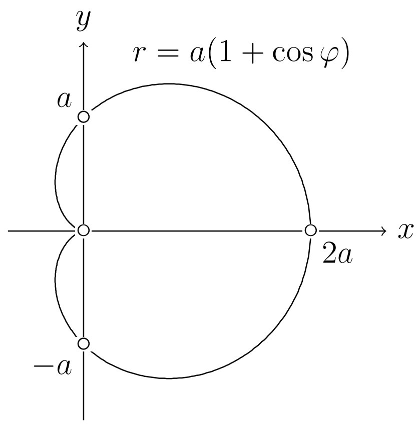
Napomena. Iz prethodnog zadatka vidimo kako se u polarnom sustavu skiciraju sljedeće krivulje:

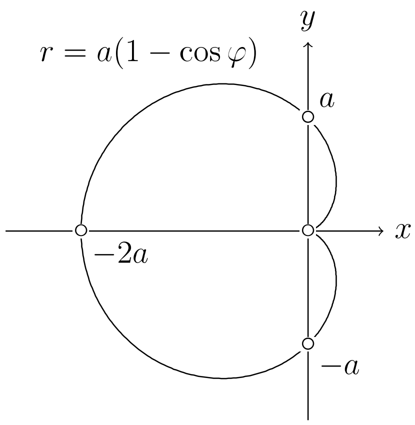


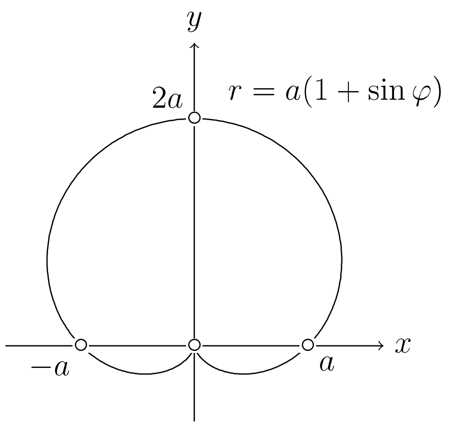


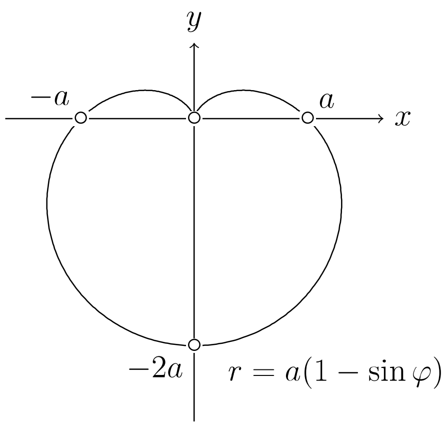
Na gornjim slikama pretpostavili smo da je $a > 0$. Međutim, analogne slike dobiju se lako i kad je $a < 0$.

Imamo i tzv. *kardioide* (samo za $a > 0$):









Zadatak 3.12. Izračunajte sljedeće integrale:

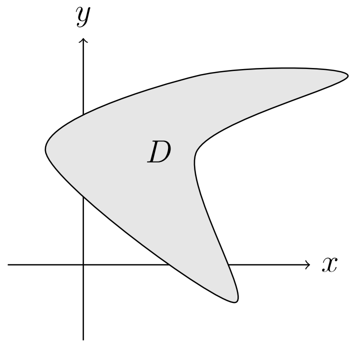
(a) $\displaystyle\int_0^{2\pi}\int_1^2 (3r - r^2\sin\varphi)\,dr\,d\varphi,$

(b) $\displaystyle\int_0^{\frac{\pi}{3}}\int_0^{2\cos\varphi} (r^2\sin\varphi)\,dr\,d\varphi,$

Rješenje: Zadaća. □

3.1.2 Površina pomoću dvostrukog integrala

Neka je zadan omeđeni lik $D \subseteq \mathbb{R}^2$ u ravnini.

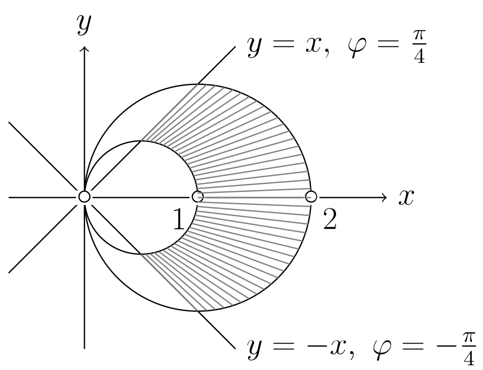


Površinu lika D računamo pomoću formula:

$$P(D) = \iint_{D(x,y)} dx\,dy = \iint_{D(r,\varphi)} r\,dr\,d\varphi.$$

Zadatak 3.13. Izračunajte površinu lika omeđenog krivuljama $x^2+y^2=x$, $x^2+y^2=2x$, $y=x$ i $y=-x$. Skicirajte lik.

Rješenje:



Unutarnja kružnica je $r = \cos\varphi$,

a vanjska kružnica $r = 2\cos\varphi$.

Opisujemo lik D u polarnim

koordinatama: $-\frac{\pi}{4} \leqslant \varphi \leqslant \frac{\pi}{4}$

$\cos\varphi \leqslant r \leqslant 2\cos\varphi$

Tražena površina je

$$P=\iint_{D(r,\varphi)} r\,dr\,d\varphi=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{\cos\varphi}^{2\cos\varphi} r\,dr\,d\varphi=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{r^2}{2}\Big|_{r=\cos\varphi}^{r=2\cos\varphi}\,d\varphi$$

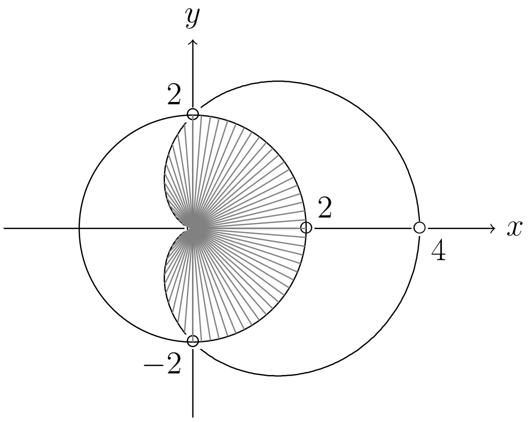
$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{3\cos^2\varphi}{2}\,d\varphi=\frac{3}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1+\cos(2\varphi)}{2}\,d\varphi$$

$$=\frac{3}{2}\cdot\frac{\varphi+\frac{\sin(2\varphi)}{2}}{2}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{3\pi+6}{8}.$$

□

Zadatak 3.14. Izračunajte površinu lika koji je omeđen krivuljama $x^2+y^2=4$ i $r=2(1+\cos\varphi)$, a nalazi se unutar obje krivulje. Skicirajte lik.

Rješenje: Ovaj lik simetričan je u odnosu na os x. Njegov desni dio je polovica kruga, a lijevi se sastoji od dva manja dijela iste površine.



Dio lika u II. kvadrantu:

$\frac{\pi}{2}\leqslant\varphi\leqslant\pi$

$0\leqslant r\leqslant 2(1+\cos\varphi)$

$$P=2\cdot\underbrace{\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{2(1+\cos\varphi)} r\,dr\,d\varphi}_{\text{površina dijela lika u II. kvadrantu}}+\underbrace{2\pi}_{\text{pola kruga}}$$

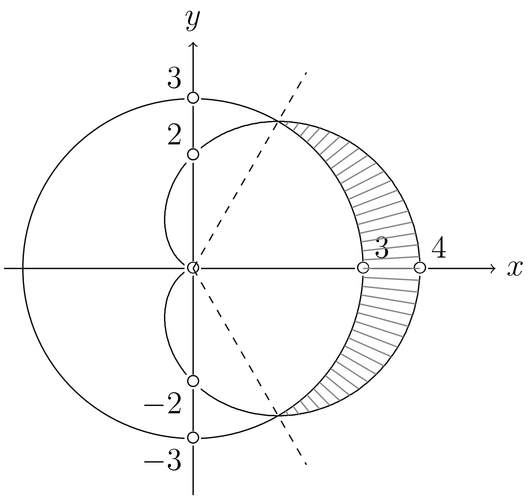
$$=2\cdot\int_{\frac{\pi}{2}}^{\pi}\frac{r^2}{2}\Big|_{r=0}^{r=2(1+\cos\varphi)}\,d\varphi+2\pi=4\int_{\frac{\pi}{2}}^{\pi}(1+\cos\varphi)^2\,d\varphi+2\pi$$

$$=4\int_{\frac{\pi}{2}}^{\pi}(1+2\cos\varphi+\underbrace{\cos^2\varphi}_{\frac{1+\cos(2\varphi)}{2}})\,d\varphi+2\pi\overset{\text{DZ}}{=}5\pi-8.$$

$\square$

Zadatak 3.15. Izračunajte površinu lika koji je omeđen krivuljama $r = 2(1+\cos\varphi)$ i $r = 3$, te nalazi se unutar prve, a izvan druge krivulje. Skicirajte lik.

Rješenje:



Računamo presjek krivulja:

$$2(1+\cos\varphi) = 3$$
$$\Rightarrow \cos\varphi = \frac{1}{2} \Rightarrow \varphi = \pm\frac{\pi}{3}.$$

Opisujemo lik:

$$-\frac{\pi}{3} \leqslant \varphi \leqslant \frac{\pi}{3}$$
$$3 \leqslant r \leqslant 2(1+\cos\varphi).$$

$$P = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{3}^{2(1+\cos\varphi)} r\, dr\, d\varphi = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{r^2}{2}\bigg|_{r=3}^{r=2(1+\cos\varphi)} d\varphi$$
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (4 + 8\cos\varphi + 4\underbrace{\cos^2\varphi}_{\frac{1+\cos(2\varphi)}{2}} - 9)\, d\varphi \overset{\text{DZ}}{=} \frac{9\sqrt{3}}{2} - \pi.$$

$\square$