Rješenja zadataka za vježbu

(II kolokvij - dvostruki i trostruki integrali)

1. a)

$$I = \int_{-1}^{1} dy \int_{0}^{1-y^2} f(x,y)dx$$

b)

$$I = \int_{-1}^{1} dx \int_{x^2}^{\frac{2}{1+x^2}} f(x,y)dy$$

c)

$$I = \int_{0}^{1} dx \int_{0}^{x} f(x,y)dy + \int_{1}^{2} dx \int_{0}^{-x^2+2x} f(x,y)dy$$

ili

$$y = -x^2 + 2x$$

$$y - 1 = -x^2 + 2x - 1$$

$$y - 1 = -(x-1)^2$$

$$(x-1)^2 = 1 - y$$

$$x - 1 = \sqrt{1-y}$$

$$x = 1 + \sqrt{1-y}$$

$$I = \int_{0}^{1} dy \int_{y}^{1+\sqrt{1-y}} f(x,y)dx$$

d)

$$I = \int_{0}^{1} dx \int_{x}^{3x} f(x,y)dy + \int_{1}^{2} dx \int_{x}^{4-x} f(x,y)dy$$

e)

$$I = \int_{0}^{\frac{\sqrt{2}}{2}} dx \int_{x}^{2x} f(x,y)dy + \int_{\frac{\sqrt{2}}{2}}^{1} dx \int_{x}^{\frac{1}{x}} f(x,y)dy$$

2. a)

$$I = \int_{1}^{e} dy \int_{\ln y}^{1} f(x,y)dx$$

b)

$$I = \int_{0}^{1} dy \int_{1}^{e^y} f(x,y)dx$$

c)

$$I = \int_{1}^{4} dx \int_{\frac{x}{2}}^{\sqrt{x}} f(x,y)dy$$

d)

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 f(x,y)dy$$

3.

$$\int_0^\pi dx \int_0^{\sin x} (3+y)dy = \int_0^\pi (3y + \frac{y^2}{2})\Big|_0^{\sin x} dx = \int_0^\pi (3\sin x + \frac{\sin^2 x}{2})dx =$$

$$3\int_0^\pi \sin x dx + \frac{1}{4}\int_0^\pi (1-\cos 2x)dx = 6 + \frac{\pi}{4}$$

4.

$$\int_0^{\frac{\pi}{2}} dy \int_y^{2y} \cos(x+y)dx = \int_0^{\frac{\pi}{2}} \sin(x+y)\Big|_y^{2y} dy = \int_0^{\frac{\pi}{2}} (\sin 3y - \sin 2y)dy = -\frac{2}{3}$$

5.

$$P_1 = \int_0^\pi d\varphi \int_0^{4(1+\cos\varphi)} rdr = \frac{1}{2}\int_0^\pi r^2\Big|_0^{4(1+\cos\varphi)} d\varphi = 8\int_0^\pi (1+\cos\varphi)^2 d\varphi =$$

$$8\int_0^\pi (1 + 2\cos\varphi + \cos^2\varphi)d\varphi = 8(\frac{3}{2}\varphi + 2\sin\varphi + \frac{\sin 2\varphi}{4})\Big|_0^\pi = 12\pi$$

$$P = 2P_1 = 24\pi$$

6.

$$r = 4\cos\varphi$$

$$r^2 = 4r\cos\varphi$$

$$x^2 + y^2 = 4x$$

$$x^2 - 4x + y^2 = 0$$

$$(x-2)^2 + y^2 = 4$$

$$\mathrm{tg}\varphi = \frac{\mathrm{y}}{\mathrm{x}} = \sqrt{3} \Longrightarrow \varphi = \frac{\pi}{3}$$

$$\mathrm{tg}\varphi = \frac{\mathrm{y}}{\mathrm{x}} = 1 \Longrightarrow \varphi = \frac{\pi}{4}$$

$$P = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^{4\cos\varphi} rdr = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} r^2\Big|_0^{4\cos\varphi} d\varphi = 8\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos\varphi^2 d\varphi = 8(\frac{\varphi}{2} + \frac{\sin 2\varphi}{4})\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

7.

$$0 \le \varphi \le \pi$$

$$0 \le r \le 1 + \cos\varphi$$

$$r = 1 + \cos\varphi$$

8.

$$0 \le \varphi \le \frac{\pi}{4} \Longrightarrow y = 0 \quad i \quad \mathrm{tg}\frac{\pi}{4} = 1 = \frac{\mathrm{y}}{\mathrm{x}} \Longrightarrow \mathrm{y} = \mathrm{x}$$

$$r = 6\cos\varphi$$
$$r^2 = 6r\cos\varphi$$
$$x^2 + y^2 = 6x$$
$$x^2 - 6x + y^2 = 0$$
$$(x-3)^2 + y^2 = 9$$

Područje omeđeno kružnicom $(x-3)^2 + y^2 = 9$ i pravcima $y = 0$ i $y = x$.

9.

$$I = \int\int_\Omega e^{1-x^2-y^2}dxdy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 e^{1-r^2} rdr = \left| \begin{array}{c} 1-r^2 = t \\ -2rdr = dt \end{array} \right| = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi \int_0^1 e^t dt = \frac{\pi}{4}(e-1)$$

10.

$$I = \int\int_\Omega \sqrt{4-x^2-y^2}dxdy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_1^2 \sqrt{4-r^2}rdr = \frac{\pi}{4}(-\frac{1}{2}\int_1^2 \sqrt{4-r^2}d(4-r^2) = \frac{\sqrt{3}\pi}{4}$$

11.

$$\int_0^{2\pi} d\varphi \int_1^2 (3r - r^2\sin\varphi)dr = 3\int_0^{2\pi} d\varphi \int_1^2 rdr - \int_0^{2\pi} \sin\varphi d\varphi \int_1^2 r^2 dr = 6\pi\frac{r^2}{2}\Big|_1^2 + \cos\varphi\Big|_0^{2\pi}\frac{r^3}{3}\Big|_1^2 = 9\pi$$

12.

$$\int_0^{\frac{\pi}{3}} d\varphi \int_0^{2\cos\varphi} r^2\sin\varphi dr = \int_0^{\frac{\pi}{3}} \sin\varphi \frac{r^3}{3}\Big|_0^{2\cos\varphi} d\varphi = -\frac{8}{3}\int_0^{\frac{\pi}{3}} \cos^3\varphi d(\cos\varphi) = \frac{5}{8}$$

13.

$$8 - x^2 - y^2 = 4 \Longrightarrow 4 = x^2 + y^2 \Longrightarrow 0 \le r \le 2, \quad 0 \le \varphi \le 2\pi$$
$$V = \int_0^{2\pi} d\varphi \int_0^2 (8 - r^2 - 4)dr = 8\pi$$

14.

$$V = \int_0^{\pi} d\varphi \int_{\cos\varphi}^{2\cos\varphi} 2rdr = \int_0^{\pi} d\varphi r^2\Big|_{\cos\varphi}^{2\cos\varphi} = 3\int_0^{\pi} \cos^2\varphi d\varphi = \frac{3\pi}{2}$$

15.

$$V = \int\int_\Omega 1dxdy = \int_{-2}^2 dx \int_0^{4-x^2} dy = \int_{-2}^2 (4-x^2)dx = \frac{32}{3}$$

16.

$$\int_0^1 dx \int_0^1 ((1-y^2) - (1-y))dy = \frac{1}{6}$$

17.

$$V = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r\cos\varphi r\sin\varphi rdr = \frac{1}{8}$$

18.

$$\int_0^1 xdx \int_0^x dy \int_0^{x-y} dz = \int_0^1 xdx \int_0^x z\Big|_0^{x-y} dy = \int_0^1 x(xy - \frac{y^2}{2})\Big|_0^x dx = \int_0^1 x(x^2 - \frac{x^2}{2})dx = \frac{1}{8}$$

19.

$$\int\int\int_V ydxdydz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^3 rdr \int_0^5 r\sin\varphi dz = \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^3 r^2 dr \int_0^5 dz = 45$$

20.

$$\int\int\int_V ydxdydz = \int_0^{2\pi} d\varphi \int_0^2 rdr \int_0^4 zdz = 32\pi$$

21.

$$\int_0^{2\pi} d\varphi \int_0^2 \rho d\rho \int_0^{\rho^2} f(\rho\cos\varphi, \rho\sin\varphi, z)dz$$

22.

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 \rho d\rho \int_0^{8-\rho^2} f(\rho\cos\varphi, \rho\sin\varphi, z)dz$$

23.

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin\vartheta d\vartheta \int_0^3 f(r\cos\varphi\sin\vartheta, r\sin\varphi\sin\vartheta, r\cos\vartheta)r^2 dr$$

24.

$$\int_0^{\pi} d\varphi \int_0^{\pi} \sin\vartheta d\vartheta \int_0^2 f(r\cos\varphi\sin\vartheta, r\sin\varphi\sin\vartheta, r\cos\vartheta)r^2 dr$$