Poglavlje 6

Plošni integrali

6.1 Plošni integrali prve vrste

Neka je ploha Σ zadana eksplicitnom jednadžbom $z = f(x, y)$ za $(x, y) \in D \subseteq \mathbb{R}^2$ i neka je $F : \Omega \to \mathbb{R}$, $\Sigma \subseteq \Omega \subseteq \mathbb{R}^3$, dovoljno glatka funkcija tri varijable. Plošni integral 1. vrste funkcije F po plohi Σ se računa po formuli:

$$\iint_\Sigma F(x, y, z)\,\mathrm{d}S = \iint_D F(x, y, f(x, y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y,$$

gdje je D projekcija plohe Σ na xy-ravninu.

Površina plohe $\Sigma \ldots z = f(x, y)$, $(x, y) \in D$, je dana formulom:

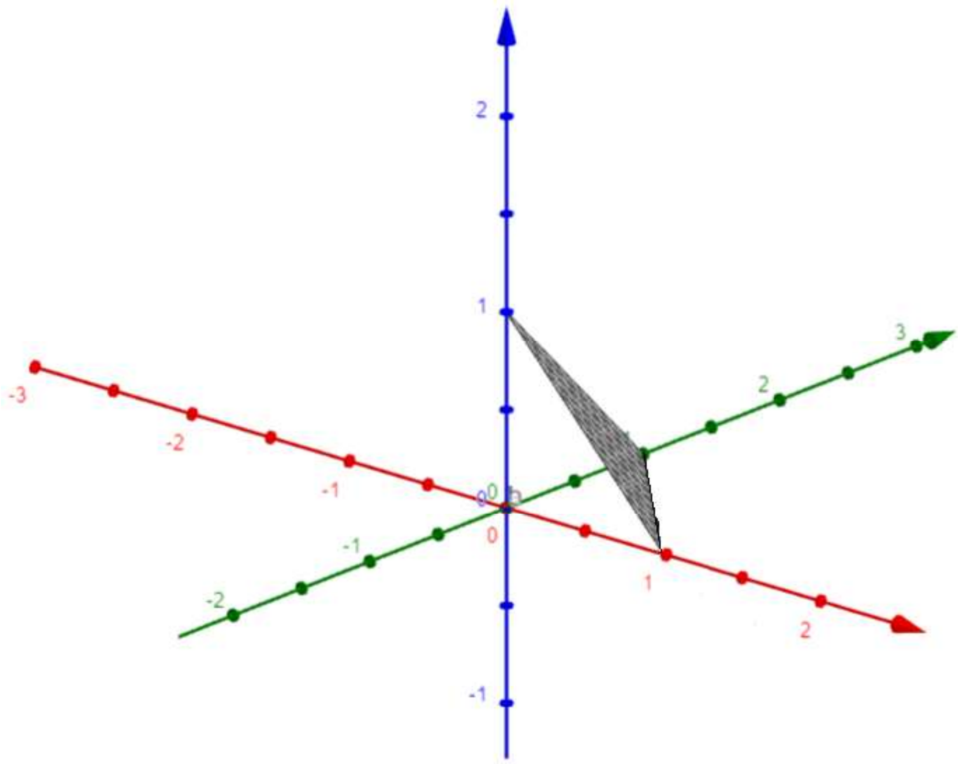
$$P(\Sigma) = \iint_\Sigma \mathrm{d}S = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y,$$

Ukoliko je ploha Σ okomita na xy-ravninu, njenu jednadžbu ćemo napisati u obliku $\Sigma \ldots y = f(x, z)$, a plošni integral 1. vrste funkcije F po plohi Σ ćemo računati po formuli:

$$\iint_\Sigma F(x, y, z)\,\mathrm{d}S = \iint_D F(x, f(x, z), z)\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,\mathrm{d}x\,\mathrm{d}z,$$

gdje je D projekcija plohe Σ na xz-ravninu.

Zadatak 6.1. Izračunajte $\iint_\Sigma \frac{dS}{(1+x+z)^2}$ ako je Σ dio ravnine $x + y + z = 1$ u 1. oktantu. Skicirajte plohu.



Slika 6.1: Ploha Σ

Rješenje: Napišimo eksplicitnu jednadžbu ravnine:

$$z = f(x,y) = 1 - x - y \Rightarrow \frac{\partial f}{\partial x} = -1\,, \frac{\partial f}{\partial y} = -1.$$

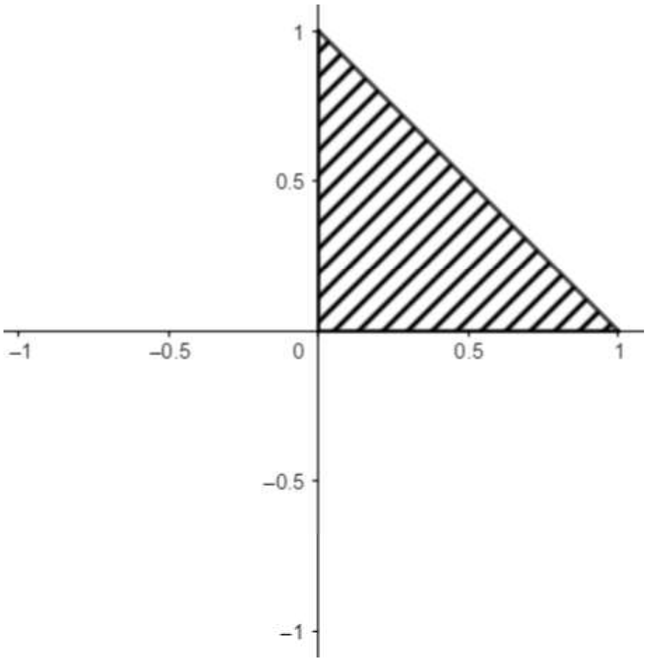
Projekcija D plohe Σ na xy-ravninu je pravokutan trokut čije katete leže na koordinatnim osima, a hipotenuza na pravcu $y = 1 - x$ (pogledajte sliku 6.2). Sada računamo integral:

$$\iint_\Sigma \frac{\mathrm{d}S}{(1+x+z)^2} = \iint_D \frac{1}{(1+x+1-x-y)^2}\sqrt{1^2+(-1)^2+(-1)^2}\,\mathrm{d}x\mathrm{d}y$$

$$= \int_0^1 \mathrm{d}x \int_0^{1-x} \frac{\sqrt{3}}{(2-y)^2}\,\mathrm{d}y = \left\{ \begin{array}{c} t = 2-y\,, \mathrm{d}t = -\mathrm{d}y \\ 0 \mapsto 2, 1-x \mapsto 1+x \end{array} \right\}$$

$$= -\sqrt{3}\int_0^1 \mathrm{d}x \int_2^{1+x} \frac{1}{t^2}\,\mathrm{d}t = \sqrt{3}\int_0^1 \frac{1}{t}\Big|_2^{1+x}\,\mathrm{d}x$$

$$= \sqrt{3}\int_0^1 \Big(\frac{1}{1+x} - \frac{1}{2}\Big)\mathrm{d}x = \sqrt{3}\Big(\ln|1+x| - \frac{x}{2}\Big)\Big|_0^1$$



Slika 6.2: Projekcija D na xy-ravninu

$$= \sqrt{3}\Big(\ln 2 - \frac{1}{2} - \ln 1 + 0\Big) = \sqrt{3}\Big(\ln 2 - \frac{1}{2}\Big).$$

$\square$

Zadatak 6.2. Izračunajte $\iint_\Sigma \Big(z + 2x + \frac{4}{3}y\Big)\mathrm{d}S$ ako je Σ dio ravnine $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ u 1. oktantu. Skicirajte plohu.

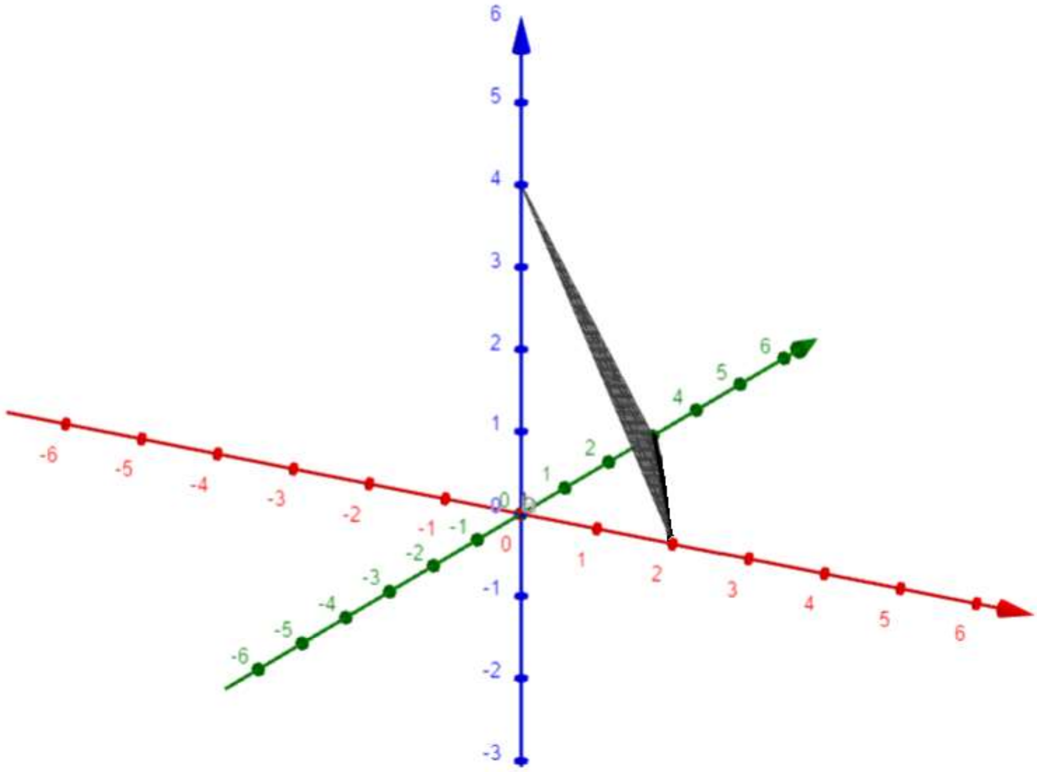
Rješenje: Napišimo eksplicitnu jednadžbu plohe:

$$z = f(x, y) = 4\Big(1 - \frac{x}{2} - \frac{y}{3}\Big) \Rightarrow \frac{\partial f}{\partial x} = -2\,, \frac{\partial f}{\partial y} = -\frac{4}{3}.$$

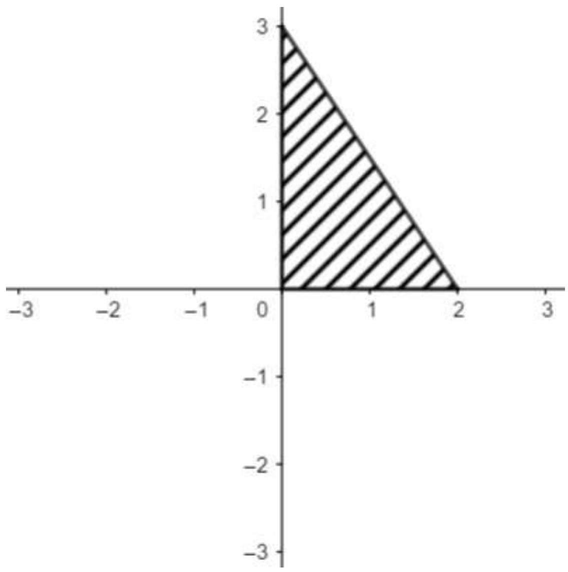
Projekcija D plohe na xy-ravninu je pravokutan trokut s katetama koje leže na koordinatnim osima, a hipotenuza na pravcu $y = 3 - \frac{3}{2}x$ (pogledajte sliku 6.4). Računamo integral:

$$\iint_\Sigma \Big(z + 2x + \frac{4}{3}y\Big)\mathrm{d}S$$

$$= \iint_D \Big(4 - 2x - \frac{4}{3}y + 2x + \frac{4}{3}y\Big)\sqrt{1 + (-2)^2 + \Big(-\frac{4}{3}\Big)^2}\,\mathrm{d}x\mathrm{d}y$$



Slika 6.3: Ploha Σ



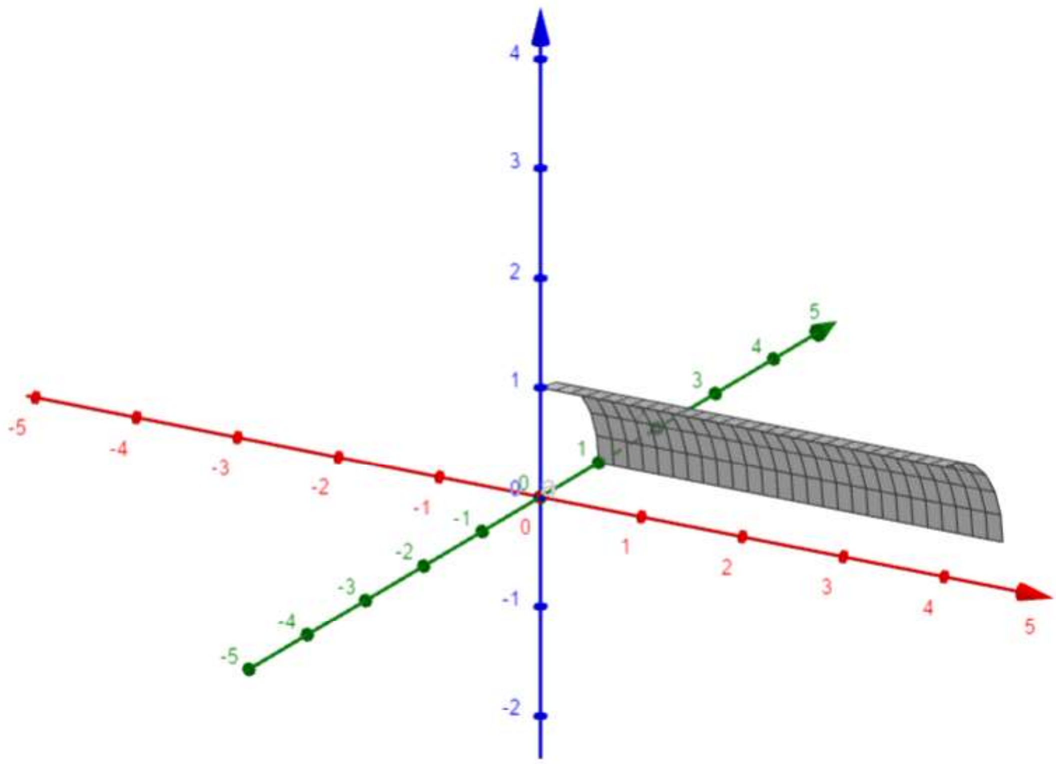
Slika 6.4: Projekcija D na xy-ravninu

$$
\begin{aligned}
&= \int_0^2 \mathrm{d}x \int_0^{3-\frac{3}{2}x} 4\sqrt{5+\frac{16}{9}}\mathrm{d}y = \frac{4\sqrt{61}}{3}\int_0^2 y\Big|_0^{3-\frac{3}{2}x}\mathrm{d}x \\
&= \frac{4\sqrt{61}}{3}\int_0^2 \Big(3-\frac{3}{2}x\Big)\mathrm{d}x = \frac{4\sqrt{61}}{3}\Big(3x-\frac{3}{2}x^2\Big)\Big|_0^2 \\
&= \frac{4\sqrt{61}}{3}(6-3) = 4\sqrt{61}.
\end{aligned}
$$

□

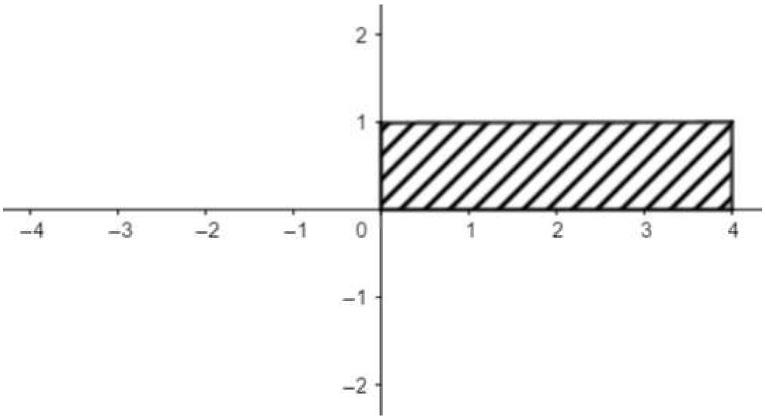
Zadatak 6.3. Izračunajte $\iint_\Sigma y\,\mathrm{d}S$, gdje je Σ dio plohe $z=\sqrt{1-y^2}$ u 1. oktantu omeđen ravninom $x=4$. Skicirajte plohu.

Rješenje: Ploha Σ je dio kružnog cilindra $y^2+z^2=1$. Budući da je:



Slika 6.5: Ploha Σ

$$
z=f(x,y)=\sqrt{1-y^2} \Rightarrow \frac{\partial f}{\partial x}=0\,,\ \frac{\partial f}{\partial y}=-\frac{y}{\sqrt{1-y^2}}.
$$



Slika 6.6: Projekcija D na xy-ravninu

Projekcija D plohe na xy-ravninu je pravokutnik $[0,4] \times [0,1]$ (pogledajte sliku 6.6). Odredimo vrijednost integrala:

$$\iint_\Sigma y \, \mathrm{d}S = \iint_D y\sqrt{1+0+\frac{y^2}{1-y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$\int_0^4 \mathrm{d}x \int_0^1 y\sqrt{\frac{1}{1-y^2}}\mathrm{d}y = \left\{ \begin{array}{l} t = 1-y^2\,, \mathrm{d}t = -2y\mathrm{d}y \\ 0 \mapsto 1,\ 1 \mapsto 0 \end{array} \right\}$$
$$= \int_0^4 \mathrm{d}x \int_1^0 -\frac{dt}{2\sqrt{t}} = \int_0^4 \sqrt{t}\Big|_0^1 \mathrm{d}x = \int_0^4 \mathrm{d}x = 4.$$

□

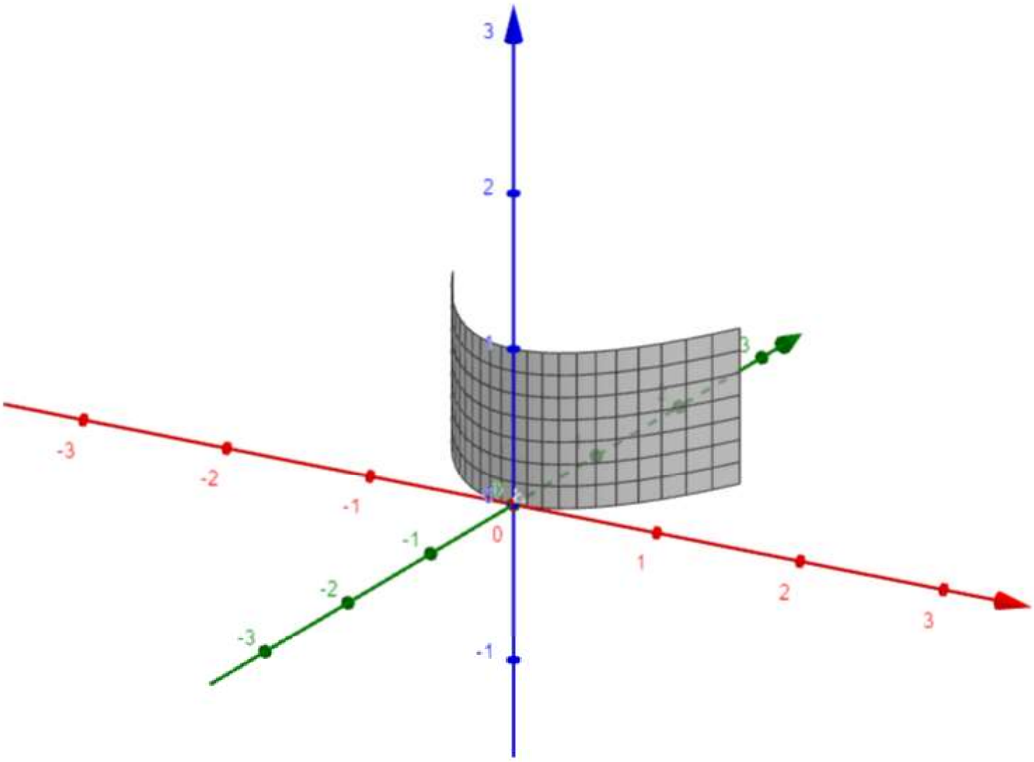
Zadatak 6.4. Izračunajte $\iint_\Sigma xy \, \mathrm{d}S$ ako je Σ dio plohe $y = x^2$ za $y \leq 1$ između ravnina $z = 0$ i $z = 1$. Skicirajte plohu.

Rješenje: Ploha Σ je okomita na xy-ravninu i stoga ju projiciramo na xz-ravninu. Jednadžba plohe glasi:

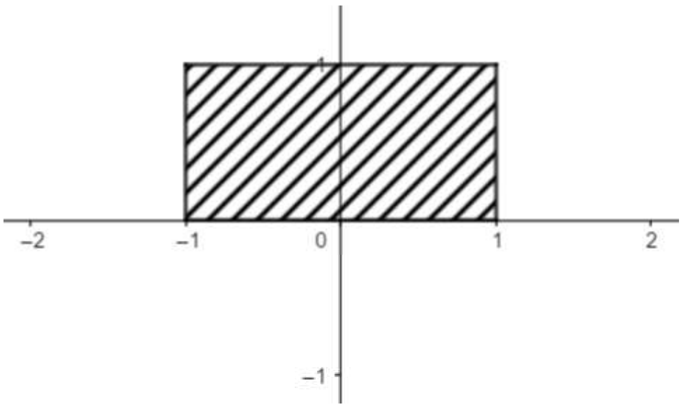
$$y = f(x,z) = x^2 \Rightarrow \frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 0.$$

Projekcija plohe Σ na xz-ravninu je pravokutnik $[-1,1] \times [0,1]$ (pogledajte sliku 6.8). Izračunajmo integral:

$$\iint_\Sigma xy \, \mathrm{d}S = \iint_D x \cdot x^2\sqrt{1+(2x)^2+0}\,\mathrm{d}x\,\mathrm{d}y$$



Slika 6.7: Ploha Σ



Slika 6.8: Projekcija D na xz-ravninu

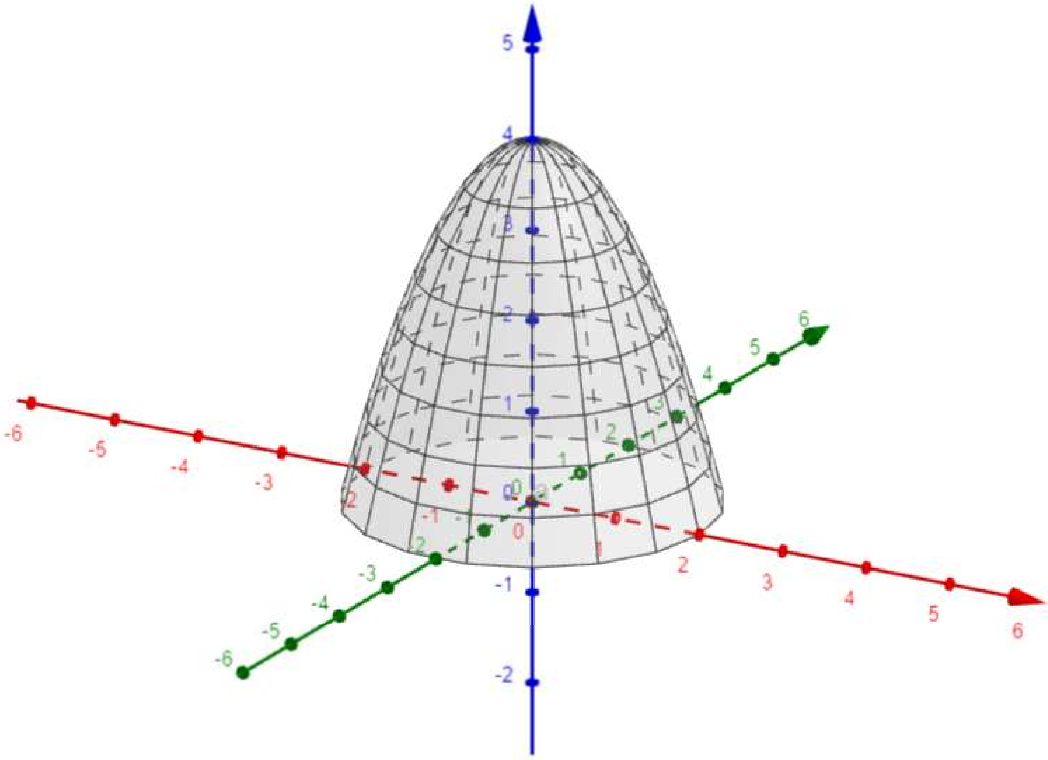
$$= \int_{-1}^{1} \mathrm{d}x \int_{0}^{1} x^3\sqrt{1+4x^2}\,\mathrm{d}z = \int_{-1}^{1} x^3\sqrt{1+4x^2}\,\mathrm{d}x$$

$$= (\text{integral neparne funkcije po simetričnoj domeni}) = 0.$$

$\square$

Zadatak 6.5. Izračunajte $\iint_\Sigma \frac{dS}{\sqrt{x^2+y^2+z}}$ ako je Σ dio rotacionog paraboloida $z = 4 - x^2 - y^2$ iznad ravnine $z = 0$. Skicirajte plohu.

Rješenje: Vidimo da je:



Slika 6.9: Ploha Σ

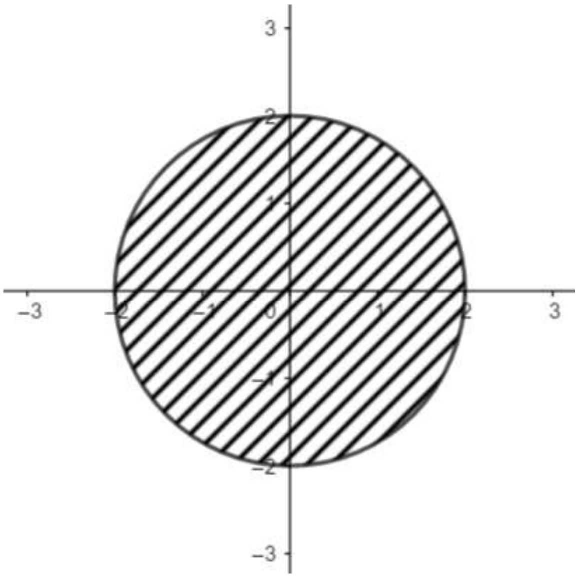
$$z = f(x,y) = 4 - x^2 - y^2 \Rightarrow \frac{\partial f}{\partial x} = -2x,\ \frac{\partial f}{\partial y} = -2y.$$

Projekcija D plohe Σ na xy-ravninu je krug sa središtem u ishodištu radijusa $r = 2$ (pogledajte sliku 6.10). Računamo vrijednost integrala:

$$\iint_\Sigma \frac{dS}{\sqrt{x^2+y^2+z}} = \iint_D \frac{1}{\sqrt{x^2+y^2+4-x^2-y^2}}\sqrt{1+4x^2+4y^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\iint_D \sqrt{1+4r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\varphi = \frac{1}{2}\int_0^{2\pi}\mathrm{d}\varphi\int_0^2 r\sqrt{1+4r^2}\,\mathrm{d}r$$

$$= \left\{\begin{array}{l} t = 1+4r^2\,,\mathrm{d}t = 8r\mathrm{d}r \\ 0\mapsto 1,\ 2\mapsto 17 \end{array}\right\} = \frac{1}{2}\int_0^{2\pi}\mathrm{d}\varphi\int_1^{17}\frac{1}{8}\sqrt{t}\,\mathrm{d}t$$



Slika 6.10: Projekcija D na xy-ravninu

$$= \frac{1}{16} \cdot 2\pi \cdot \left. \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^{17} = \frac{\pi}{12}(17\sqrt{17} - 1).$$

$\square$

Zadatak 6.6. Izračunajte $\iint_\Sigma z^2 \mathrm{d}S$ po dijelu plohe cilindra $x^2 + z^2 = 1$ koji iz njega isijeca cilindar $x^2 + y^2 = 1$ uz pretpostavku da je $z \geq 0$. Skicirajte plohu.

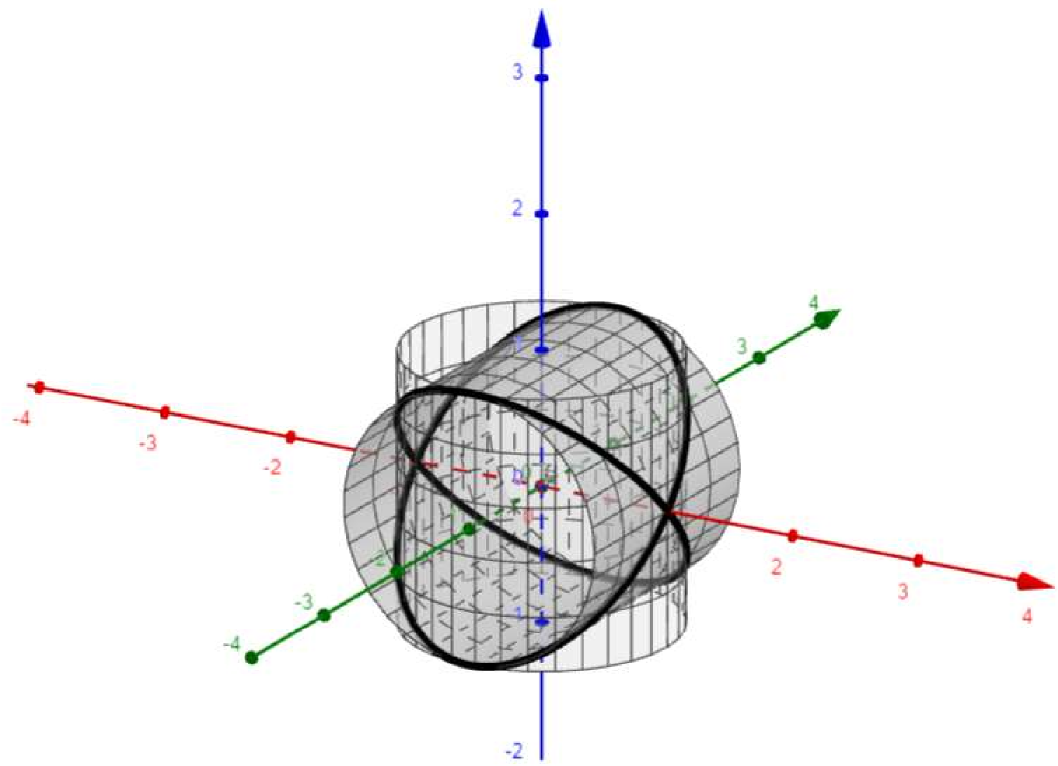
Rješenje: Eksplicitna jednadžba plohe Σ je oblika:

$$z = f(x, y) = \sqrt{1 - x^2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{x}{\sqrt{1 - x^2}}, \ \frac{\partial f}{\partial y} = 0.$$

Projekcija plohe Σ na xy-ravninu je krug sa središtem u ishodištu radijusa $r = 1$ (pogledajte sliku 6.12). Zadani integral računamo u Kartezijevom koordinatnom sustavu:

$$\iint_\Sigma z^2 \mathrm{d}S = \iint_D (1 - x^2)\sqrt{1 + \frac{x^2}{1 - x^2} + 0}\, \mathrm{d}x\, \mathrm{d}y$$

$$= \int_{-1}^{1} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2)\sqrt{\frac{1}{1 - x^2}}\, \mathrm{d}y = \int_{-1}^{1} \sqrt{1 - x^2} \cdot y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$



Slika 6.11: Ploha Σ

$$= \int_{-1}^{1} 2(1-x^2)\mathrm{d}x = 2\Big(x - \frac{x^3}{3}\Big)\Big|_{-1}^{1}$$
$$= 2\Big(1 - \frac{1}{3} + 1 - \frac{1}{3}\Big) = 2 \cdot \frac{4}{3} = \frac{8}{3}.$$

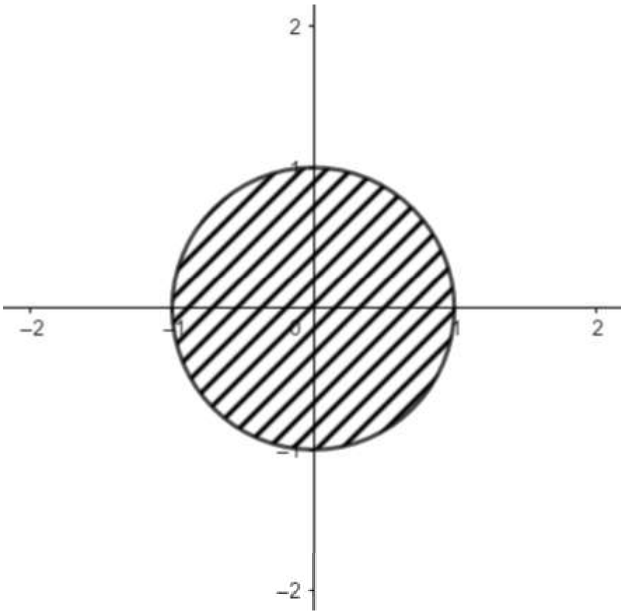
□

Zadatak 6.7. (DZ) Izračunajte površinu dijela plohe $z = \sqrt{x^2+y^2}$ koji iz nje isijeca cilindar $x^2 + (y-1)^2 = 1$. Skicirajte plohu.

Rješenje: Kružni cilindar $x^2+(y-1)^2 = 1$ isijeca dio iz stošca $z = \sqrt{x^2+y^2}$, tako da vrijedi:

$$z = f(x,y) = \sqrt{x^2+y^2} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \ \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}.$$

Projekcija D plohe Σ na xy-ravninu je krug sa središtem u točki $(0,1)$ radijusa $r = 1$, čija jednadžba u polarnim koordinatama ima oblik $r = 2\sin\varphi$ (pogledajte sliku 6.14). Površina plohe Σ se računa na sljedeći način:



Slika 6.12: Projekcija D na xy-ravninu

$$P(\Sigma) = \iint_{\Sigma} \mathrm{d}S = \iint_D \sqrt{1 + \Big(\frac{\partial f}{\partial x}\Big)^2 + \Big(\frac{\partial f}{\partial y}\Big)^2}\, \mathrm{d}x\, \mathrm{d}y$$

$$= \iint_D \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, \mathrm{d}x\, \mathrm{d}y = \sqrt{2}\int_0^{\pi} \mathrm{d}\varphi \int_0^{2\sin\varphi} r\, \mathrm{d}r$$

$$= \sqrt{2}\int_0^{\pi} \frac{r^2}{2}\Big|_0^{2\sin\varphi} \mathrm{d}\varphi = 2\sqrt{2}\int_0^{\pi} \sin^2\varphi\, \mathrm{d}\varphi$$

$$= 2\sqrt{2}\int_0^{\pi} \frac{1-\cos 2\varphi}{2}\, \mathrm{d}\varphi = \sqrt{2}\Big(\varphi - \frac{\sin 2\varphi}{2}\Big)\Big|_0^{\pi} = \pi\sqrt{2}.$$

□

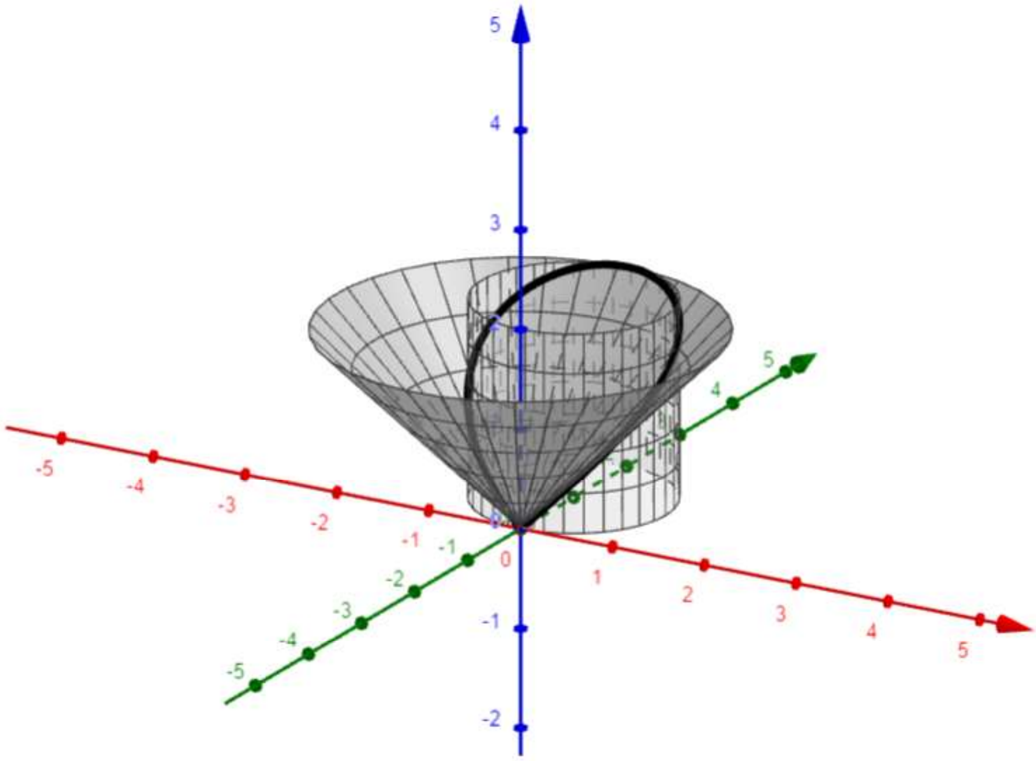
Zadatak 6.8. (DZ) Izračunajte $\iint_{\Sigma}(xy+yz+xz)\mathrm{d}S$, gdje je Σ dio stožaste plohe $z = \sqrt{x^2+y^2}$ isječen plohom $x^2+y^2 = 2ax$, $(a \geq 0)$. Skicirajte plohu.

Rješenje: Budući da je $z = f(x,y) = \sqrt{x^2+y^2}$ vrijedi:

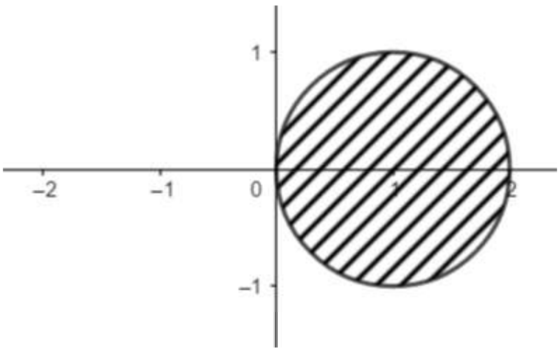
$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}.$$

Također znamo da je:

$$F(x,y,z) = xy + yz + xz = xy + y\sqrt{x^2+y^2} + x\sqrt{x^2+y^2}.$$



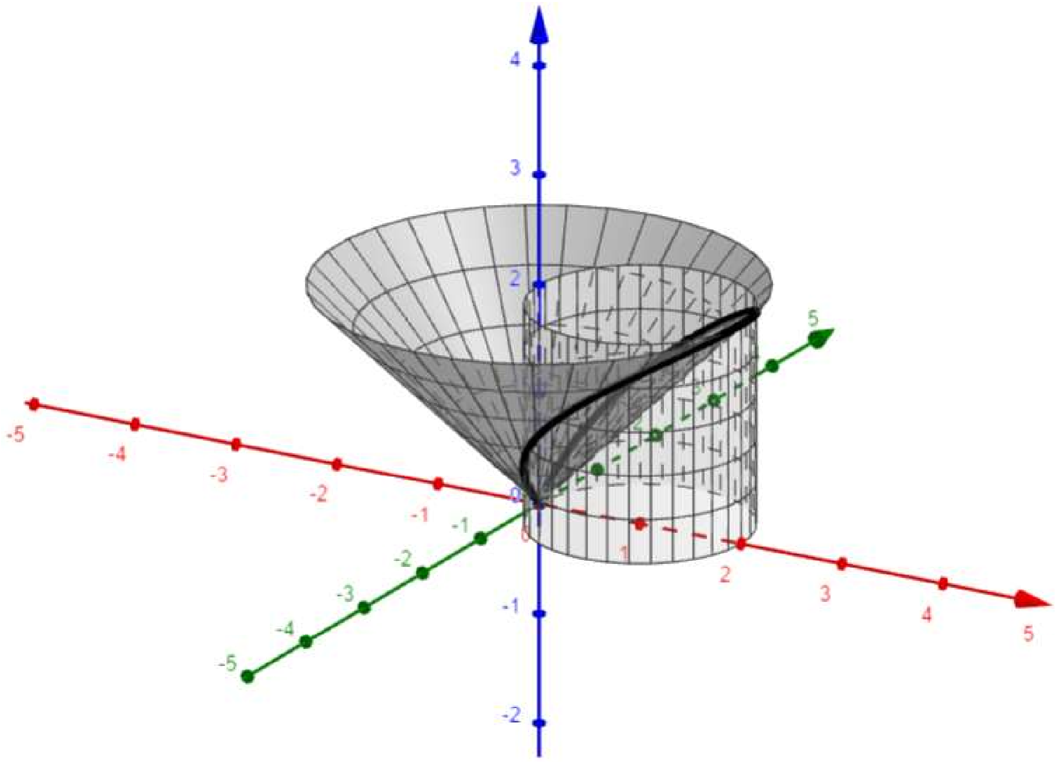
Slika 6.13: Ploha Σ



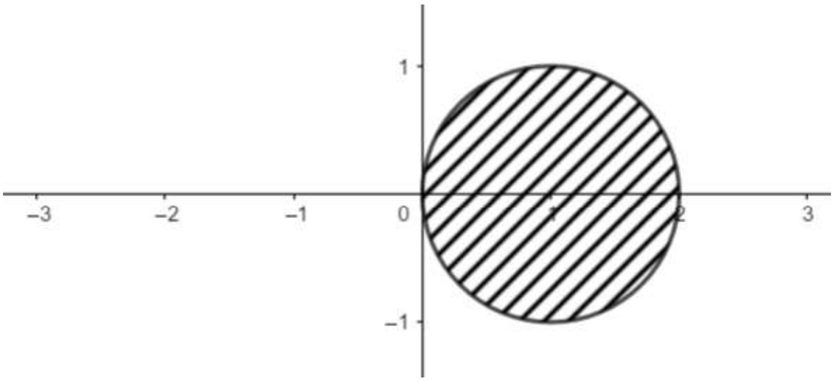
Slika 6.14: Projekcija D na xy-ravninu

Projekcija plohe Σ na xy-ravninu je krug sa središtem u točki $(a, 0)$ radijusa $r = a$, čija jednadžba u polarnim koordinatama ima oblik $r = 2a\cos\varphi$ (pogledajte sliku 6.16). Računamo:

$$\iint_{\Sigma}(xy + yz + xz)\mathrm{d}S$$



Slika 6.15: Ploha Σ



Slika 6.16: Projekcija D na xy-ravninu

$$= \iint_D (xy + y\sqrt{x^2+y^2} + x\sqrt{x^2+y^2} \cdot \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{2a\cos\varphi} (r^2 \sin\varphi\cos\varphi + r^2\sin\varphi + r^2\cos\varphi) r\,\mathrm{d}r$$

$$
\begin{aligned}
&= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{2a\cos\varphi} r^3(\sin\varphi\cos\varphi + \sin\varphi + \cos\varphi)\,\mathrm{d}r \\
&= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\varphi\cos\varphi + \sin\varphi + \cos\varphi)\cdot \frac{16a^4\cos^4\varphi}{4}\,\mathrm{d}\varphi \\
&= 4\sqrt{2}a^4\Big[\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^5\varphi + \cos^4\varphi)\mathrm{d}\varphi}_{\substack{\text{integral neparne funkcije na}\\ \text{simetričnoj domeni je jednak nuli}}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\varphi\,\mathrm{d}\varphi\Big] \\
&= 4\sqrt{2}a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\varphi)^2\cos\varphi\,\mathrm{d}\varphi = \left\{\begin{array}{l} t = \sin\varphi\,,\mathrm{d}t = \cos\varphi\mathrm{d}\varphi \\ -\frac{\pi}{2}\mapsto -1,\ \frac{\pi}{2}\mapsto 1 \end{array}\right\} \\
&= 4\sqrt{2}a^4\int_{-1}^{1}(1-t^2)^2\mathrm{d}t = 8\sqrt{2}a^4\int_0^1(1-t^2)^2\mathrm{d}t = 8\sqrt{2}a^4\int_0^1(1-2t^2+t^4)\mathrm{d}t \\
&= 8\sqrt{2}a^4\Big(t - \frac{2t^3}{3} + \frac{t^5}{5}\Big)\Big|_0^1 = 8\sqrt{2}a^4\Big(1-\frac{2}{3}+\frac{1}{5}\Big) = \frac{64\sqrt{2}a^4}{15}.
\end{aligned}
$$

□

Zadatak 6.9. (DZ) Izračunajte $\iint_\Sigma (x^2+y^2)\mathrm{d}S$ gdje je Σ dio površine paraboloida $x^2+y^2=2z$ što je isijeca ravnina $z=1$. Skicirajte plohu.

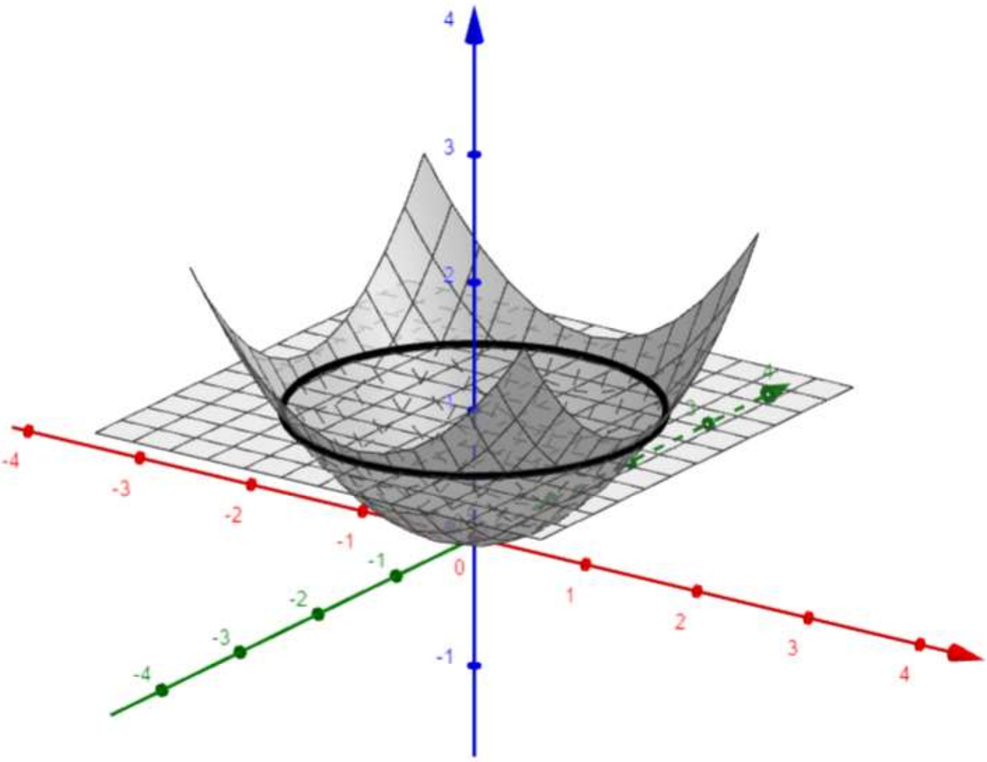
Rješenje: Ploha Σ je zadana jednadžbom:

$$z = f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} \Rightarrow \frac{\partial f}{\partial x} = x,\ \frac{\partial f}{\partial y} = y.$$

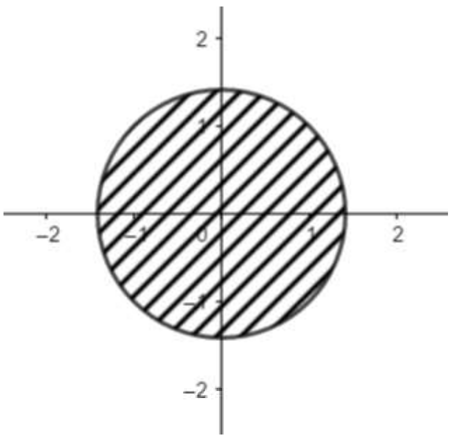
Jednadžba ruba projekcije plohe Σ na xy-ravninu glasi $x^2+y^2=2$, dakle radi se o krugu sa središtem u ishodištu radijusa $r=\sqrt{2}$ (pogledajte sliku 6.18). Odredimo vrijednost integrala:

$$
\begin{aligned}
&\iint_\Sigma (x^2+y^2)\mathrm{d}S = \iint_D (x^2+y^2)\sqrt{1+x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\sqrt{2}} r^2\sqrt{1+r^2}\,r\,\mathrm{d}r = \left\{\begin{array}{l} t = 1+r^2,\ \mathrm{d}t = 2r\,\mathrm{d}r \\ 0\mapsto 1,\ \sqrt{2}\mapsto 3 \end{array}\right\} \\
&= \int_0^{2\pi}\mathrm{d}\varphi\int_1^3 (t-1)\sqrt{t}\,\frac{\mathrm{d}t}{2} = \pi\int_1^3\big(t^{\frac{3}{2}} - t^{\frac{1}{2}}\big)\mathrm{d}t \\
&= \pi\left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^3 = \pi\Big[\frac{2}{5}(9\sqrt{3}-1) - \frac{2}{3}(3\sqrt{3}-1)\Big] = \pi\Big(\frac{8\sqrt{3}}{5} + \frac{4}{15}\Big).
\end{aligned}
$$

□



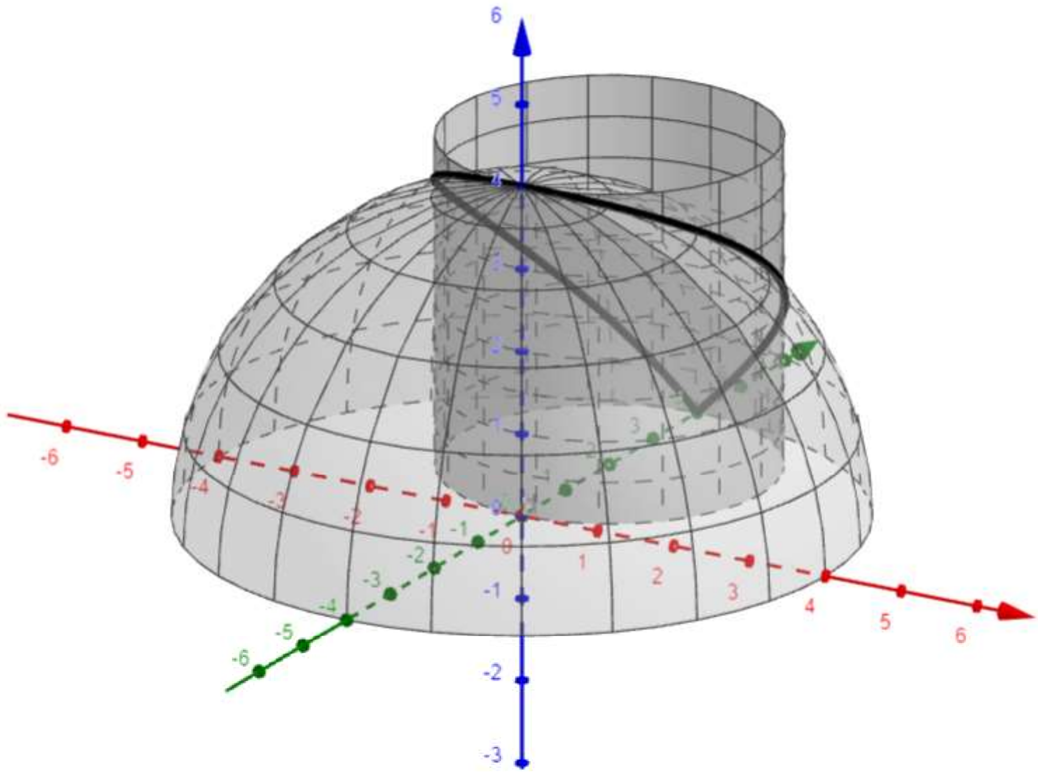
Slika 6.17: Ploha Σ



Slika 6.18: Projekcija D na xy-ravninu

Zadatak 6.10. Odredite površinu dijela sfere $x^2 + y^2 + z^2 = 16$ koju isijeca kružni cilindar $x^2 + y^2 = 4y$ ako je $z \geq 0$. Skicirajte plohu.

Rješenje: Ploha Σ je zadana eksplicitnom jednadžbom



Slika 6.19: Ploha Σ

$z = f(x, y) = \sqrt{16 - x^2 - y^2}$. Prema tome,

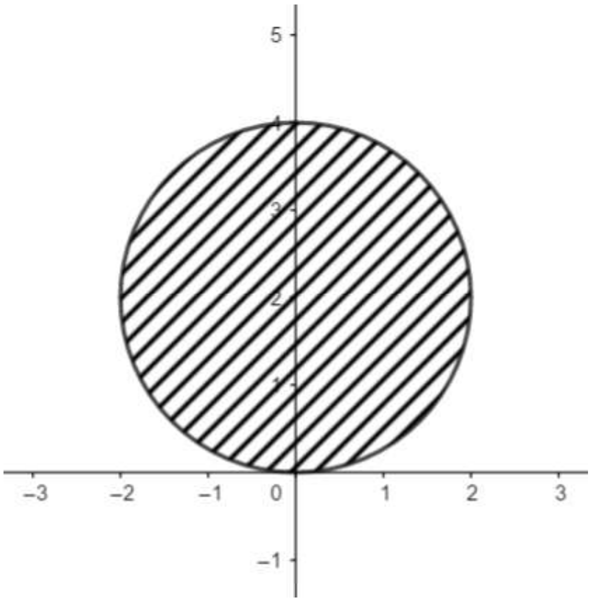
$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{16 - x^2 - y^2}} \quad \text{i} \quad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{16 - x^2 - y^2}}.$$

Projekcija D plohe Σ na xy-ravninu je krug sa središtem u točki $(0, 2)$ radijusa $r = 2$, koja u polarnom koordinatnom sustavu ima jednadžbu oblika $r = 4 \sin \varphi$ (pogledajte sliku 6.20). Računamo površinu plohe Σ:

$$P(\Sigma) = \iint_{\Sigma} \mathrm{d}S = \iint_{D} \sqrt{1 + \frac{x^2}{16 - x^2 - y^2} + \frac{y^2}{16 - x^2 - y^2}} \,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_{D} \frac{4}{\sqrt{16 - x^2 - y^2}} \,\mathrm{d}x\,\mathrm{d}y = 4 \int_0^{\pi} \mathrm{d}\varphi \int_0^{4 \sin \varphi} \frac{1}{\sqrt{16 - r^2}} \, r \,\mathrm{d}r$$

$$= -4 \int_0^{\pi} \mathrm{d}\varphi \int_0^{4 \sin \varphi} \frac{1}{2} \frac{-2r dr}{\sqrt{16 - r^2}} \,\mathrm{d}r = -4 \int_0^{\pi} \sqrt{16 - r^2}\Big|_0^{4 \sin \varphi} \mathrm{d}\varphi$$



Slika 6.20: Projekcija D na xy-ravninu

$$
\begin{aligned}
&= -4\int_0^{\pi}\Big(\sqrt{16-16\sin^2\varphi}-4\Big)\mathrm{d}\varphi = -4\int_0^{\pi}(4|\cos\varphi|-4)\mathrm{d}\varphi \\
&= -8\int_0^{\frac{\pi}{2}}(4\cos\varphi-4)\mathrm{d}\varphi = -32(\sin\varphi-\varphi)\Big|_0^{\frac{\pi}{2}} \\
&= -32\Big(1-\frac{\pi}{2}\Big) = 32\Big(\frac{\pi}{2}-1\Big).
\end{aligned}
$$

□

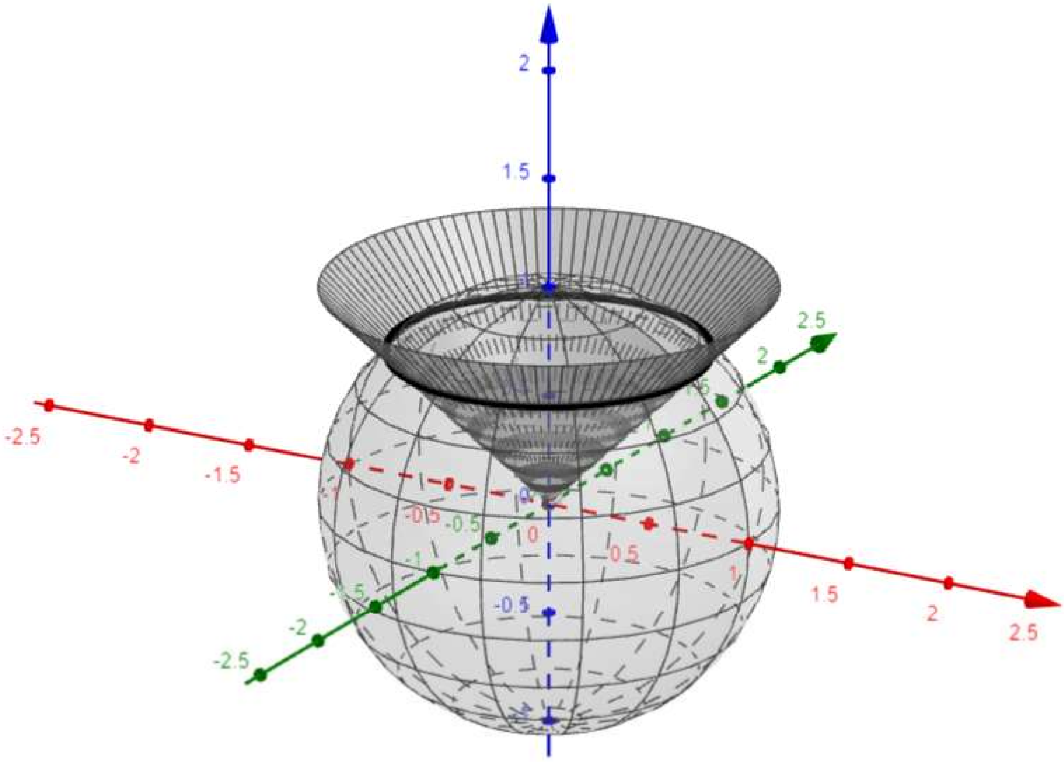
Zadatak 6.11. (DZ) Izračunajte površinu manjeg dijela sfere $x^2+y^2+z^2 = a^2$, $(a > 0)$ kojeg iz nje isijeca stožac $z = \sqrt{x^2+y^2}$. Skicirajte plohu.

Rješenje: Eksplicitna jednadžba plohe Σ ima oblik
$z = f(x, y) = \sqrt{a^2-x^2-y^2}$. Stoga je

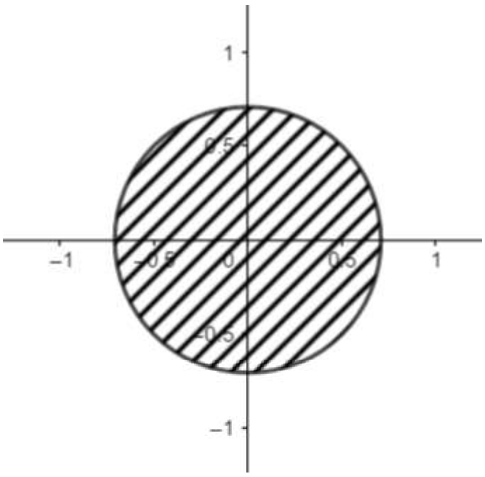
$$
\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{a^2-x^2-y^2}} \quad \text{i} \quad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{a^2-x^2-y^2}}.
$$

Projekcija D plohe Σ na xy-ravninu je krug $x^2+y^2 \le \frac{a^2}{2}$ sa središtem u ishodištu radijusa $r = \frac{a}{\sqrt{2}}$ (pogledajte sliku 6.22). Odredimo površinu plohe Σ:

$$
P(\Sigma) = \iint_{\Sigma}\mathrm{d}S = \iint_D\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y
$$



Slika 6.21: Ploha Σ



Slika 6.22: Projekcija D na xy-ravninu

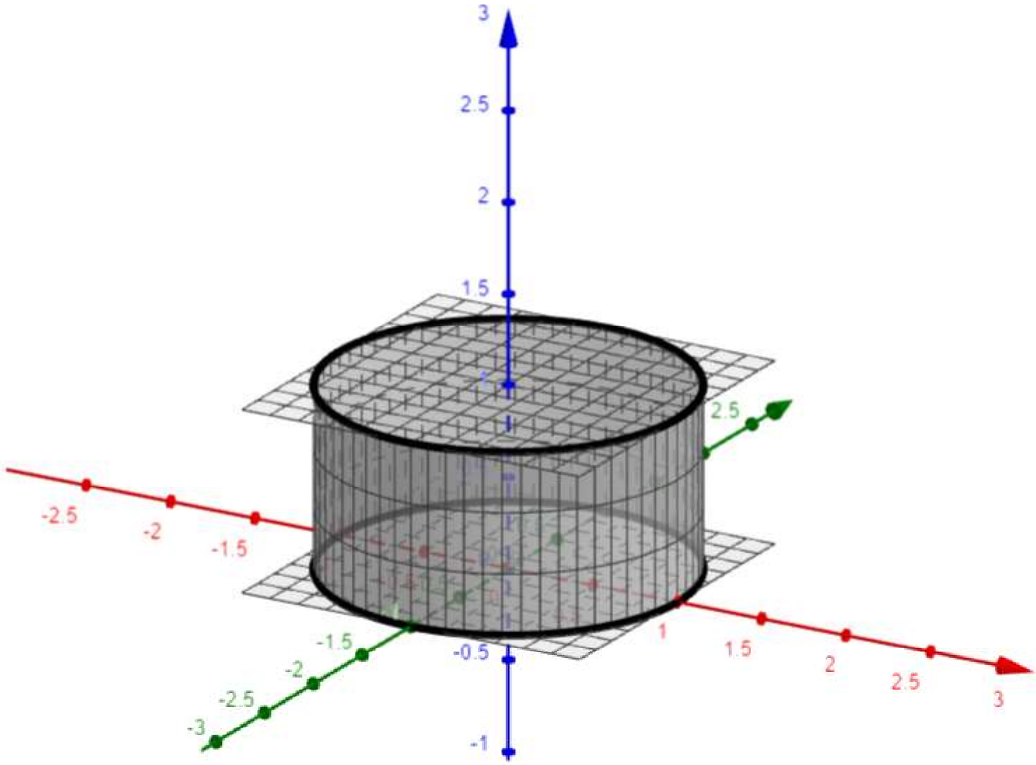
$$= a \iint_D \frac{1}{\sqrt{a^2 - x^2 - y^2}} \,\mathrm{d}x\,\mathrm{d}y = a \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{a}{\sqrt{2}}} \frac{1}{\sqrt{a^2 - r^2}}\, r\,\mathrm{d}r$$

$$= -a\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{a}{\sqrt{2}}} \frac{-2rdr}{2\sqrt{a^2-r^2}} = -a\int_0^{2\pi} \sqrt{a^2-r^2}\Big|_0^{\frac{a}{\sqrt{2}}} \mathrm{d}\varphi$$
$$= -2a\pi\Big(\sqrt{a^2-\frac{a^2}{2}}-\sqrt{a^2}\Big) = -2a\pi\Big(\frac{a}{\sqrt{2}}-a\Big)$$
$$= 2a^2\pi\Big(1-\frac{1}{\sqrt{2}}\Big) = a^2\pi(2-\sqrt{2}).$$

□

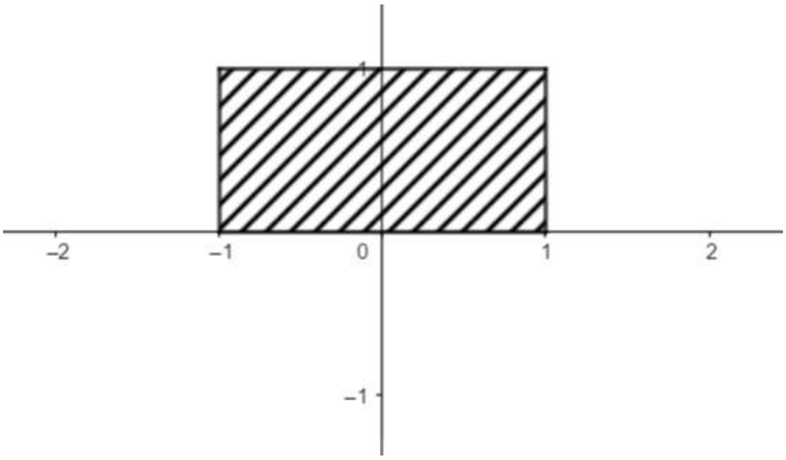
Zadatak 6.12. Izračunajte $\iint_\Sigma (y+z+\sqrt{a^2-x^2})\mathrm{d}S$ ako je Σ dio plašta valjka $x^2+y^2=a^2$, $(a>0)$, omeđen sa $z=0$ i $z=k$, $(k>0)$. Skicirajte plohu.

Rješenje: Budući da je ploha Σ okomita na xy-ravninu, projiciramo ju na xz-



Slika 6.23: Ploha Σ

ravninu (pogledajte sliku 6.24). Imamo da je $y^2=a^2-x^2$, što znači da plohu moramo podijeliti xz-ravninom na desni komad s jednadžbom $y=f_1(x,z)=\sqrt{a^2-x^2}$ i na lijevi komad koji ima jednadžbu $y=f_2(x,z)=-\sqrt{a^2-x^2}$, a



Slika 6.24: Projekcija D na xz-ravninu

zatim integriramo po svakom dijelu posebno, te zbrojimo rezultate. Parcijalne derivacije funkcija f_1 i f_2 su redom

$$\frac{\partial f_1}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2}} \quad \text{i} \quad \frac{\partial f_2}{\partial x} = \frac{x}{\sqrt{a^2 - x^2}}, \quad \text{te} \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_2}{\partial z} = 0.$$

Zajednička projekcija D oba komada plohe na xz-ravninu je pravokutnik $[-a, a] \times [0, k]$. Odredimo:

$$\iint_\Sigma (y + z + \sqrt{a^2 - x^2}) \mathrm{d}S$$

$$= \iint_D (\sqrt{a^2 - x^2} + z + \sqrt{a^2 - x^2}) \sqrt{1 + \frac{x^2}{a^2 - x^2} + 0} \, \mathrm{d}x \, \mathrm{d}z$$

$$+ \iint_D (-\sqrt{a^2 - x^2} + z + \sqrt{a^2 - x^2}) \sqrt{1 + \frac{x^2}{a^2 - x^2} + 0} \, \mathrm{d}x \, \mathrm{d}z$$

$$= \iint_D 2(\sqrt{a^2 - x^2} + z) \frac{a}{\sqrt{a^2 - x^2}} \, \mathrm{d}x \, \mathrm{d}z = 2a \int_0^k \mathrm{d}z \int_{-a}^a \left(1 + \frac{z}{\sqrt{a^2 - x^2}}\right) \mathrm{d}x \, \mathrm{d}z$$

$$= 2a \int_0^k \left(x + z \arcsin \frac{x}{a}\right)\Big|_{-a}^a \mathrm{d}z = 2a \int_0^k \left[a + z\frac{\pi}{2} - \Big(-a + z\Big(-\frac{\pi}{2}\Big)\Big)\right] \mathrm{d}z$$

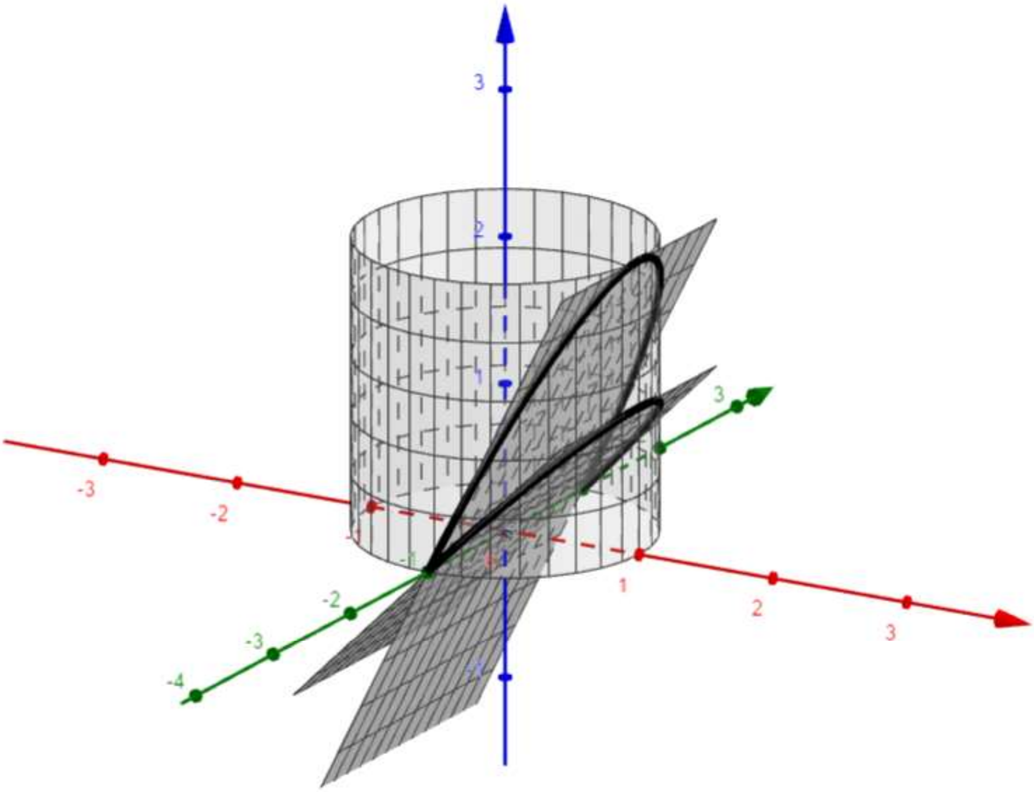
$$= 2a \int_0^k (2a + z\pi) \mathrm{d}z = 2a\Big(2az + \pi\frac{z^2}{2}\Big)\Big|_0^k$$

$$= 2a\Big(2ak + \pi\frac{k^2}{2}\Big) = ak(4a + \pi k^2).$$

□

Zadatak 6.13. (DZ) Nađite površinu dijela plohe valjka $x^2 + y^2 = R^2$, $(z \geq 0)$, koja se nalazi medju ravninama $z = \alpha x$ i $z = \beta x$ $(\alpha > \beta > 0)$.

Rješenje: Ploha Σ je okomita na xy-ravninu i stoga ju projiciramo na xz-



Slika 6.25: Ploha Σ

ravninu (pogledajte sliku 6.26). Plohu podijelimo xz-ravninom na lijevi i desni komad čije su jednadžbe redom

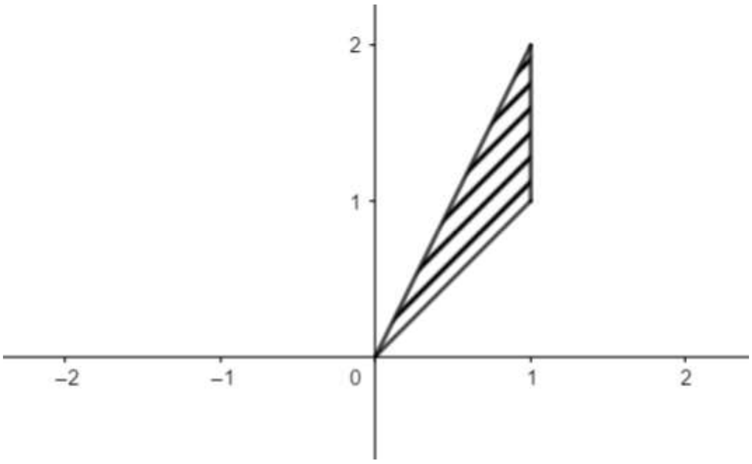
$$\Sigma_1 \ldots y = f_1(x, z) = \sqrt{R^2 - x^2} \quad \text{i} \quad \Sigma_2 \ldots y = f_2(x, z) = -\sqrt{R^2 - x^2}.$$

Parcijalne derivacije funkcija f_1 i f_2 su dane sa

$$\frac{\partial f_1}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2}} \quad \text{i} \quad \frac{\partial f_2}{\partial x} = \frac{x}{\sqrt{R^2 - x^2}}, \quad \text{te} \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_2}{\partial z} = 0.$$

Zajednička projekcija D ploha Σ_1 i Σ_2 na xz-ravninu je dio te ravnine u 1. kvadrantu između pravaca $z = \beta x$ i $z = \alpha x$. Površina plohe Σ je zbroj površina ploha Σ_1 i Σ_2:

$$P(\Sigma) = \iint_{\Sigma_1} \mathrm{d}S + \iint_{\Sigma_2} \mathrm{d}S$$



Slika 6.26: Projekcija D na xz-ravninu

$$
\begin{aligned}
&= \iint_D \sqrt{1 + \frac{x^2}{R^2 - x^2} + 0}\,\mathrm{d}x\,\mathrm{d}z + \iint_D \sqrt{1 + \frac{x^2}{R^2 - x^2} + 0}\,\mathrm{d}x\,\mathrm{d}z \\
&= 2R \iint_D \frac{1}{\sqrt{R^2 - x^2}}\,\mathrm{d}x\,\mathrm{d}z = 2R \int_0^R \mathrm{d}x \int_{\beta x}^{\alpha x} \frac{1}{\sqrt{R^2 - x^2}}\,\mathrm{d}z \\
&= 2R \int_0^R (\alpha x - \beta x) \frac{1}{\sqrt{R^2 - x^2}}\,\mathrm{d}x = 2R(\alpha - \beta) \int_0^R \frac{x}{\sqrt{R^2 - x^2}}\,\mathrm{d}x \\
&= -2R(\alpha - \beta)\sqrt{R^2 - x^2}\Big|_0^R = 2R^2(\alpha - \beta).
\end{aligned}
$$

$\square$