MATEMATIKA II 9.9.2020.

1. Riješite diferencijalne jednadžbe:

   a) (8 bodova) $\sin y y' = 2\cos x + 4$ uz početni uvjet $y(0) = \pi$,

   b) (10 bodova) $y'' + y = x^2 + 3$.

2. a) (10 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \ln(2^{24-2x^2+4xy-y^2} - 4^{2x^2+2xy+y^2}).$$

   b) (10 bodova) Odredite $\frac{\partial^2 f}{\partial x \partial y}$ u točki $T(1,0)$ ako je $f(x,y) = \ln(\frac{x+2y}{2x-y})$.

3. a) (12 bodova) Izračunajte površinu lika omeđenog krivuljama $x^2 + y^2 = 1$ i $r = 1 + \sin\phi$ koji se nalazi unutar prve, a izvan druge krivulje. Skicirajte lik.

   b) (10 bodova) Izračunajte integral

$$\int_0^{\pi} \mathrm{d}\varphi \int_0^1 \mathrm{d}\rho \int_0^{\sqrt{1-\rho^2}} \rho^3 \sin^2\varphi \, \mathrm{d}z.$$

4. a) (6 bodova) Odredite gradijent polja $u(x,y,z) = x^4 - y\sin z - xy^4 + \frac{z^2}{2}$.

   b) (14 bodova) Izračunajte

$$\int_\Gamma xy\mathrm{d}s,$$

   ako je krivulja Γ dio presjeka cilindra $x^2+y^2 = 9$ i ravnine $z = y$ od točke $A(3,0,0)$ do točke $B(0,3,3)$. Skicirajte krivulju.

5. (20 bodova) Izračunajte $\int\int_{\overrightarrow{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$, ako je $\vec{a} = (z - x^2)\vec{i} - \sqrt{1-z}\,\vec{k}$, a $\overrightarrow{\Sigma} = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2,\ 0 \leq z \leq 1\}$ orijentirana normalom koja zatvara tupi kut s vektorom $\vec{k}$. Skicirajte plohu.

Zadaci 1,2 i 3 čine 1. dio gradiva, dok se zadaci 4 i 5 odnose na 2. dio gradiva.
Za polaganje ispita treba sakupiti 50 bodova (od toga barem 24 boda iz prvog dijela i barem 16 bodova iz drugog dijela).

$\boxed{9.9.2020.}$

1. (a) (8b) $\sin y\, y' = 2\cos x + 4$, $y(0) = \pi$

$\sin y\, dy = (2\cos x + 4)\, dx \quad / \int$ (2)

$-\cos y = 2\sin x + 4x + C$ (3)

$y(0) = \pi: \quad -\cos\pi = C \Rightarrow \boxed{C = 1}$ (2)

$\Rightarrow -\cos y = 2\sin x + 4x + 1 \Rightarrow y = \arccos(-2\sin x - 4x - 1)$ (1)

(b) (10b) $y'' + y = x^2 + 3$

$\boxed{1.}$ HOMOGENO (1)

$\cdot\ \lambda^2 + 1 = 0$ (1) $\Rightarrow \lambda_{1,2} = \pm i$ (1) $\Rightarrow y_H = C_1\cos x + C_2\sin x$ (1) $C_1, C_2 \in \mathbb{R}$

$\boxed{2.}$ PARTIKULARNO

$y_p = Ax^2 + Bx + C$ (1) $y_p'' = 2A$ (1)

$Ax^2 + Bx + C + 2A = x^2 + 3$ (1) $\Rightarrow \boxed{A = 1}$, (1) $2A + C = 3 \Rightarrow \boxed{C = 1}$ (1)

$\Rightarrow y_p = x^2 + 1$

$\Rightarrow y = y_H + y_p = C_1\cos x + C_2\sin x + x^2 + 1$ (1)

2. (a) (10b) $f(x,y) = \ln\left(2^{24 - 2x^2 + 4xy - y^2} - 4^{2x^2 + 2xy + y^2}\right)$

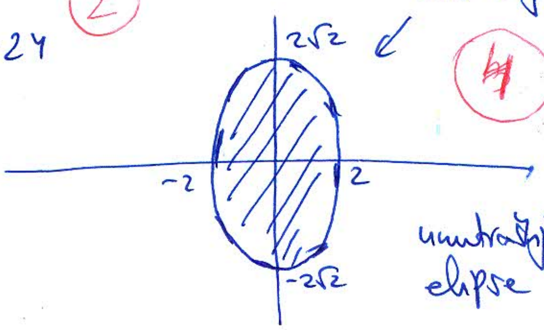
$\cdot\ D(\ln) = \langle 0, +\infty\rangle$

$2^{24 - 2x^2 + 4xy - y^2} > 4^{2x^2 + 2xy + y^2} = 2^{4x^2 + 4xy + 2y^2}$ (2) $\quad / \log_2$

$24 - 2x^2 + \cancel{4xy} - y^2 > 4x^2 + \cancel{4xy} + 2y^2$ (2)

$6x^2 + 3y^2 < 24 \quad / : 24$

$\frac{x^2}{4} + \frac{y^2}{8} < 1$ (2)



(4)

(1) (10b) $\dfrac{\partial^2 f}{\partial x \partial y}$ u točki $T(1,0)$ $\quad f(x,y) = \ln\left(\dfrac{x+2y}{2x-y}\right)$ $\qquad \dfrac{\partial f}{\partial y} = \dfrac{5x}{2x^2+3xy-2y^2}$

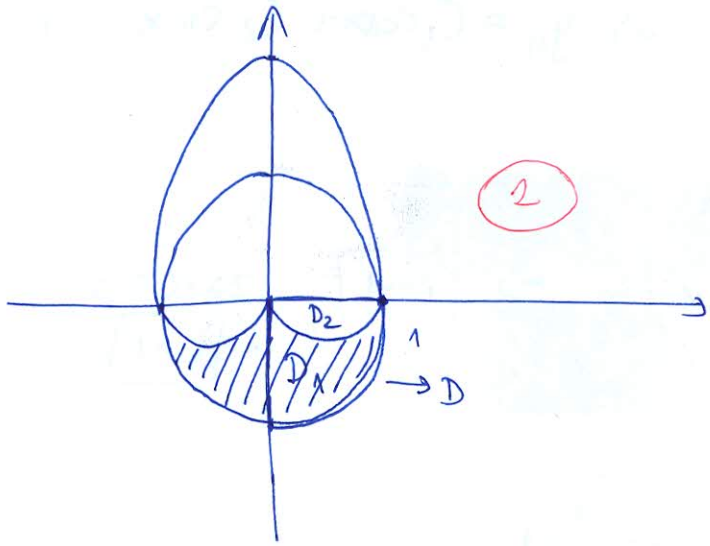
$$\frac{\partial f}{\partial x} = \frac{\cancel{2x-y}}{x+2y} \cdot \frac{1\cdot(2x-y) - (x+2y)\cdot 2}{(2x-y)^2} = \frac{2x-y-2x-4y}{(x+2y)(2x-y)} = \frac{-5y}{2x^2+3xy-2y^2} \quad 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-5(2x^2+3xy-2y^2) + 5y\cdot(3x-4y)}{(2x^2+3xy-2y^2)^2} = \frac{-10x^2 - \cancel{15xy} + 10y^2 + \cancel{15xy} - 20y^2}{(2x^2+3xy-2y^2)^2}$$

$$= \frac{-10x^2-10y^2}{(2x^2+3xy-2y^2)^2} \quad 4$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,0) = \frac{-10}{2^2} = -\frac{10}{4} = \boxed{-\frac{5}{2}} \quad 2$$

3. (a) (12b) $x^2+y^2=1$, $\quad r = 1+\sin\varphi$



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$D_2 \ldots \quad \frac{3\pi}{2} \le \varphi \le 2\pi$

$\qquad 0 \le r \le 1+\sin\varphi$

Tražimo $\quad 2\cdot P(D_1) = 2\left(P(D) - P(D_2)\right) = 2\left(\frac{\pi}{4} - \frac{3\pi}{8} + 1\right) = \boxed{2 - \frac{\pi}{4}}$ 1

· $P(D) = \frac{\pi}{4}$ $\leadsto$ četvrtina kruga

$$P(D_2) = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{1+\sin\varphi} r\,dr\,d\varphi = \int_{\frac{3\pi}{2}}^{2\pi} \frac{r^2}{2}\Big|_0^{1+\sin\varphi} = \frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (1+2\sin\varphi+\sin^2\varphi)\,d\varphi$$

3 $\quad$ 1

$$= \frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}\left(1+2\sin\varphi + \frac{1-\cos 2\varphi}{2}\right)d\varphi = \frac{1}{2}\left(\varphi + 2(-\cos\varphi) + \frac{1}{2}\varphi - \frac{\sin 2\varphi}{2}\right)\Big|_{\frac{3\pi}{2}}^{2\pi}$$

1 $\quad$ 1

$$= \frac{1}{2}\left(3\pi - 2 - \frac{9\pi}{4}\right) = \frac{3\pi}{8} - 1 \quad 1$$

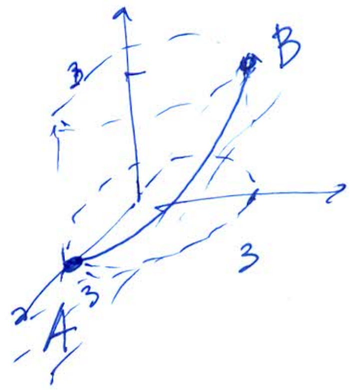
(1.) (6b)

(a) $u(x,y,z) = x^4 - y\sin z - xy^4 + \frac{z^2}{2}$

$$\operatorname{grad} u = \underbrace{(4x^3 - y^4)}_{2}\vec{i} + \underbrace{(-\sin z - 4xy^3)}_{2}\vec{j} + \underbrace{(-y\cos z + z)}_{2}\vec{k}$$

(b) (14b) $\int_p xy\, ds$ $\quad$ p... presjek $x^2+y^2=9$ i $z=y$

od $A(3,0,0)$ do $B(0,3,3)$



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p... $x(t) = 3\cos t$ $\qquad x'(t) = -3\sin t$

$y(t) = 3\sin t$ $\qquad y'(t) = 3\cos t$

$z(t) = 3\sin t$ $\qquad z'(t) = 3\cos t$

$t \in [0, \frac{\pi}{2}]$ $\qquad$ 3

$$\int_p xy\, ds = \int_0^{\frac{\pi}{2}} 9\cos t \sin t \cdot \sqrt{\underbrace{9\sin^2 t + 18\cos^2 t}_{=9(1+\cos^2 t)}}\, dt \quad 3$$

$$= 27\int_0^{\frac{\pi}{2}} \cos t \sin t \sqrt{1+\cos^2 t}\, dt = \left\{\begin{array}{l} u = 1+\cos^2 t \\ du = 2\cos t\cdot(-\sin t)dt \\ 0 \mapsto 2 \\ \frac{\pi}{2} \mapsto 1 \end{array}\right\} \quad 2$$

$$= \frac{27}{2}\int_1^2 \sqrt{u}\, du = \frac{27}{\cancel{2}}\cdot\frac{\cancel{2}}{3} u\sqrt{u}\Big|_1^2 = 9(2\sqrt{2}-1)$$

2 $\qquad$ 2

b) $$\int_0^{\pi} d\varphi \int_0^1 d\rho \int_0^{\sqrt{1-\rho^2}} \rho^3 \, dz$$

$0 \leq \varphi \leq \pi$ $\qquad$ $0 \leq z \leq \sqrt{1-\rho^2}$

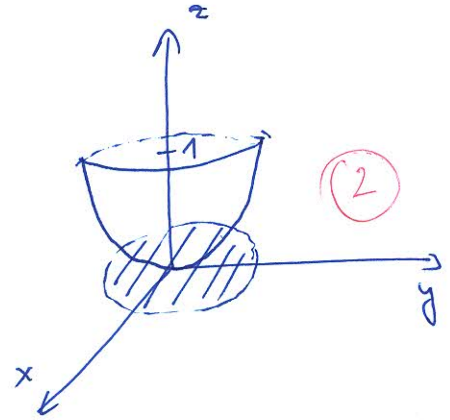
$0 \leq \rho \leq 1$

5. (20b)

$$\iint_{\vec{\Sigma}} \vec{a}\, d\vec{S}, \quad \vec{a} = (z-x^2)\vec{i} - \sqrt{1-z}\,\vec{k}$$

$\vec{\Sigma} = \{(x,y,z) \in \mathbb{R}^3 : z = x^2+y^2, \ 0 \le z \le 1\}$ → tupi kut s vektorom $\vec{k}$

Rj:

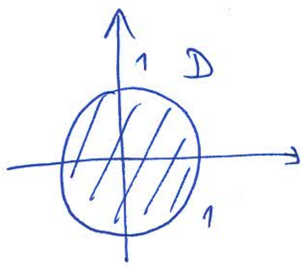


(2)

$$z = x^2+y^2 = f(x,y)$$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$$

$$\vec{n} = 2x\vec{i} + 2y\vec{j} - \vec{k} \quad (2)$$



$D \ldots \ 0 \le \varphi \le 2\pi$

$ 0 \le r \le 1$ (2)

- $\vec{a} \cdot \vec{n} = (z-x^2)\cdot 2x + \sqrt{1-z} = 2xy^2 + \sqrt{1-x^2-y^2}$ (2)

$$\iint_{\Sigma} \vec{a}\, d\vec{S} = \iint_D \left(2xy^2 + \sqrt{1-x^2-y^2}\right) dx\,dy = \int_0^{2\pi} d\varphi \int_0^1 \left(2r^3 \cos\varphi \sin^2\varphi + \sqrt{1-r^2}\right) r\,dr$$

(1) (2)

$$= \int_0^{2\pi} d\varphi \int_0^1 \left(2r^4 \cos\varphi \sin^2\varphi + r\sqrt{1-r^2}\right) dr = (*)$$

- $\int r \cdot \sqrt{1-r^2}\, dr = \left\{\begin{array}{l} t = 1-r^2 \\ dt = -2r\,dr \end{array}\right\} = \int \sqrt{t} \cdot \frac{dt}{-2} = -\frac{1}{3} t\sqrt{t} = -\frac{1}{3}(1-r^2)\sqrt{1-r^2}$

(2) (1) (1) (1)

$$\Rightarrow (*) = \int_0^{2\pi} \left[\frac{2r^5}{5}\cos\varphi \sin^2\varphi - \frac{1}{3}(1-r^2)\sqrt{1-r^2}\right]\Bigg|_0^1 d\varphi = \int_0^{2\pi} \left(\frac{2}{5}\cos\varphi \sin^2\varphi + \frac{1}{3}\right) d\varphi$$

$$= \underbrace{\frac{2}{5}\int_0^{2\pi} \sin^2\varphi \cos\varphi\, d\varphi}_{} + \frac{1}{3}\varphi \Bigg|_0^{2\pi} = \boxed{\frac{2\pi}{3}} \quad (1)$$

$$\left\{\begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \end{array} \quad \begin{array}{l} 0 \mapsto 0 \\ 2\pi \mapsto 0 \end{array}\right\} = 0 \quad (2)$$

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3.b) (10 b)

$$\int_0^{\pi} d\varphi \int_0^1 d\rho \int_0^{\sqrt{1-\rho^2}} \rho^3 \sin^2\varphi \, dz = \int_0^{\pi} d\varphi \int_0^1 \rho^3 \sin^2\varphi \, z \Big|_0^{\sqrt{1-\rho^2}} d\rho \quad 1$$

$$= \int_0^{\pi} \sin^2\varphi \, d\varphi \int_0^1 \rho^3 \sqrt{1-\rho^2}\, d\rho \quad 1 \quad = \frac{1}{2}\int_0^{\pi} (1-\cos 2\varphi)\, d\varphi \quad 1 \quad \int_0^1 \rho^3 \sqrt{1-\rho^2}\, d\rho$$

$$= \left| \begin{array}{ll} t^2 = 1-\rho^2 & \rho = 0 \Rightarrow t = 1 \\ \cancel{2}t\,dt = -\cancel{2}\rho\, d\rho & \rho = 1 \Rightarrow t = 0 \\ \rho^2 = 1-t^2 & \end{array} \right| \quad 2 \quad = \frac{1}{2}\left(\varphi - \frac{1}{2}\sin 2\varphi\right)\Big|_0^{\pi} \quad 1 \quad \cdot \int_0^1 (1-t^2)\,t\cdot t\,dt \quad 2$$

$$= \frac{\pi}{2}\int_0^1 (t^2 - t^4)\,dt = \frac{\pi}{2}\left(\frac{t^3}{3} - \frac{t^5}{5}\right)\Big|_0^1 = \frac{\pi}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{\pi}{2}\cdot\frac{2}{15}$$

$$= \boxed{\frac{\pi}{15}} \quad 2$$