MATEMATIKA II 9.9.2020.

1. Riješite diferencijalne jednadžbe:

   a) (8 bodova) $\sin y y' = 2\cos x + 4$ uz početni uvjet $y(0) = \pi$,

   b) (10 bodova) $y'' + y = x^2 + 3$.

2. a) (10 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \ln(2^{24-2x^2+4xy-y^2} - 4^{2x^2+2xy+y^2}).$$

   b) (10 bodova) Odredite $\frac{\partial^2 f}{\partial x \partial y}$ u točki $T(1,0)$ ako je $f(x,y) = \ln(\frac{x+2y}{2x-y})$.

3. a) (12 bodova) Izračunajte površinu lika omeđenog krivuljama $x^2 + y^2 = 1$ i $r = 1 + \sin\phi$ koji se nalazi unutar prve, a izvan druge krivulje. Skicirajte lik.

   b) (10 bodova) Izračunajte integral

$$\int_0^{\pi} \mathrm{d}\varphi \int_0^1 \mathrm{d}\rho \int_0^{\sqrt{1-\rho^2}} \rho^3 \sin^2\varphi \, \mathrm{d}z.$$

4. a) (6 bodova) Odredite gradijent polja $u(x,y,z) = x^4 - y\sin z - xy^4 + \frac{z^2}{2}$.

   b) (14 bodova) Izračunajte

$$\int_{\Gamma} xy \mathrm{d}s,$$

   ako je krivulja Γ dio presjeka cilindra $x^2+y^2 = 9$ i ravnine $z = y$ od točke $A(3,0,0)$ do točke $B(0,3,3)$. Skicirajte krivulju.

5. (20 bodova) Izračunajte $\int\int_{\overrightarrow{\Sigma}} \vec{a}\, \mathrm{d}\vec{S}$, ako je $\vec{a} = (z - x^2)\vec{i} - \sqrt{1-z}\vec{k}$, a $\overrightarrow{\Sigma} = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2, 0 \leq z \leq 1\}$ orijentirana normalom koja zatvara tupi kut s vektorom $\vec{k}$. Skicirajte plohu.

Zadaci 1,2 i 3 čine 1. dio gradiva, dok se zadaci 4 i 5 odnose na 2. dio gradiva.
Za polaganje ispita treba sakupiti 50 bodova (od toga barem 24 boda iz prvog dijela i barem 16 bodova iz drugog dijela).

$\boxed{9.9.2020.}$

1. (a) (8b) $\sin y\, y' = 2\cos x + 4 \quad , \; y(0) = \pi$

$\sin y\, dy = (2\cos x + 4)\, dx \quad / \int$ (2)

$-\cos y = 2\sin x + 4x + C$ (3)

$y(0) = \pi : \; -\cos\pi = C \Rightarrow \boxed{C = 1}$ (2)

$\Rightarrow -\cos y = 2\sin x + 4x + 1 \Rightarrow y = \arccos(-2\sin x - 4x - 1)$ (1)

(b) (10b) $y'' + y = x^2 + 3$

$\boxed{1.}$ HOMOGENO (1)

$\cdot\; \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$ (1) $\Rightarrow y_H = C_1\cos x + C_2\sin x$ (2) $C_1, C_2 \in \mathbb{R}$

$\boxed{2.}$ PARTIKULARNO

$y_p = Ax^2 + Bx + C$ (1) $y_p'' = 2A$ (1)

$Ax^2 + Bx + C + 2A = x^2 + 3$ (1) $\Rightarrow \boxed{A = 1}$ (1), $2A + C = 3 \Rightarrow \boxed{C = 1}$ (1)

$\Rightarrow y_p = x^2 + 1$

$\Rightarrow y = y_H + y_p = C_1\cos x + C_2\sin x + x^2 + 1$ (1)

2. (a) (10b) $f(x,y) = \ln\left(2^{24 - 2x^2 + 4xy - y^2} - 4^{2x^2 + 2xy + y^2}\right)$

$\cdot\; D(\ln) = \langle 0, +\infty \rangle$

$2^{24 - 2x^2 + 4xy - y^2} > 4^{2x^2 + 2xy + y^2} = 2^{4x^2 + 4xy + 2y^2}$ (2) $\quad / \log_2$

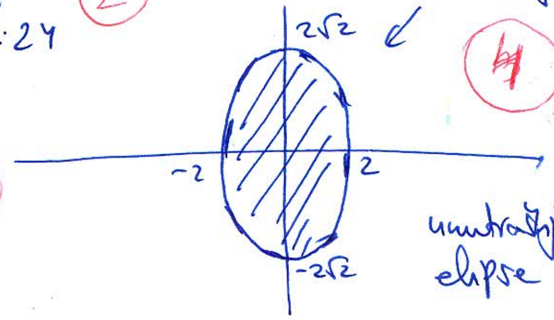
$24 - 2x^2 + \cancel{4xy} - y^2 > 4x^2 + \cancel{4xy} + 2y^2$ (2)

$6x^2 + 3y^2 < 24 \quad / : 24$

$\frac{x^2}{4} + \frac{y^2}{8} < 1$ (2)

Rub isključen

(4)



unutrašnjost elipse

(10b)

(1) $\dfrac{\partial^2 f}{\partial x \partial y}$ u točki $T(1,0)$ $\quad f(x,y)=\ln\left(\dfrac{x+2y}{2x-y}\right)$ $\quad \dfrac{\partial f}{\partial y} = \dfrac{5x}{2x^2+3xy-2y^2}$

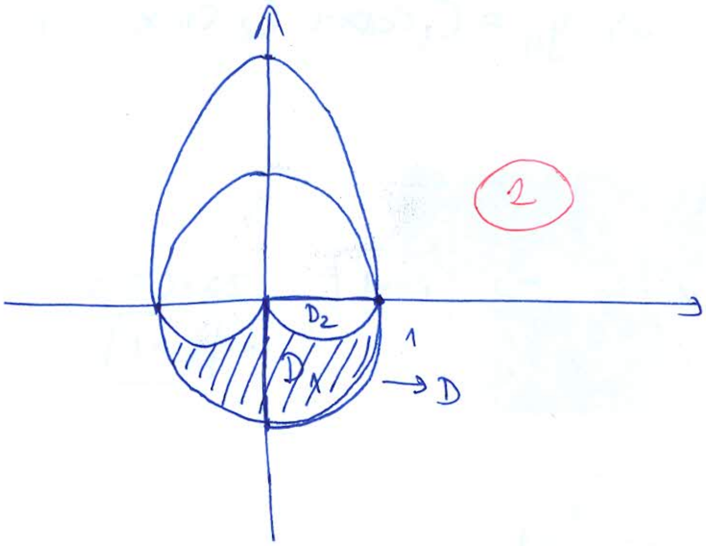
$$\frac{\partial f}{\partial x} = \frac{\cancel{2x-y}}{x+2y}\cdot\frac{1\cdot(2x-y)-(x+2y)\cdot 2}{(2x-y)^2} = \frac{2x-y-2x-4y}{(x+2y)(2x-y)} = \frac{-5y}{2x^2+3xy-2y^2} \quad (4)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-5(2x^2+3xy-2y^2)+5y\cdot(3x-4y)}{(2x^2+3xy-2y^2)^2} = \frac{-10x^2-\cancel{15xy}+10y^2+\cancel{15xy}-20y^2}{(2x^2+3xy-2y^2)^2}$$

$$= \frac{-10x^2-10y^2}{(2x^2+3xy-2y^2)^2} \quad (4)$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,0) = \frac{-10}{2^2} = -\frac{10}{4} = \boxed{-\frac{5}{2}} \quad (2)$$

(3.) (a) (12b) $x^2+y^2=1$, $r=1+\sin\varphi$



(2)

$D_2 \ldots \quad \dfrac{3\pi}{2} \le \varphi \le 2\pi$

$\quad 0 \le r \le 1+\sin\varphi$

Tražimo $2\cdot P(D_1) = 2\left(P(D)-P(D_2)\right) = 2\left(\dfrac{\pi}{4}-\dfrac{3\pi}{8}+1\right) = \boxed{2-\dfrac{\pi}{4}}$ (1)

$\cdot\ P(D) = \dfrac{\pi}{4}$ ⇝ četvrtina kruga

$$P(D_2) = \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{1+\sin\varphi} r\,dr\,d\varphi = \int_{\frac{3\pi}{2}}^{2\pi} \frac{r^2}{2}\Big|_0^{1+\sin\varphi} = \frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}(1+2\sin\varphi+\sin^2\varphi)\,d\varphi$$

(3) (1)

$$= \frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}\left(1+2\sin\varphi+\frac{1-\cos 2\varphi}{2}\right)d\varphi = \frac{1}{2}\left(\varphi+2(-\cos\varphi)+\frac{1}{2}\varphi-\frac{\sin 2\varphi}{2}\right)\Big|_{\frac{3\pi}{2}}^{2\pi}$$

(1) (1)

$$= \frac{1}{2}\left(3\pi-2-\frac{9\pi}{4}\right) = \frac{3\pi}{8}-1 \quad (1)$$

1. (a) (6b) $u(x,y,z) = x^4 - y\sin z - xy^4 + \frac{z^2}{2}$

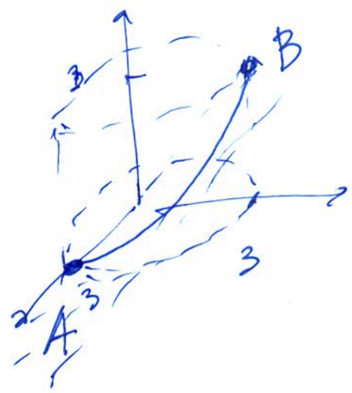
$$\operatorname{grad} u = \underbrace{(4x^3 - y^4)}_{2}\vec{i} + \underbrace{(-\sin z - 4xy^3)}_{2}\vec{j} + \underbrace{(-y\cos z + z)}_{2}\vec{k}$$

(b) (14b) $\int_P xy\,ds$ $\quad P\ldots$ presjek $x^2+y^2=9$ i $z=y$

od $A(3,0,0)$ do $B(0,3,3)$

$$P\ldots\quad x(t) = 3\cos t \qquad x'(t) = -3\sin t$$
$$y(t) = 3\sin t \qquad y'(t) = 3\cos t$$
$$z(t) = 3\sin t \qquad z'(t) = 3\cos t$$
$$t \in \left[0, \frac{\pi}{2}\right]$$

3



2

$$\int_P xy\,ds = \int_0^{\frac{\pi}{2}} 9\cos t\sin t \cdot \sqrt{\underbrace{9\sin^2 t + 18\cos^2 t}_{=9(1+\cos^2 t)}}\,dt \quad 3$$

$$= 27\int_0^{\frac{\pi}{2}} \cos t\sin t\sqrt{1+\cos^2 t}\,dt = \left\{\begin{array}{l} u = 1+\cos^2 t \\ du = 2\cos t\cdot(-\sin t)\,dt \\ 0 \mapsto 2 \\ \frac{\pi}{2} \mapsto 1 \end{array}\right\} \quad 2$$

$$= \frac{27}{2}\int_1^2 \sqrt{u}\,du = \frac{27}{2}\cdot\frac{2}{3}u\sqrt{u}\Big|_1^2 = 9\left(2\sqrt{2}-1\right)$$

2 $\qquad$ 2

b)
$$\int_0^{\pi} d\varphi \int_0^{1} d\rho \int_0^{\sqrt{1-\rho^2}} \rho^3 \, dz$$

$0 \le \varphi \le \pi$ $\qquad$ $0 \le z \le \sqrt{1-\rho^2}$

$0 \le \rho \le 1$