Poglavlje 5

Krivuljni integrali

Skalarna i vektorska polja integrirat ćemo po krivuljama u prostoru i ravnini. Neka je Γ krivulja u $\mathbb{R}^3$, te neka je njena parametrizacija $r([a,b],\vec{r})$. Parametrizaciju ćemo pisati po komponentama: $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$.

Integrale skalarnih polja po krivuljama nazivamo krivuljnim integralima prve vrste, a integrale vektorskih polja krivuljnim integralima druge vrste.

5.1 Krivuljni integrali prve vrste

Za funkciju $f\colon \Gamma \to \mathbb{R}$, *krivuljni integral prve vrste funkcije f po krivulji* Γ je

$$\int_\Gamma f ds = \int_a^b f(r(t)) \cdot r'(t)\, dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt. \tag{5.1}$$

Krivuljni integral ne ovisi o parametrizaciji.

Ako je Γ krivulja u $\mathbb{R}^2$ s parametrizacijom $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $t \in [a,b]$, a funkcija f definirana na njoj ($f\colon \Gamma \subseteq \mathbb{R}^2 \to \mathbb{R}$), krivuljni integral f po Γ računamo analogno:

$$\int_\Gamma f\, ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{5.2}$$

Krivuljni integrali imaju razne primjene, npr. duljina krivulje Γ jednaka je $l(\Gamma) = \int_\Gamma ds$, a masa krivulje Γ s linijskom gustoćom $\rho\colon \Gamma \to \mathbb{R}$ jednaka je $m(\Gamma) = \int_\Gamma \rho\, ds$.

Zadatak 5.1. Izračunajte $\int_\Gamma (x^2 + 2y^2)\,ds$ ako je Γ kružnica $x^2 + y^2 = 1$.

Rješenje: Općenito, kružnicu $x^2 + y^2 = R^2$ parametriziramo kao $x(t) = R\cos t, y(t) = R\sin t$, pri čemu je $t \in [0, 2\pi]$. Ovdje je onda moguća parametrizacija

$$\begin{aligned} x(t) = \cos t &\Rightarrow x'(t) = -\sin t \\ y(t) = \sin t &\Rightarrow y'(t) = \cos t. \end{aligned}$$

Sada računamo integral po formuli 5.2:

$$\begin{aligned} \int_\Gamma (x^2 + 2y^2)\,ds &= \int_0^{2\pi} (\cos^2 t + 2\sin^2 t) \cdot \sqrt{(-\sin t)^2 + \cos^2 t}\,dt = \\ &= \int_0^{2\pi} (1 + \sin^2 t)\,dt = \int_0^{2\pi} \left(1 + \frac{1 - \cos 2t}{2}\right) dt \\ &= t\Big|_0^{2\pi} + \left(\frac{1}{2}t - \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = 2\pi + \pi = 3\pi. \end{aligned}$$

□

Općenitiju kružnicu $(x - x_0)^2 + (y - y_0)^2 = R^2$ možemo parametrizirati na sličan način:

$$\begin{aligned} x - x_0 = R\cos t &\Rightarrow x(t) = x_0 + R\cos t \\ y - y_0 = R\sin t &\Rightarrow y(t) = y_0 + R\sin t \\ t &\in [0, 2\pi]. \end{aligned}$$

Zadatak 5.2. Izračunajte $\int_\Gamma x^4\,ds$, ako je Γ kružnica $x^2 + y^2 = 2y$.

Rješenje: Jednadžba $x^2 + y^2 = 2y$ ekvivalentna je jednadžbi $x^2 + (y-1)^2 = 1$, koja opisuje kružnicu sa središtem u $(0, 1)$ radijusa 1. Nju možemo parametrizirati kao

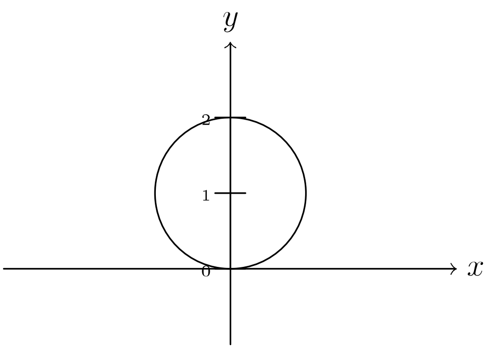
$$\begin{aligned} x(t) = \cos t &\Rightarrow x'(t) = -\sin t \\ y(t) = 1 + \sin t &\Rightarrow y'(t) = \cos t \\ t &\in [0, 2\pi]. \end{aligned}$$

Računamo

$$\int_\Gamma x^4\,ds = \int_0^{2\pi} \cos^4 t \cdot \sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{2\pi} \cos^4 t\,dt = \int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$
$$= \int_0^{2\pi} \frac{1 + 2\cos 2t + \frac{1+\cos 4t}{2}}{4}\,dt = \left(\frac{3}{8}t + \frac{\sin 2t}{4} + \frac{\sin 4t}{32}\right)\Bigg|_0^{2\pi} =$$
$$= \frac{3}{8}\cdot 2\pi = \frac{3\pi}{4}.$$

□

Osim na ovaj način, kružnicu $x^2 + y^2 = 2y$ mogli smo parametrizirati i u polarnim koordinatama:



Slika 5.1: Kružnica $x^2 + y^2 = 2y$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$x^2 + y^2 = 2y \Rightarrow r^2 = 2r\sin\varphi$$
$$\Rightarrow r = 2\sin\varphi$$

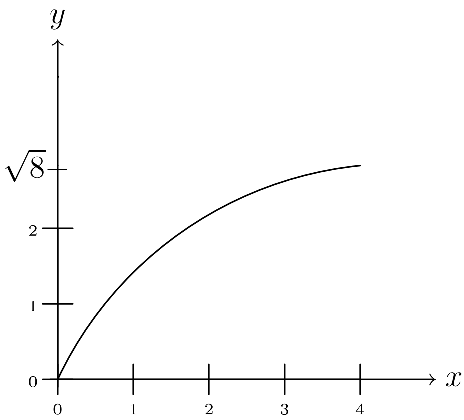
Iz toga dobivamo sljedeću parametrizaciju:

$$x(\varphi) = 2\sin\varphi\cos\varphi$$
$$y(\varphi) = 2\sin^2\varphi$$
$$\varphi \in [0,\pi]$$

Integral u zadatku 5.2 izračunat po ovoj parametrizaciji bit će jednak kao i po onoj korištenoj u rješenju.

Zadatak 5.3. Izračunajte $\int_\Gamma y\,ds$, gdje je Γ luk parabole $x = \frac{y^2}{2}$ od točke $A(0,0)$ do točke $B(4,\sqrt{8})$.

Rješenje:



Slika 5.2: Dio parabole $x = \frac{y^2}{2}$ od $(0,0)$ do $(4,\sqrt{8})$.

$y > 0 \Rightarrow y = \sqrt{2x}$ pa lako parametriziramo krivulju Γ:

$$x(t) = t \Rightarrow x'(t) = 1$$
$$y(t) = \sqrt{2t} \Rightarrow y'(t) = \frac{1}{2\sqrt{2t}}\cdot 2 = \frac{1}{\sqrt{2t}}$$
$$t \in [0,4]$$

Računamo

$$
\begin{aligned}
\int_\Gamma y\,ds &= \int_0^4 \sqrt{2t}\cdot\sqrt{1+\frac{1}{2t}}\,dt = \int_0^4 \sqrt{2t+1}\,dt = \\
&= \left\{ \begin{array}{l} u=\sqrt{2t+1} \\ du = \frac{1}{\sqrt{2t+1}}dt \Rightarrow dt = udu \\ 0\mapsto 1, 4\mapsto 3 \end{array} \right\} = \\
&= \int_1^3 u\cdot u\,du = \left.\frac{u^3}{3}\right|_1^3 = \frac{1}{3}(27-1) = \frac{26}{3}.
\end{aligned}
$$

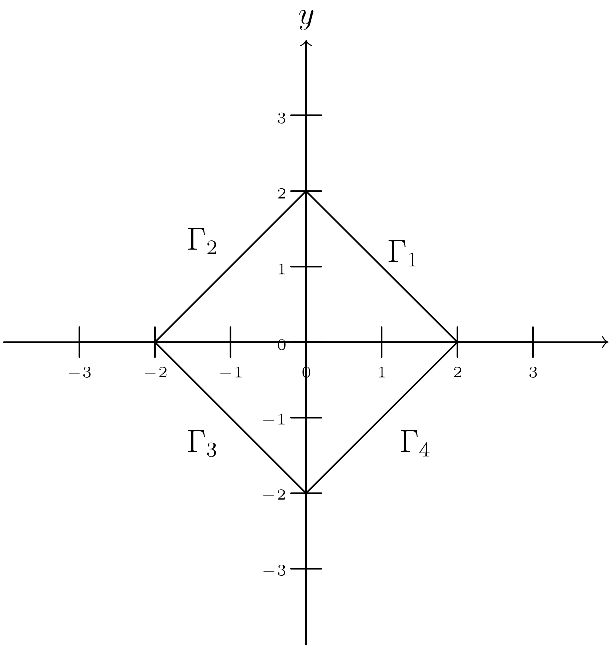
□

Zadatak 5.4. (DZ) Izračunajte $\int_\Gamma \frac{y}{\sqrt{x}}\,ds$, ako je Γ luk polukubne parabole $y^2 = \frac{4}{9}x^3$ od točke $A(3, 2\sqrt{3})$ do točke $B(8, \frac{32}{3}\sqrt{2})$.

Rješenje: Zadaća ($\frac{2152}{45}$). □

Zadatak 5.5. Izračunajte $\int_\Gamma xy\,ds$, ako je Γ opseg kvadrata $|x|+|y|=2$.

Rješenje:



Slika 5.3: $\Gamma ... |x|+|y|=2$.

Ovisno o predznacima x i y dobivamo 4 dijela kvadrata Γ:
$\Gamma_1 \dots x\geq 0, y\geq 0$
$\Rightarrow x+y=2$
$\Rightarrow y=2-x$
$\Gamma_2 \dots x<0, y\geq 0$
$\Rightarrow -x+y=2$
$\Rightarrow y=2+x$
$\Gamma_3 \dots x<0, y<0$
$\Rightarrow -x-y=2$
$\Rightarrow y=-x-2$
$\Gamma_4 \dots x\geq 0, y<0$
$\Rightarrow x-y=2$
$\Rightarrow y=x-2$.

$\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$
$\Rightarrow \int_\Gamma f\,ds = \int_{\Gamma_1} f\,ds +$

$\int_{\Gamma_2} f\,ds + \int_{\Gamma_3} f\,ds + \int_{\Gamma_4} f\,ds$ budući da je i krivuljni integral prve vrste aditivan po području integracije.

Dakle, potrebno je parametrizirati svaku od 4 stranice kvadrata te po svakoj posebno izračunati zadani integral, a onda to pozbrajati.

Budući da smo odredili jednadžbe svih pravaca na kojima leže stranice kvadrata, parametrizacije lako dobivamo:

$$\begin{aligned}\Gamma_1 \ldots \quad & x(t) = t \Rightarrow x'(t) = 1\\ & y(t) = 2 - t \Rightarrow y'(t) = -1\\ & t \in [0, 2].\end{aligned}$$

Sad računamo

$$\int_{\Gamma_1} f\,ds = \int_0^2 t(2-t)\sqrt{1+1}\,dt = \sqrt{2}\left(t^2 - \frac{t^3}{3}\right)\Big|_0^2 = \sqrt{2}\left(4 - \frac{8}{3}\right) = \frac{4\sqrt{2}}{3}.$$

Analogno dobivamo ostale parametrizacije te računamo po njima:

$$\begin{aligned}\Gamma_2 \ldots \quad & x(t) = t \Rightarrow x'(t) = 1\\ & y(t) = t + 2 \Rightarrow y'(t) = 1\\ & t \in [-2, 0].\end{aligned}$$

$$\Rightarrow \int_{\Gamma_2} f\,ds = \int_{-2}^0 t(t+2)\sqrt{2}\,dt = \sqrt{2}\left(\frac{t^3}{3} + t^2\right)\Big|_{-2}^0 = \sqrt{2}\left(\frac{8}{3} - 4\right) = -\frac{4\sqrt{2}}{3},$$

$$\begin{aligned}\Gamma_3 \ldots \quad & x(t) = t \Rightarrow x'(t) = 1\\ & y(t) = -t - 2 \Rightarrow y'(t) = -1\\ & t \in [-2, 0].\end{aligned}$$

$$\Rightarrow \int_{\Gamma_3} f\,ds = \int_{-2}^0 t(-t-2)\sqrt{2}\,dt = \frac{4\sqrt{2}}{3},$$

$$\begin{aligned}\Gamma_4 \ldots \quad & x(t) = t \Rightarrow x'(t) = 1\\ & y(t) = t - 2 \Rightarrow y'(t) = 1\\ & t \in [0, 2].\end{aligned}$$

$$\Rightarrow \int_{\Gamma_4} f\,ds = \int_0^{-2} t(t-2)\sqrt{2}\,dt = -\frac{4\sqrt{2}}{3},$$

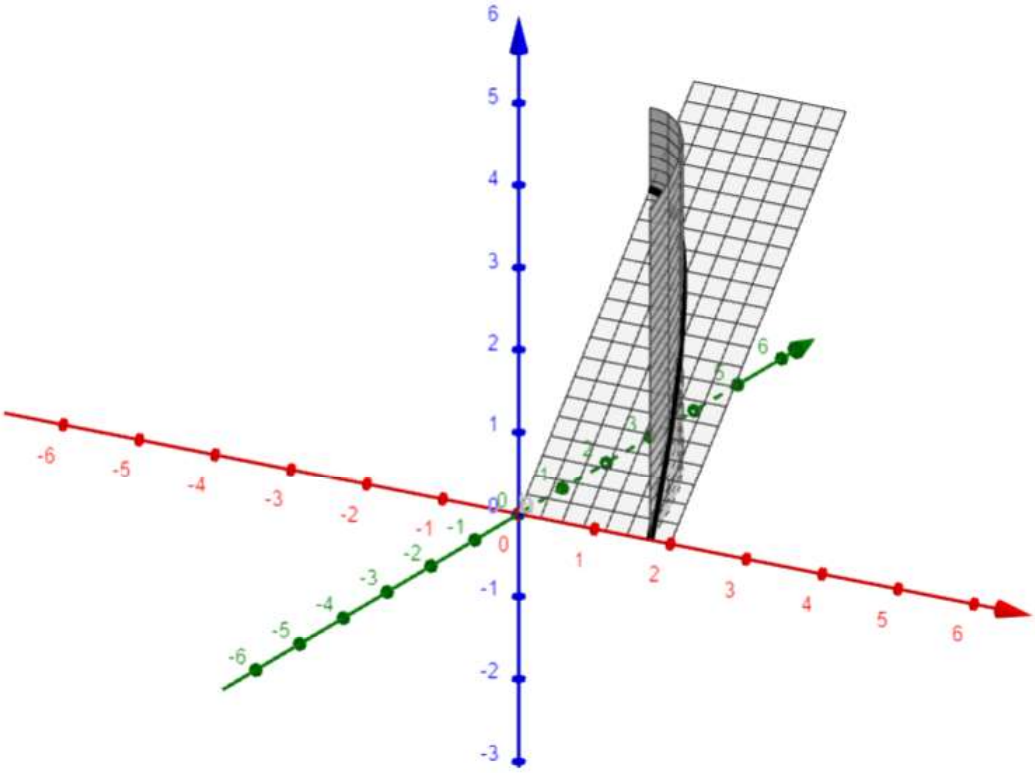
iz čega onda slijedi

$$\int_\Gamma xy\,ds = \frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{3} = 0.$$

$\square$

Zadatak 5.6. Izračunajte $\int_\Gamma x\,ds$, ako je Γ presječnica ploha $y = 3 - x^2$ i $z = y$ u prvom oktantu. Skicirajte krivulju.

Rješenje: Krivulja Γ je presječnica paraboličkog cilindra $y = 3 - x^2$ i ravnine $z = y$ u 1. oktantu (pogledajte sliku 5.4).



Slika 5.4: Krivulja Γ

Parametrizirajmo krivulju:

$$\begin{aligned}\Gamma \ldots \quad & x(t) = t \Rightarrow x'(t) = 1\\ & y(t) = 3 - t^2 \Rightarrow y'(t) = -2t\\ & z(t) = y(t) = 3 - t^2 \Rightarrow z'(t) = -2t\\ & t \in [0, \sqrt{3}].\end{aligned}$$

Sada računamo

$$\int_\Gamma x\,\mathrm{d}s = \int_0^{\sqrt{3}} t\sqrt{1+4t^2+4t^2}\,\mathrm{d}t = \int_0^{\sqrt{3}} t\sqrt{1+8t^2}\,\mathrm{d}t$$

$$= \left\{ \begin{array}{l} u = 1+8t^2 \\ du = 16t\mathrm{d}t \Rightarrow t\mathrm{d}t = \frac{du}{16} \\ 0 \mapsto 1, \sqrt{3} \mapsto 25 \end{array} \right\} = \frac{1}{16}\int_1^{25} \sqrt{u}\,\mathrm{d}u = \frac{1}{16}\,\frac{u\sqrt{u}}{\frac{3}{2}}\Big|_1^{25}$$

$$= \frac{1}{24}(125-1) = \frac{124}{24} = \frac{31}{6}.$$

□

Zadatak 5.7. Izračunajte $\int_\Gamma \frac{y}{x+3z}\,ds$, ako je Γ luk krivulje $x = t, y = \frac{t^2}{\sqrt{2}}, z = \frac{t^3}{3}$ od točke $A(0,0,0)$ do $B(\sqrt{2}, \sqrt{2}, \frac{2\sqrt{2}}{3})$.

Rješenje: Zadaća ($\frac{1}{\sqrt{2}}$). □

Zadatak 5.8. Izračunajte duljinu luka kardioide $r = 1 + \cos\varphi$.

Rješenje: Kardioidu Γ ćemo parametrizirati pomoću polarnih koordinata:

$$\begin{aligned} \Gamma \ldots \quad & x(\varphi) = r(\varphi)\cos\varphi \Rightarrow x'(t) = r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi \\ & y(t) = r(\varphi)\sin\varphi \Rightarrow y'(t) = r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi. \end{aligned}$$

Duljinu kardioide računamo po formuli:

$$l(\Gamma) = \int_0^{2\pi} \sqrt{x'(\varphi)^2 + y'(\varphi)^2}\mathrm{d}\varphi$$

$$= \int_0^{2\pi} \sqrt{(r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi)^2 + (r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi)^2}\mathrm{d}\varphi$$

$$= \int_0^{2\pi} \sqrt{r(\varphi)^2 + r'(\varphi)^2}\mathrm{d}\varphi = \int_0^{2\pi} \sqrt{(1+\cos\varphi)^2 + (-\sin\varphi)^2}\mathrm{d}\varphi$$

$$= \int_0^{2\pi} \sqrt{2}\sqrt{\underbrace{1+\cos\varphi}_{2\cos^2\frac{\varphi}{2}}}\mathrm{d}\varphi = 2\int_0^{2\pi} \left|\cos\frac{\varphi}{2}\right|\mathrm{d}\varphi$$

$$= 2\int_0^{\pi} \cos\frac{\varphi}{2}\mathrm{d}\varphi - 2\int_\pi^{2\pi} \cos\frac{\varphi}{2}\mathrm{d}\varphi$$

$$= 2\cdot 2\cdot \sin\frac{\varphi}{2}\Big|_0^{\pi} - 4\cdot\sin\frac{\varphi}{2}\Big|_\pi^{2\pi} = 4 - 4\cdot(0-1) = 4+4 = 8.$$

□

Zadatak 5.9. Izračunajte $\int_\Gamma (x^2+y^2+z^2)\,\mathrm{d}s$, gdje je Γ helikoida $x(t) = \cos t, y(t) = \sin t, z(t) = t$ od $A(1,0,0)$ do $B(0,1,\frac{\pi}{2})$.

Rješenje: Raspišimo parametrizaciju helikoide Γ:

$$\begin{aligned}\Gamma \ldots \quad & x(t) = \cos t \Rightarrow x'(t) = -\sin t\\ & y(t) = \sin t \Rightarrow y'(t) = \cos t\\ & z(t) = y(t) = t \Rightarrow z'(t) = 1\\ & t \in [0, \frac{\pi}{2}].\end{aligned}$$

Sada računamo integral:

$$\int_\Gamma (x^2+y^2+z^2)\,\mathrm{d}s = \int_0^{\frac{\pi}{2}} (1+t^2)\sqrt{2}\,\mathrm{d}t$$

$$= \sqrt{2}\Big(t+\frac{t^3}{3}\Big)\Big|_0^{\frac{\pi}{2}} = \sqrt{2}\Big(\frac{\pi}{2}+\frac{\pi^3}{24}\Big) = \frac{\sqrt{2}\pi}{24}(12+\pi^2).$$

□

Zadatak 5.10. Izračunajte duljinu luka krivulje Γ od točke $A(0,0,0)$ do točke $B(3,3,2)$, gdje je Γ presjek $y = \frac{x^2}{3}$ i $z = \frac{2}{9}xy$.

Rješenje: Parametrizirajmo krivulju Γ:

$$\begin{aligned}\Gamma \ldots \quad & x(t) = t \Rightarrow x'(t) = 1\\ & y(t) = \frac{t^2}{3} \Rightarrow y'(t) = \frac{2}{3}t\\ & z(t) = \frac{2}{9}t \cdot \frac{t^2}{3} = \frac{2t^3}{27} \Rightarrow z'(t) = \frac{2t^2}{9}\\ & t \in [0,3].\end{aligned}$$

Duljina luka krivulje Γ iznosi:

$$l(\Gamma) = \int_0^3 \sqrt{1+\frac{4}{9}t^2+\frac{4}{81}t^4}\,\mathrm{d}t == \int_0^3 \sqrt{\Big(1+\frac{2t^2}{9}\Big)^2}\,\mathrm{d}t$$

$$= \int_0^3 \Big(1+\frac{2t^2}{9}\Big)\,\mathrm{d}t = \Big(t+\frac{2t^3}{27}\Big)\Big|_0^3 = 3+2 = 5.$$

□

Zadatak 5.11. Izračunajte $\int_\Gamma e^{x+y-z}\, ds$, ako je Γ dio pravca $\frac{x}{2} = \frac{y-1}{2} = \frac{z}{3}$ od točke $A(0,1,0)$ do $B(2,3,3)$.

Rješenje: Napišimo parametrizaciju krivulje Γ:

$$\begin{aligned} \Gamma \ldots \quad & x(t) = 2t \Rightarrow x'(t) = 2 \\ & y(t) = 1 + 2t \Rightarrow y'(t) = 2 \\ & z(t) = 3t \Rightarrow z'(t) = 3 \\ & t \in [0,1]. \end{aligned}$$

Sada odredimo integral:

$$\int_\Gamma e^{x+y-z}\, \mathrm{d}s = \int_0^1 e^{2t+1+2t-3t}\sqrt{4+4+9}\, \mathrm{d}t$$

$$= \sqrt{17}\int_0^1 e^{t+1}\mathrm{d}t = \sqrt{17}e^{t+1}\Big|_0^1 = \sqrt{17}(e^2 - e).$$

$\square$

Zadatak 5.12. Izračunajte $\int_\Gamma xy\, ds$, ako je Γ dio krivulje nastale presjekom cilindra $x^2 + y^2 = 1$ i ravnine $z = y$ od $A(1,0,0)$ do $B(0,1,1)$.

Rješenje: Parametrizirajmo krivulju Γ:

$$\begin{aligned} \Gamma \ldots \quad & x(t) = \cos t \Rightarrow x'(t) = -\sin t \\ & y(t) = \sin t \Rightarrow y'(t) = \cos t \\ & z(t) = y(t) = \sin t \Rightarrow z'(t) = \cos t \\ & t \in [0, \frac{\pi}{2}]. \end{aligned}$$

Sada računamo integral:

$$\int_\Gamma xy\, \mathrm{d}s = \int_0^{\frac{\pi}{2}} \cos t \sin t \sqrt{1 + \cos^2 t}\, \mathrm{d}t$$

$$= \left\{ \begin{array}{l} u = 1 + \cos^2 t \\ du = 2\cos t(-\sin t)\mathrm{d}t \\ 0 \mapsto 2, \frac{\pi}{2} \mapsto 1 \end{array} \right\} = -\frac{1}{2}\int_2^1 \sqrt{u}\, \mathrm{d}u$$

$$= \frac{1}{2}\int_1^2 \sqrt{u}\, \mathrm{d}u = \frac{1}{2} \cdot \frac{u\sqrt{u}}{\frac{3}{2}}\Big|_1^2 = \frac{1}{3}(2\sqrt{2} - 1).$$

$\square$

Zadatak 5.13. Izračunajte $\int_\Gamma y\,ds$, ako je Γ dio presječne krivulje paraboloida $z = 6 - x^2 - y^2$ i ravnine $z = 2$ koji se nalazi u prvom oktantu.

Rješenje: Zadaća (4). □

Zadatak 5.14. Izračunajte duljinu i masu žice u obliku logaritamske spirale $r = e^{\frac{\varphi}{2\pi}}$ od točke $A(1,0)$ do točke $B(e,0)$, ako je linijska gustoća $\rho(x,y) = x^2 + y^2$.

Rješenje: Znamo da je $x^2+y^2 = r^2 = e^{\frac{\varphi}{\pi}}$, što ćemo često koristiti u računima koji slijede. Logaritamsku spiralu Γ ćemo parametrizirati tako da u vezu između kartezijevih i polarnih koordinata uvrstimo jednadžbu spirale:

$$\Gamma \ldots \quad \begin{aligned} &x(\varphi) = r(\varphi)\cos\varphi \Rightarrow x(\varphi) = e^{\frac{\varphi}{2\pi}}\cos\varphi \\ &y(\varphi) = r(\varphi)\sin\varphi \Rightarrow y(\varphi) = e^{\frac{\varphi}{2\pi}}\sin\varphi \\ &\varphi \in [0, 2\pi]. \end{aligned}$$

Interval parametrizacije slijedi iz činjenice da uvrštavanjem točke $\varphi = 0$ u parametrizaciju dolazimo do točke A, dok uvrštavajući $\varphi = 2\pi$ stižemo do točke B. Koristeći izvod iz rješenja zadatka u kojem smo računali duljinu kardioide imamo da je:

$$x'(\varphi)^2 + y'(\varphi)^2 = r(\varphi)^2 + r'(\varphi)^2 = e^{\frac{\varphi}{\pi}} + \frac{1}{4\pi^2} \cdot e^{\frac{\varphi}{\pi}}.$$

Odredimo duljinu spirale:

$$\begin{aligned} l(\Gamma) &= \int_0^{2\pi} \sqrt{e^{\frac{\varphi}{\pi}} + \frac{1}{4\pi^2} \cdot e^{\frac{\varphi}{\pi}}}\,\mathrm{d}\varphi = \sqrt{1 + \frac{1}{4\pi^2}} \int_0^{2\pi} \sqrt{e^{\frac{\varphi}{\pi}}}\,\mathrm{d}\varphi \\ &= \sqrt{1 + \frac{1}{4\pi^2}} \int_0^{2\pi} e^{\frac{\varphi}{2\pi}}\,\mathrm{d}\varphi = \sqrt{1 + \frac{1}{4\pi^2}} \cdot \frac{1}{\frac{1}{2\pi}} \cdot e^{\frac{\varphi}{2\pi}}\Big|_0^{2\pi} \\ &= 2\pi\sqrt{\frac{4\pi^2+1}{4\pi^2}}(e-1) = \sqrt{1+4\pi^2}(e-1). \end{aligned}$$

Sada izračunajmo masu spirale:

$$m = \int_\Gamma (x^2+y^2)\mathrm{d}s = \int_0^{2\pi} e^{\frac{\varphi}{\pi}} \cdot e^{\frac{\varphi}{2\pi}} \cdot \sqrt{1 + \frac{1}{4\pi^2}}\,\mathrm{d}\varphi$$

$$= \frac{\sqrt{1+4\pi^2}}{2\pi} \int_0^{2\pi} e^{\frac{3\varphi}{2\pi}}\,\mathrm{d}\varphi = \frac{\sqrt{1+4\pi^2}}{2\pi} \cdot \frac{1}{\frac{3}{2\pi}} \cdot e^{\frac{3\varphi}{2\pi}}\Big|_0^{2\pi} = \frac{\sqrt{1+4\pi^2}}{3}\,(e^3-1).$$

□