# MATEMATIKA 2, 15.7.2020.

1. Riješite diferencijalne jednadžbe:

   (a) (10 bodova) $x(e^{2y}+1)\mathrm{d}x = e^{3y}(2+x^2)\mathrm{d}y$,

   (b) (10 bodova) $y'' + 4y' + 4y = e^{-2x}$.

2. a) (12 bodova) Skicirajte prirodnu domenu funkcije

$$f(x,y) = \sqrt{x^2+y^2-4} - \ln(x-1)$$

   i odredite $\frac{\partial^2 f}{\partial y \partial x}(2,1)$.

   b) (8 bodova) Odredite tangencijalnu ravninu na graf funkcije $f(x,y) = x^3 + 2y^2 - 3xy$ u točki $T(1,1,?)$.

3. a) (10 bodova) Izračunajte površinu manjeg lika koji je omeđen krivuljama $x^2+y^2=2y$ i $y=-x$. Skicirajte lik.

   b) (10 bodova) Prelaskom na cilindrične koordinate izračunajte

$$\int_0^2 \mathrm{d}x \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \mathrm{d}y \int_0^{\sqrt{4-x^2}} \sqrt{4-x^2}\,\mathrm{d}z.$$

---

4. a) (12 bodova) Izračunajte

$$\int_\Gamma xy\,ds,$$

   gdje je $\Gamma$ dio krivulje nastale presjekom cilindra $x^2+y^2=1$ i ravnine $z=y$ u 1. oktantu. Skicirajte krivulju.

   b) (8 bodova) Izračunajte $\int_{\vec{\Gamma}} z\,\mathrm{d}x - x\,\mathrm{d}y + y\,\mathrm{d}z$ ako je $\vec{\Gamma}$ zadana parametrizacijom $x(t)=t^2$, $y(t)=t^3$, $z(t)=t$ za $t\in[0,1]$.

5. (20 bodova) Izračunajte $\int\int_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$, ako je $\vec{a} = (z-x^2)\,\vec{j}$, a ploha $\vec{\Sigma} = \{(x,y,z)\in\mathbb{R}^3 : z = x^2+y^2, y\geq 0, 0\leq z\leq 4\}$ je orijentirana normalom koja zatvara tupi kut s vektorom $\vec{k}$. Skicirajte plohu.

**Prvi dio** čine prva tri zadatka. **Drugi dio** čine 4. i 5. zadatak.

Za polaganje ispita treba skupiti 50 bodova (od tog barem 24 boda iz prvog dijela i barem 16 bodova iz drugog dijela).

MAT2 15. 7. 2020.

1. a) (10b)

$$x(e^{2y}+1)dx = e^{3y}(2+x^2)dy$$

Separiramo varijable i integriramo:

$$\int \frac{e^{3y}dy}{e^{2y}+1} = \int \frac{x\,dx}{2+x^2} \quad 2$$

$$\int \frac{e^{3y}dy}{e^{2y}+1} = \left| \begin{matrix} t = e^y \\ dt = e^y dy \end{matrix} \right| = \int \frac{t^2 dt}{t^2+1} = \left| \begin{matrix} t^2 : (t^2+1) = 1 \\ \underline{-t^2 \mp 1} \\ -1 \end{matrix} \right|$$

$$= \int \left(1 - \frac{1}{t^2+1}\right)dt = t - \operatorname{arctg} t = \underline{e^y - \operatorname{arctg} e^y}, \quad 5$$

$$\int \frac{x\,dx}{2+x^2} = \left| \begin{matrix} t = 2+x^2 \\ dt = 2x\,dx \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \underline{\frac{1}{2}\ln(2+x^2)+C}, \quad 3$$

Rješenje diferencijalne jednadžbe je:

$$\boxed{e^y - \operatorname{arctg} e^y = \frac{1}{2}\ln(2+x^2) + C}$$

1. b) (10b) $y'' + 4y' + 4y = e^{-2x}$

Rješenja karakteristične jednadžbe $\lambda^2 + 4\lambda + 4 = 0$ su $\lambda_{1,2} = -2$. 1

Rješenje pripadne homogene jednadžbe ima oblik

$\underline{y_H = C_1 e^{-2x} + C_2 x e^{-2x}}$. 1

Partikularno rješenje ima oblik $\underline{y_P = Ax^2 e^{-2x}}$, 2

zbog $\alpha = -2 = \lambda_{1,2}$.

$$y_P' = 2Axe^{-2x} - 2Ax^2e^{-2x} \quad 1$$

$$y_P'' = 2Ae^{-2x} - 4Axe^{-2x} - 4Axe^{-2x} + 4Ax^2e^{-2x} = 2Ae^{-2x} - 8Axe^{-2x} + 4Ax^2e^{-2x} \quad 1$$

$$2Ae^{-2x} - \cancel{8Axe^{-2x}} + \cancel{4Ax^2e^{-2x}} + \cancel{8Axe^{-2x}} - \cancel{8Ax^2e^{-2x}} + \cancel{4Ax^2e^{-2x}} = e^{-2x} \quad 1$$

$$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2} \quad 1 \quad \Rightarrow \underline{y_P = \frac{1}{2}x^2e^{-2x}}, \quad 1$$

Rješenje diferencijalne jednadžbe je:

$$\boxed{y = y_H + y_P = C_1e^{-2x} + C_2xe^{-2x} + \frac{1}{2}x^2e^{-2x}} \quad 1$$

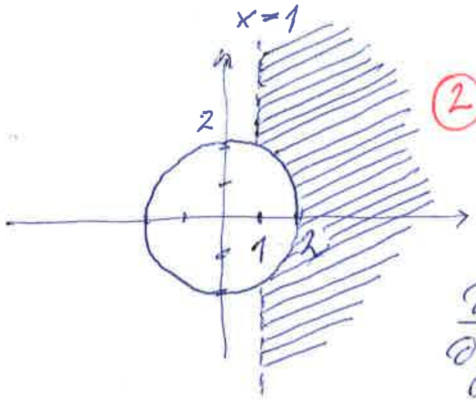
2. a) (12 b) $f(x,y) = \sqrt{x^2+y^2-4} - \ln(x-1)$

uvjeti na točke iz prirodne domene:

$1^\circ \quad x^2+y^2-4 \geq 0$ i $2^\circ \quad x-1>0$

$\Rightarrow x^2+y^2 \geq 4$ 2 $\qquad \Rightarrow x > 1$ 2

$\boxed{D_f = \{(x,y)\in \mathbb{R}^2 : x^2+y^2 \geq 4,\ x>1\}}$ 1




2

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2-4}} - \frac{1}{x-1}$$ 2

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{xy}{\sqrt{(x^2+y^2-4)^3}}$$ 2

$$\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(2,1)} = -\frac{2\cdot 1}{\sqrt{(2^2+1^2-4)^3}} = \boxed{-2}$$ 1

2b) (8b) $f(x,y) = x^3+2y^2-3xy$ , $T(1,1,?)$

$f(1,1) = 1^3+2\cdot1^2-3\cdot1\cdot1 = 0 \Rightarrow T(1,1,0)$ 1

$\frac{\partial f}{\partial x} = 3x^2-3y$ 1 $\qquad \frac{\partial f}{\partial y} = 4y-3x$ 1

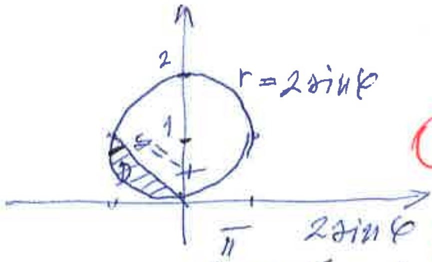
$\frac{\partial f}{\partial x}(1,1) = 3-3=0$ 1 $\qquad \frac{\partial f}{\partial y}(1,1) = 4-3=1$ 1

$\Pi_t \ldots z-0 = 0(x-1)+1(y-1)$ 2

$\boxed{\Pi_t \ldots y-z-1=0}$ tangencijalna ravnina kroz točku $T$ 1

3. a) (10b) $x^2+y^2=2y \;\Rightarrow\; x^2+(y-1)^2=1$ , $y=-x$

$r=2\sin\varphi$, jednadžba u polarnim koordinatama




1

D.. $\frac{3\pi}{4}\le\varphi\le\pi$

$0\le r\le 2\sin\varphi$ 2

$$P(D)=\int_{\frac{3\pi}{4}}^{\pi}d\varphi\int_0^{2\sin\varphi}r\,dr \;(1)\; =\int_{\frac{3\pi}{4}}^{\pi}\frac{r^2}{2}\Big|_0^{2\sin\varphi}d\varphi \;(1)\; =2\int_{\frac{3\pi}{4}}^{\pi}\sin^2\varphi\,d\varphi \;(1)$$

$$=\int_{\frac{3\pi}{4}}^{\pi}(1-\cos2\varphi)\,d\varphi \;(1)\; =\left(\varphi-\frac{1}{2}\sin2\varphi\right)\Big|_{\frac{3\pi}{4}}^{\pi}= \;(1)$$

$$=\pi-\frac{3\pi}{4}+\frac{1}{2}\underbrace{\sin\frac{3\pi}{2}}_{-1} \;(1)\; =\boxed{\frac{\pi}{4}-\frac{1}{2}} \;(1)$$

3. b) (10b)

$$\int_0^2dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dy\int_0^{\sqrt{4-x^2}}\sqrt{4-x^2}\,dz=\left|\begin{array}{ll}\text{cilindrični} & x=\rho\cos\varphi\\ \text{koordinatni} & -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\\ \text{sustav} & 0\le\rho\le2\end{array}\right|$$

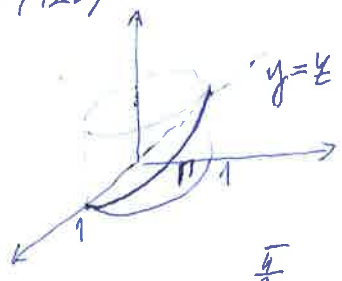
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^2\rho\,d\rho\int_0^{\sqrt{4-\rho^2\cos^2\varphi}}\sqrt{4-\rho^2\cos^2\varphi}\,dz \;(4)\; =\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^2\rho\sqrt{4-\rho^2\cos^2\varphi}\;z\Big|_0^{\sqrt{4-\rho^2\cos^2\varphi}}d\rho \;(1)$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^2(4\rho-\rho^3\cos^2\varphi)\,d\rho \;(1)\; =\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2\rho^2-\frac{\rho^4}{4}\cos^2\varphi\right)\Big|_0^2d\varphi \;(1)$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(8-4\underbrace{\cos^2\varphi}_{\frac{1+\cos2\varphi}{2}})\,d\varphi \;(1)\; =\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(6-2\cos2\varphi)\,d\varphi=(6\varphi-\sin2\varphi)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \;(1)$$

$$=3\pi+3\pi=\boxed{6\pi} \;(1)$$

4: a) (12b)




2

Parametrizacija:

$x(t) = \cos t$ $\qquad x'(t) = -\sin t$

$y(t) = \sin t$ $\qquad y'(t) = \cos t$

$z(t) = \sin t$ $\qquad z'(t) = \cos t$

$t \in [0, \frac{\pi}{2}]$

3

$$\int_\Gamma xy\,ds = \int_0^{\frac{\pi}{2}} \cos t \sin t \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + \cos^2 t}\,dt = \int_0^{\frac{\pi}{2}} \sin t \cos t \sqrt{1+\cos^2 t}\,dt$$ 2

$$= \left| \begin{array}{ll} u = 1+\cos^2 t & t=0 \Rightarrow u = 2 \\ du = -2\cos t \sin t\,dt & t = \frac{\pi}{2} \Rightarrow u = 1 \end{array} \right|$$ 2

$$= -\frac{1}{2}\int_2^1 \sqrt{u}\,du = \frac{1}{2}\int_1^2 \sqrt{u}\,du = \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^2 = \boxed{\frac{1}{3}(2\sqrt{2}-1)}$$ 1 1 1

4. b) (8b)

$\Gamma \ldots \quad x(t) = t^2 \qquad x'(t) = 2t$

$y(t) = t^3 \qquad y'(t) = 3t^2$

$z(t) = t \qquad z'(t) = 1$

$t \in [0,1]$

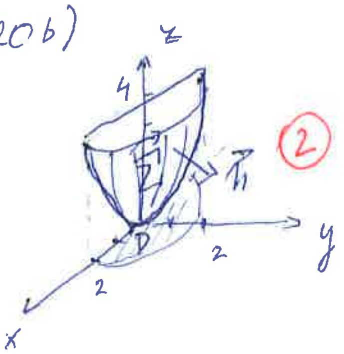
2

$$\int_\Gamma z\,dx - x\,dy + y\,dz = \int_0^1 (t\cdot 2t - t^2\cdot 3t^2 + t^3\cdot 1)\,dt$$ 3

$$= \int_0^1 (2t^2 - 3t^4 + t^3)\,dt = \left(\frac{2}{3}t^3 - \frac{3}{5}t^5 + \frac{1}{4}t^4\right)\Big|_0^1$$ 2

$$= \frac{2}{3} - \frac{3}{5} + \frac{1}{4} = \frac{40-36+15}{60} = \boxed{\frac{19}{60}}$$ 1

5. (20b)

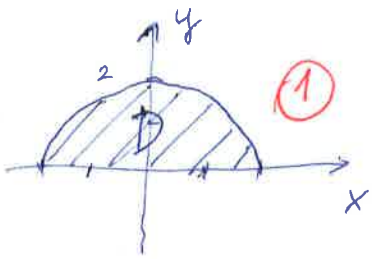



2

$\vec n = 2x\vec i + 2y\vec j - \vec k$ 2

$\vec a = (z - x^2)\vec j$

$\vec a \cdot \vec n = 2y(z - x^2) = 2y(x^2 + y^2 - x^2) = 2y^3$ 2




1

$D: \; 0 \le \varphi \le \pi$

$\phantom{D: \;} 0 \le r \le 2$ 2

$$\iint\limits_{\vec\Sigma} \vec a\, d\vec S = \iint\limits_{D} 2y^3\, dx\,dy = \int_0^{\pi} d\varphi \int_0^{2} 2r^3 \sin^3\varphi \, r\,dr$$ 1 2

$$= \int_0^{\pi} \sin^3\varphi\, d\varphi \cdot \int_0^{2} 2r^4\, dr = \int_0^{\pi} (1 - \cos^2\varphi)\sin\varphi\, d\varphi \cdot \frac{2}{5} r^5 \Big|_0^2$$ 2

$$= \left| \begin{array}{ll} t = \cos\varphi & \varphi = 0 \Rightarrow t = 1 \\ dt = -\sin\varphi\, d\varphi & \varphi = \pi \Rightarrow t = -1 \end{array} \right| = \frac{64}{5} \int_{-1}^{1} (1 - t^2)\, dt$$ 2 2

$$= \frac{128}{5} \int_0^1 (1 - t^2)\, dt = \frac{128}{5}\left(t - \frac{t^3}{3}\right)\Big|_0^1 = \frac{128}{5} \cdot \frac{2}{3} = \boxed{\frac{256}{15}}$$ 2