6.2 Plošni integrali druge vrste

Neka je ploha Σ zadana eksplicitno jednadžbom $z = f(x, y)$ za $(x, y) \in D$, gdje je D podskup xy-ravnine. Plohu Σ možemo orijentirati na dva načina:

a) pomoću normale $\vec{n} = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}$ koja zatvara šiljasti kut s vektorom $\vec{k}$,

b) pomoću normale $\vec{n} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} - \vec{k}$ koja zatvara tupi kut s vektorom $\vec{k}$.

Plohu orijentiranu na jedan od gornja dva načina označavamo sa $\vec{\Sigma}$.

Neka je $\vec{a} : \vec{\Sigma} \subseteq \mathbb{R}^3 \to X_0(E)$ vektorsko polje:

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}.$$

Plošni integral 2. vrste vektorskog polja $\vec{a}$ po plohi $\vec{\Sigma}$ jednak je:

$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}S = \iint_{\Sigma} \vec{a} \cdot \vec{n}_0\,\mathrm{d}S = \iint_{D} (\vec{a} \cdot \vec{n})(x, y, f(x, y))\mathrm{d}x\,\mathrm{d}y,$$

gdje je n_0 jedinični vektor normale na plohu $\vec{\Sigma}$, $\left(n_0 = \frac{\vec{n}}{|\vec{n}|}\right)$.

Zadatak 6.14. Izračunajte $\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$ ako je $\vec{a} = x\vec{i} + y^2\vec{j} + z^3\vec{k}$, a $\vec{\Sigma} = \{(x, y, z) \in \mathbb{R}^3 : z = 1, x^2 + y^2 \leq 1\}$ orijentirana normalom koja zatvara šiljasti kut s vektorom $\vec{k}$. Skicirajte plohu.

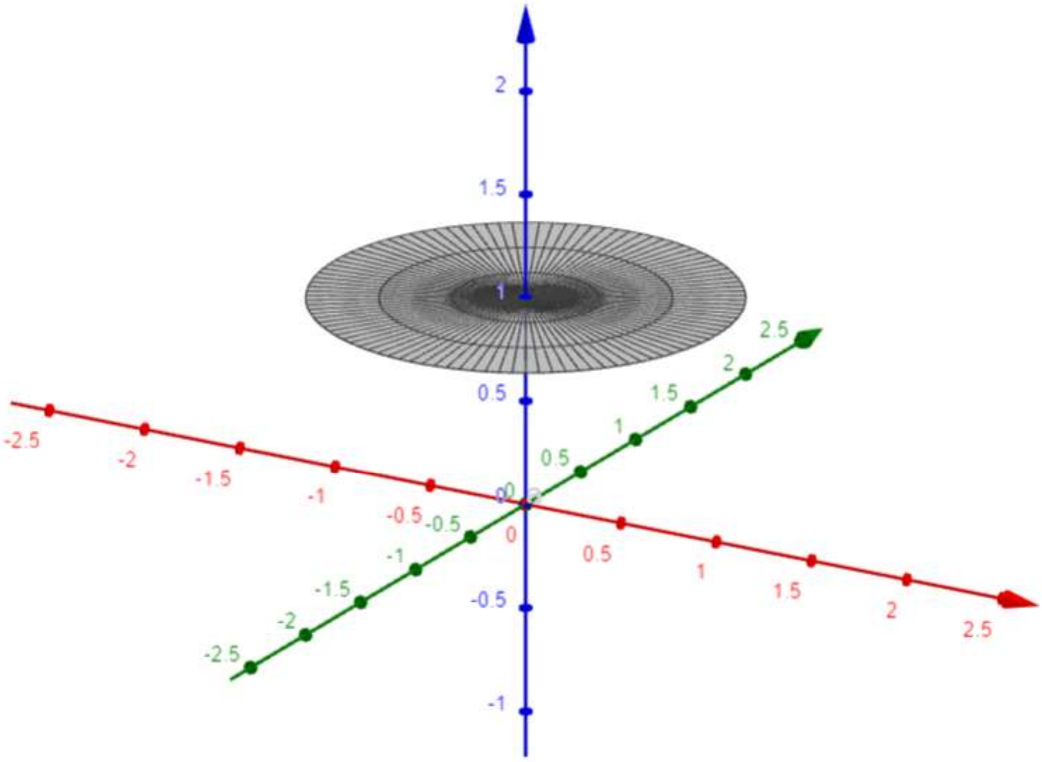
Rješenje: Ploha $\vec{\Sigma}$ je zadana jednadžbom $z = f(x, y) = 1$ (pogledajte sliku 6.27). Projekcija plohe $\vec{\Sigma}$ na xy-ravninu je krug $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, a normala koja zatvara šiljasti kut s vektorom $\vec{k}$ je vektor $n = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k} = \vec{k}$. Odredimo skalarni produkt vektorskog polja $\vec{a}$ i vektora normale $\vec{n}$:

$$\vec{a} \cdot \vec{n} = (x\vec{i} + y^2\vec{j} + z^3\vec{k}) \cdot \vec{k} = z^3 = 1^3 = 1.$$

Sada znamo da je

$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} = \iint_{D} 1\,\mathrm{d}x\,\mathrm{d}y = P(D) = \pi.$$

$\square$



Slika 6.27: Ploha $\vec{\Sigma}$

Zadatak 6.15. Izračunajte $\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$, ako je $\vec{a} = y\vec{i} - x\vec{j} + z\vec{k}$, a $\vec{\Sigma}$ dio ravnine $x + y + z = 2$ u 1. oktantu orijentiran normalom koja zatvara šiljasti kut s vektorom $\vec{k}$. Skicirajte plohu.

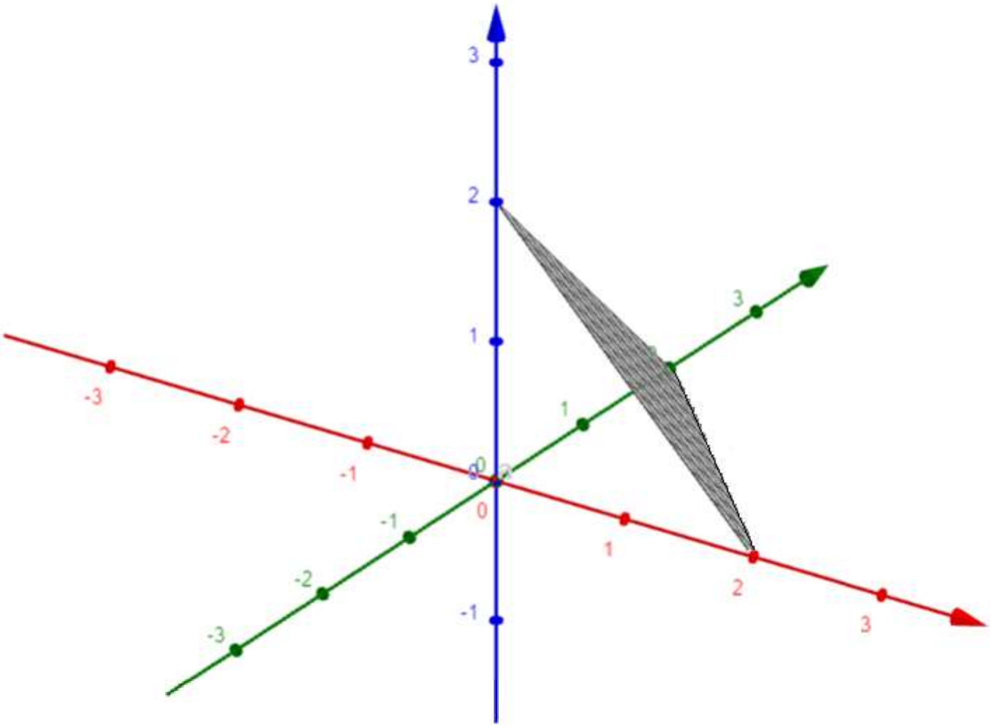
Rješenje: Eksplicitna jednadžba plohe $\vec{\Sigma}$ ima oblik $z = f(x, y) = 2 - x - y$ (pogledajte sliku 6.28) i stoga je $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = -1$, tako da je normala na plohu koja zatvara šiljasti kut s vektorom $\vec{k}$ zadana sa $\vec{n} = \vec{i} + \vec{j} + \vec{k}$. Projekcija D plohe na xy-ravninu je pravokutan trokut s katetama na koordinatnim osima i hipotenuzom na pravcu $y = 2 - x$ (pogledajte sliku 6.29). Odredimo skalarni produkt vektorskog polja $\vec{a}$ i vektora normale $\vec{n}$:

$$\vec{a} \cdot \vec{n} = y - x + z = y - x + 2 - x - y = 2 - 2x.$$

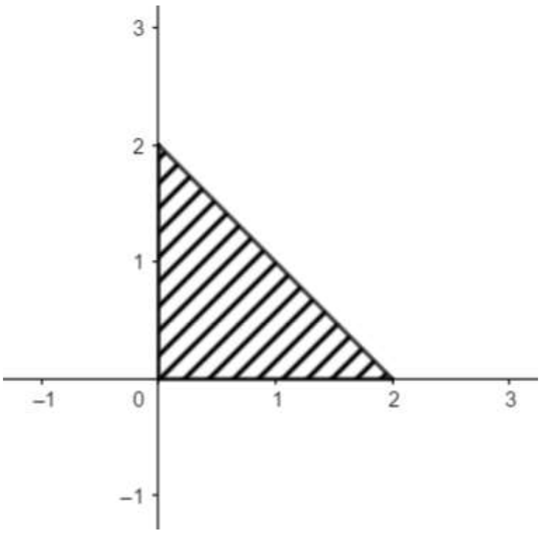
Sada računamo:

$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} = \iint_D (2 - 2x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \mathrm{d}x \int_0^{2-x} (2 - 2x)\mathrm{d}y$$

$$= \int_0^2 (2 - 2x)y\Big|_0^{2-x} \mathrm{d}x = 2\int_0^2 (1 - x)(2 - x)\mathrm{d}x$$



Slika 6.28: Ploha $\vec{\Sigma}$



Slika 6.29: Projekcija D

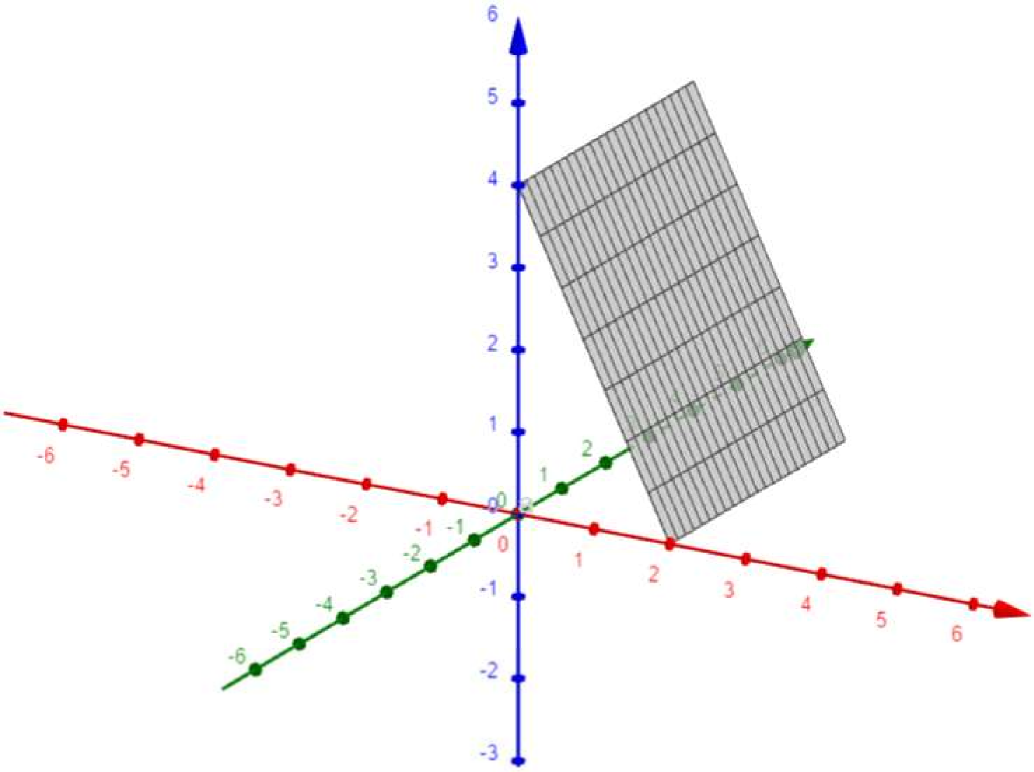
$$= 2\int_0^2 (2 - 3x + x^2)\mathrm{d}x = 2\Big(2x - \frac{3x^2}{2} + \frac{x^3}{3}\Big)\Big|_0^2$$

$$= 2\Big(4 - \frac{3 \cdot 4}{2} + \frac{8}{3}\Big) = 2\Big(-2 + \frac{8}{3}\Big) = \frac{4}{3}.$$

$\square$

Zadatak 6.16. Izračunajte $\iint_{\vec{\Sigma}} \vec{a}\, \mathrm{d}\vec{S}$, ako je $\vec{a} = y\vec{i} + x\vec{j} + z\vec{k}$, a $\vec{\Sigma}$ dio ravnine $2x + z = 4$, $0 < y < 4$, u 1. oktantu orijentiran normalom koja zatvara šiljasti kut s osi z. Skicirajte plohu.

Rješenje: Eksplicitna jednadžba plohe $\vec{\Sigma}$ ima oblik $z = f(x, y) = 4 - 2x$



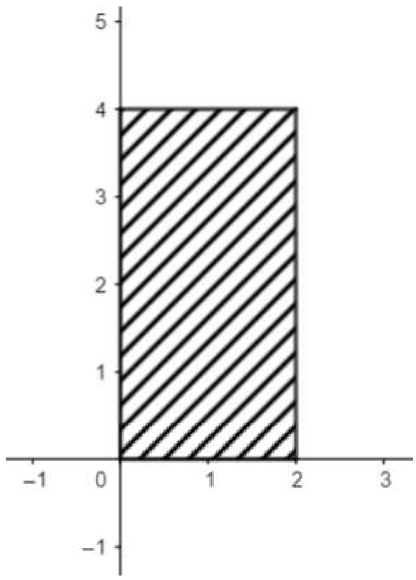
Slika 6.30: Ploha $\vec{\Sigma}$

(pogledajte sliku 6.30) i stoga je $\frac{\partial f}{\partial x} = -2$, $\frac{\partial f}{\partial y} = 0$, pa je normala na plohu koja zatvara šiljasti kut s osi z jednaka $\vec{n} = 2\vec{i} + \vec{k}$. Projekcija D plohe na xy-ravninu je pravokutnik $[0, 2] \times [0, 4]$ (pogledajte sliku 6.31). Odredimo skalarni produkt vektorskog polja $\vec{a}$ i vektora normale $\vec{n}$:

$$\vec{a} \cdot \vec{n} = 2y + z = 2y + 4 - 2x.$$

Sada računamo:

$$\iint_{\vec{\Sigma}} \vec{a}\, \mathrm{d}\vec{S} = \iint_D (2y + 4 - 2x)\, \mathrm{d}x\, \mathrm{d}y = \int_0^2 \mathrm{d}x \int_0^4 (2y + 4 - 2x) \mathrm{d}y$$



Slika 6.31: Projekcija D

$$= \int_0^2 (y^2 + 4y - 2xy)\Big|_0^4 \mathrm{d}x = \int_0^2 (16 + 16 - 8x)\mathrm{d}x$$

$$= (32x - 4x^2)\Big|_0^2 = 64 - 16 = 48.$$

$\square$

Zadatak 6.17. Izračunajte $\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$, ako je $\vec{a} = (z - x^2)\vec{j} + \sqrt{4 - z}\,\vec{k}$, a $\vec{\Sigma} = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2, 0 \leq z \leq 4\}$ orijentirana normalom koja zatvara tupi kut s vektorom $\vec{k}$. Skicirajte plohu.

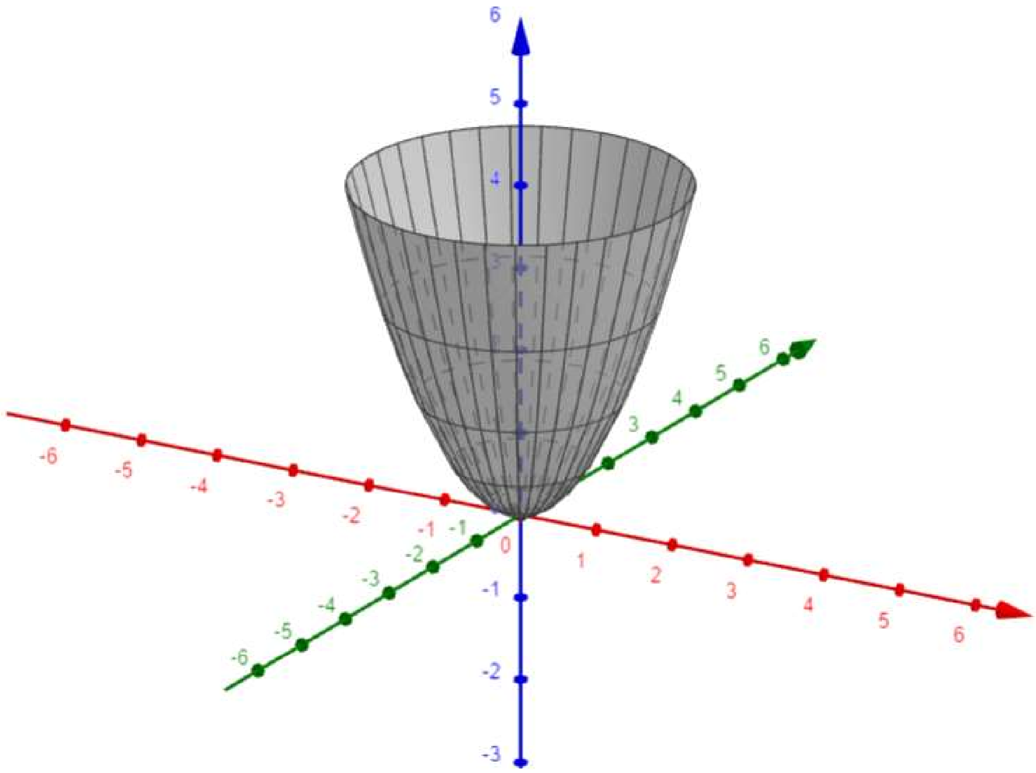
Rješenje: Znamo da je jednadžba plohe $\vec{\Sigma}$ oblika $z = f(x, y) = x^2 + y^2$ (pogledajte sliku 6.32), tako da vrijedi

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y,$$

što znači da je normala na plohu koja zatvara tupi kut s vektorom $\vec{k}$ jednaka $\vec{n} = 2x\vec{i} + 2y\vec{j} - \vec{k}$. Projekcija D plohe $\vec{\Sigma}$ na xy-ravninu je krug radijusa $r = 2$ sa središtem u ishodištu. Skalarni produkt vektorskog polja $\vec{a}$ i vektora normale $\vec{n}$ je dan sa:

$$\vec{a} \cdot \vec{n} = (z - x^2)2y - \sqrt{4 - z} = 2yz - 2x^2y - \sqrt{4 - z}$$

$$= 2y(x^2 + y^2) - 2x^2y - \sqrt{4 - x^2 - y^2} = 2y^3 - \sqrt{4 - x^2 - y^2}.$$



Slika 6.32: Ploha $\vec{\Sigma}$

Odredimo vrijednost integrala:

$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} = \iint_D (2y^3 - \sqrt{4-x^2-y^2})\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^2 (2r^3\sin^3\varphi - \sqrt{4-r^2})r\,\mathrm{d}r$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^2 2r^4\sin^3\varphi\,\mathrm{d}r - \int_0^{2\pi} \mathrm{d}\varphi \int_0^2 r\sqrt{4-r^2}\,\mathrm{d}r$$

$$= \int_0^{2\pi} \frac{2r^5}{5}\bigg|_0^2 \sin^3\varphi\,\mathrm{d}\varphi + 2\pi\cdot\frac{1}{2}\int_0^2 \sqrt{4-r^2}\,\mathrm{d}(4-r^2)$$

$$= \int_0^{2\pi} \frac{64}{5}\sin^3\varphi\,\mathrm{d}\varphi + 2\pi\cdot\frac{1}{2}\,\frac{(4-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^2$$

$$= \frac{64}{5}\int_0^{2\pi} \sin\varphi(1-\cos^2\varphi)\,\mathrm{d}\varphi + 2\pi\cdot\frac{1}{3}(4-r^2)\sqrt{4-r^2}\Bigg|_0^2$$

$$= \left\{ \begin{array}{l} t = \cos\varphi,\ \mathrm{d}t = -\sin\varphi\, d\varphi \\ 0 \mapsto 1,\ \ 2\pi \mapsto 1 \end{array} \right\} = 0 - 2\pi \cdot \frac{8}{3} = -\frac{16}{3}\pi.$$

$\square$

Teorem o divergenciji

Neka je $\vec{\Sigma} \subseteq \mathbb{R}^3$ zatvorena ploha orijentirana vanjskom normalom koja omeđuje područje $\Omega \subseteq \mathbb{R}^3$. Neka je $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ vektorsko polje. Plošni integral vektorskog polja $\vec{a}$ po zatvorenoj plohi $\vec{\Sigma}$ zovemo *tok vektorskog polja kroz plohu.* Prisjetimo se da je divergencija vektorskog polja skalarno polje koje određujemo na sljedeći način:

$$\operatorname{div} \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

Prema *teoremu o divergenciji* tok vektorskog polja $\vec{a}$ kroz zatvorenu plohu $\vec{\Sigma}$ možemo izračunati po formuli:

$$\iint_{\vec{\Sigma}} \vec{a}\, \mathrm{d}\vec{S} = \iiint_{\Omega} \operatorname{div} \vec{a}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

Zadatak 6.18. Izračunajte tok vektorskog polja $\vec{a} = \vec{j} + z^3\vec{k}$ kroz sferu $x^2 + y^2 + z^2 = 1$. Skicirajte sferu.

Rješenje: Zatvorena ploha $\vec{\Sigma}$ omeđuje kuglu Ω sa središtem u ishodištu radijusa $r = 1$ (pogledajte sliku 6.33). Stoga ćemo integral po kugli Ω računati u sfernim koordinatama. Odredimo najprije divergenciju polja $\vec{a}$:

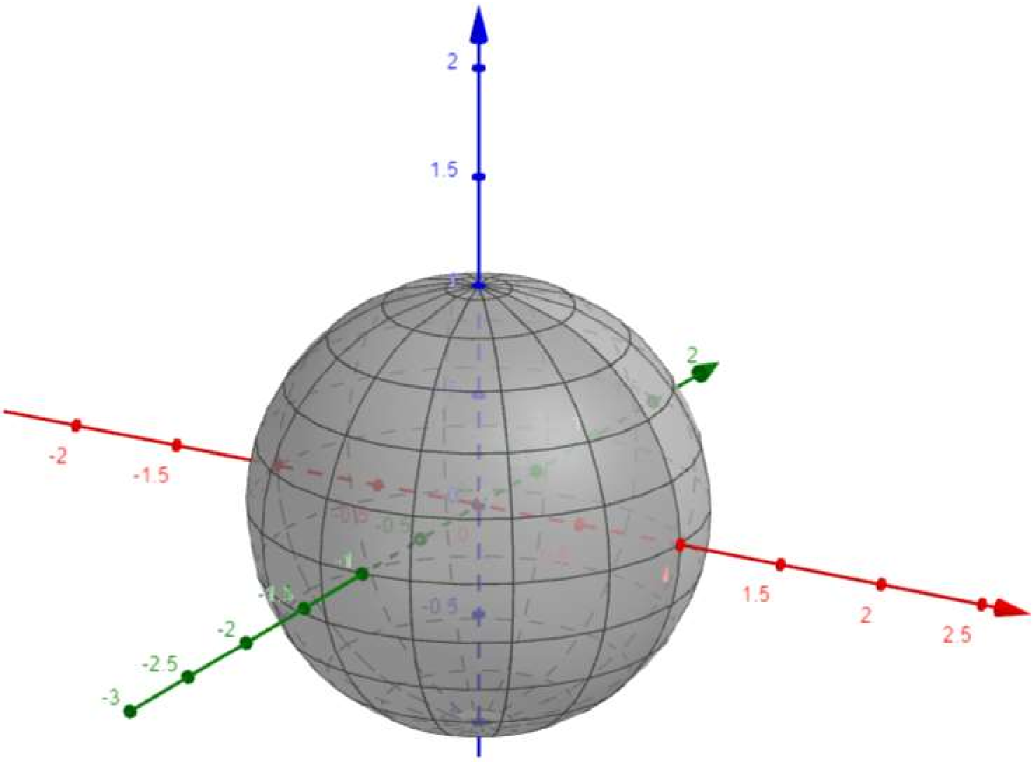
$$\operatorname{div} \vec{a} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}(z^3) = 3z^2.$$

Prema teoremu o divergenciji tok polja $\vec{a}$ kroz sferu je jednak:

$$\iint_{\vec{\Sigma}} \vec{a}\, \mathrm{d}\vec{S} = \iiint_{\Omega} \operatorname{div} \vec{a}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\vartheta \int_0^1 3r^2 \cos^2\vartheta \cdot r^2 \sin\vartheta\, \mathrm{d}r$$

$$= 2\pi \int_0^{\pi} \cos^2\vartheta \sin\vartheta \cdot \frac{3r^5}{5}\Big|_0^1 \mathrm{d}\vartheta = \frac{6\pi}{5} \int_0^{\pi} \cos^2\vartheta \sin\vartheta\, \mathrm{d}\vartheta$$



Slika 6.33: Ploha $\vec{\Sigma}$

$$= \left\{ \begin{array}{l} t = \cos\vartheta,\ \mathrm{d}t = -\sin\vartheta\, d\vartheta \\ 0 \mapsto 1,\ \pi \mapsto -1 \end{array} \right\} = \frac{6\pi}{5}\int_1^{-1} t^2(-\mathrm{d}t)$$

$$= \frac{6\pi}{5}\int_{-1}^{1} t^2\,\mathrm{d}t = \frac{6\pi}{5}\cdot\frac{t^3}{3}\Big|_{-1}^{1} = \frac{2\pi}{5}(1+1) = \frac{4\pi}{5}.$$

□

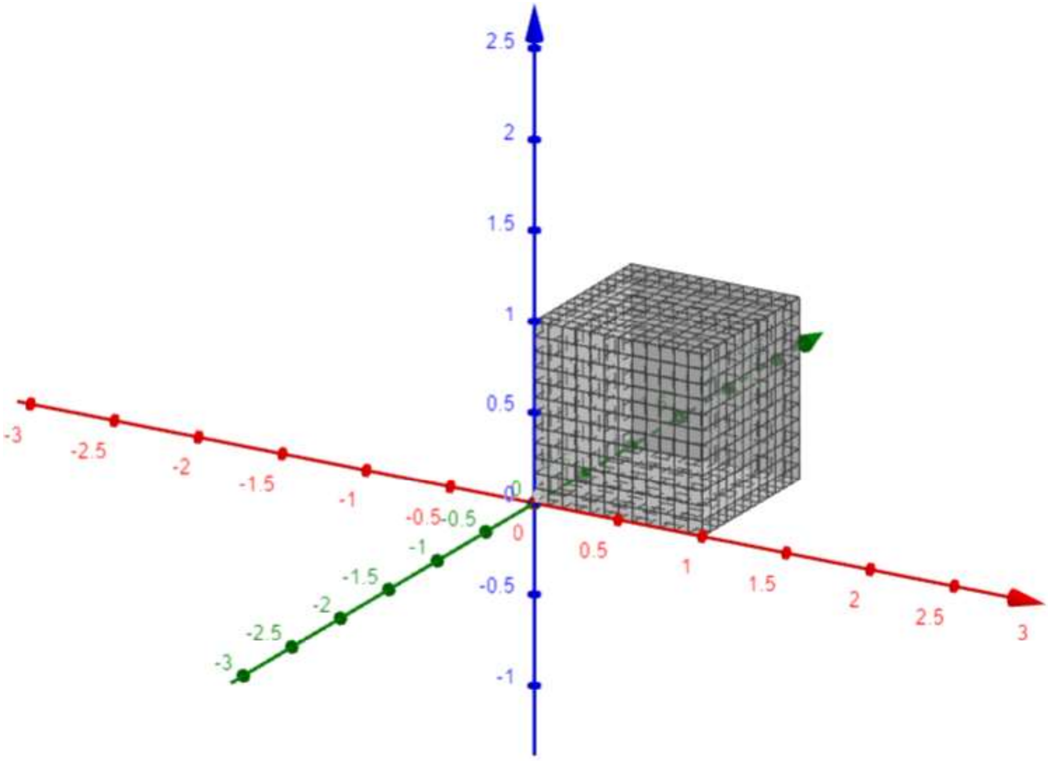
Zadatak 6.19. Izračunajte tok vektorskog polja $\vec{a} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ po oplošju kocke $[0,1]^3$. Skicirajte kocku.

Rješenje: Odredimo $\operatorname{div}\vec{a} = 2x + 2y + 2z$. Prema teoremu o divergenciji vrijedi:

$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} = \iiint_{\Omega} (2x+2y+2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 (2x+2y+2z)\mathrm{d}z = \int_0^1 \mathrm{d}x \int_0^1 (2xz+2yz+z^2)\big|_0^1 \mathrm{d}y$$

$$= \int_0^1 \mathrm{d}x \int_0^1 (2x+2y+1)\mathrm{d}y = \int_0^1 (2xy+y^2+y)\big|_0^1 \mathrm{d}x$$



Slika 6.34: Kocka Ω

$$= \int_0^1 (2x + 1 + 1)\mathrm{d}x = (x^2 + 2x)\Big|_0^1 = 3.$$

$\square$

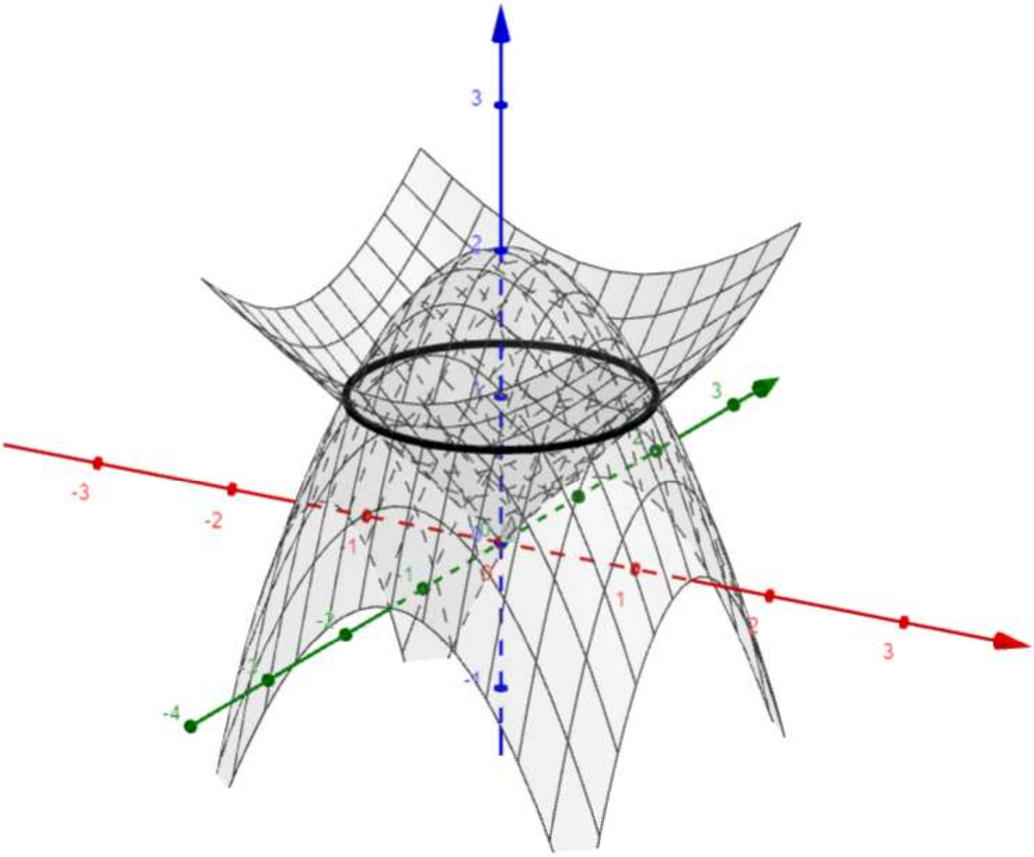
Zadatak 6.20. Izračunajte tok vektorskog polja $\vec{a} = x^2\vec{i} - 2xy\vec{j} + 3z\vec{k}$ kroz zatvorenu plohu koja se sastoji od ploha $z = 2 - x^2 - y^2$ i $z = \sqrt{x^2 + y^2}$. Skicirajte plohu.

Rješenje: Odredimo najprije presjek kružnog paraboloida $z = 2 - x^2 - y^2$ i kružnog stošca $z = \sqrt{x^2 + y^2}$:

$$2 - x^2 - y^2 = \sqrt{x^2 + y^2}\,(\rho = \sqrt{x^2 + y^2}) \Rightarrow 2 - \rho^2 = \rho$$

$$\Rightarrow \rho^2 + \rho - 2 = 0 \Rightarrow \rho = 1 \Rightarrow x^2 + y^2 = 1.$$

Vidimo dakle, da je presjek ploha kružnica radijusa $r = 1$ sa središtem u točki $(0, 0, 1)$ na osi z. Volumen Ω kojeg zatvaraju zadane plohe ćemo opisati u cilindričnim koordinatama. Znamo da je $\operatorname{div} \vec{a} = 2x - 2x + 3 = 3$. Prema



Slika 6.35: Ploha $\vec{\Sigma}$

teoremu o divergenciji računamo:

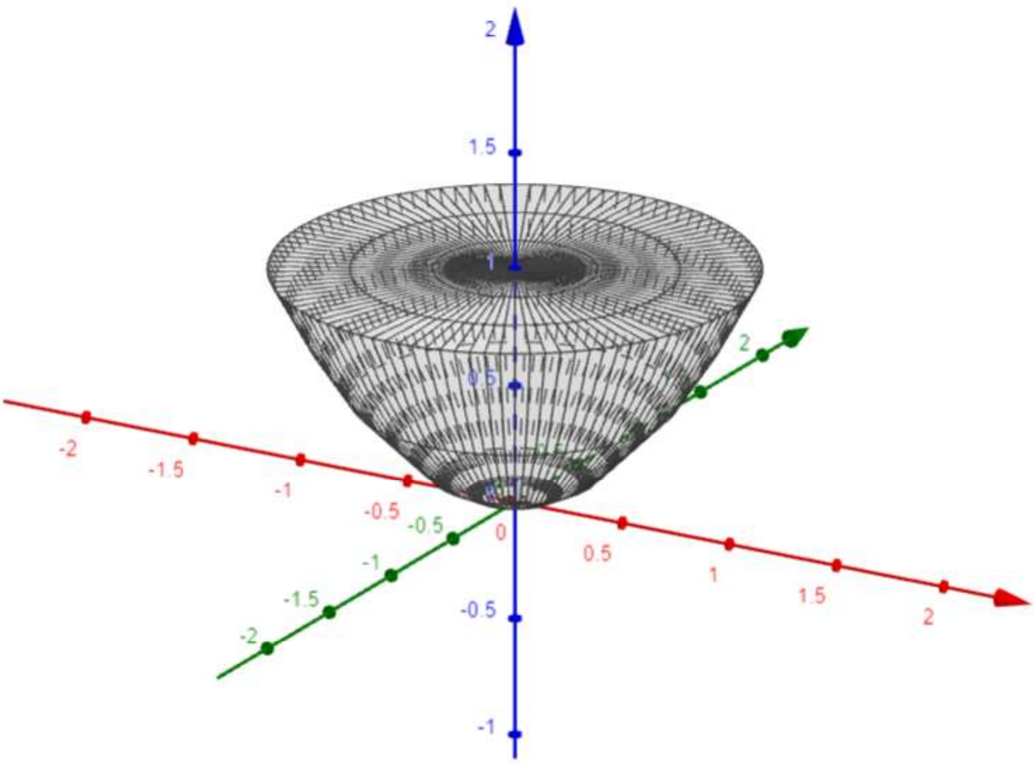
$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} = \iiint_{\Omega} 3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= 3\int_0^{2\pi}\mathrm{d}\varphi\int_0^1\mathrm{d}\rho\int_{\rho}^{2-\rho^2}\rho\,\mathrm{d}z = 3\int_0^{2\pi}\mathrm{d}\varphi\int_0^1(2\rho-\rho^3-\rho^2)\mathrm{d}\rho$$

$$= 3\int_0^{2\pi}\Big(\rho^2-\frac{\rho^4}{4}-\frac{\rho^3}{3}\Big)\Big|_0^1\mathrm{d}\varphi = 3\cdot 2\pi\Big(1-\frac{1}{4}-\frac{1}{3}\Big) = \frac{5\pi}{2}.$$

□

Zadatak 6.21. Izračunajte $\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$, gdje je $\vec{a} = xz\vec{i} + x^2y\vec{j} + y^2z\vec{k}$, ako je $\vec{\Sigma}$ zatvorena ploha orijentirana u smjeru vanjskih normala, a sastavljena od paraboloida $z = x^2 + y^2$ i ravnine $z = 1$. Skicirajte plohu.



Slika 6.36: Ploha $\vec{\Sigma}$

Rješenje: Zadane plohe zatvaraju volumen Ω koji ćemo opisati u cilindričnim koordinatama (pogledajte sliku 6.36). Odredimo div $\vec{a} = z + x^2 + y^2$. Sada prema teoremu o divergenciji računamo:

$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} = \iiint_{\Omega} (z + x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \mathrm{d}\rho \int_{\rho^2}^1 (z + \rho^2)\rho\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \Big(\rho \cdot \frac{z^2}{2} + \rho^3 z\Big)\Big|_{\rho^2}^1 \mathrm{d}\rho$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \Big(\frac{\rho}{2} + \rho^3 - \frac{3}{2}\rho^5\Big)\mathrm{d}\rho = 2\pi\Big(\frac{\rho^2}{4} + \frac{\rho^4}{4} - \frac{\rho^6}{4}\Big)\Big|_0^1$$

$$= 2\pi\Big(\frac{1}{4} + \frac{1}{4} - \frac{1}{4}\Big) = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}.$$

$\square$

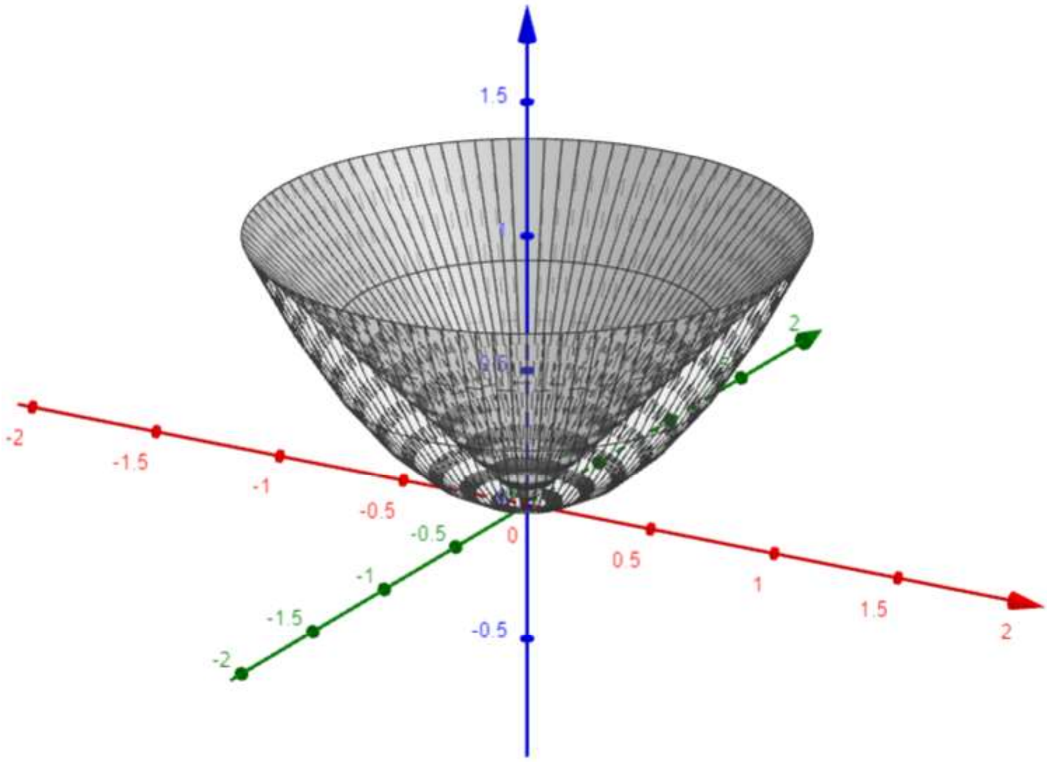
Zadatak 6.22. Izračunajte $\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S}$, gdje je $\vec{a} = xz\vec{i} + yz\vec{j} + z\vec{k}$, ako je $\vec{\Sigma}$ zatvorena ploha orijentirana u smjeru vanjskih normala sastavljena od paraboloida $z = x^2 + y^2$ i stošca $z = \sqrt{x^2 + y^2}$

a) direktno,

b) pomoću teorema o divergenciji.

Skicirajte plohu.

Rješenje:



Slika 6.37: Ploha $\vec{\Sigma}$

a) Presjek stošca i paraboloida je očito kružnica radijusa $r = 1$ sa središtem u točki $(0, 0, 1)$ na osi z. Prema tome, projekcija D plohe $\vec{\Sigma}$ na xy-ravninu je krug sa središtem u ishodištu radijusa $r = 1$.

   Stožac $z = f_1(x, y) = \sqrt{x^2 + y^2}$ je orijentiran normalom $\vec{n}_1$ koja zatvara šiljasti kut s vektorom $\vec{k}$:

$$\frac{\partial f_1}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial f_1}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow n_1 = -\frac{x}{\sqrt{x^2+y^2}}\,\vec{i} - \frac{y}{\sqrt{x^2+y^2}}\,\vec{j} + \vec{k}.$$

Odredimo:

$$\vec{a} \cdot \vec{n}_1 = -\frac{x^2 z}{\sqrt{x^2+y^2}} - \frac{y^2 z}{\sqrt{x^2+y^2}} + z = -x^2 - y^2 + \sqrt{x^2+y^2}.$$

Paraboloid $z = f_2(x,y) = x^2 + y^2$ je orijentiran normalom $\vec{n}_2$ koja zatvara tupi kut s vektorom $\vec{k}$:

$$\frac{\partial f_2}{\partial x} = 2x,\ \frac{\partial f_2}{\partial y} = 2y \Rightarrow n_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}.$$

Nađimo:

$$\vec{a}\cdot\vec{n}_2 = 2x^2z + 2y^2z - z = 2x^4 + 2y^4 + 4x^2y^2 - x^2 - y^2 = 2(x^2+y^2)^2 - x^2 - y^2.$$

Sada zbog aditivnosti krivuljnog integrala po području integracije vrijedi:

$$\begin{aligned}
\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} &= \iint_D \vec{a}\cdot\vec{n}_1\,\mathrm{d}x\,\mathrm{d}y + \iint_D \vec{a}\cdot\vec{n}_2\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_D (-2x^2 - 2y^2 + \sqrt{x^2+y^2} + 2(x^2+y^2)^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 (-2r^2 + r + 2r^4)\,r\,\mathrm{d}r = 2\pi\Big(-\frac{2r^4}{4} + \frac{r^3}{3} + \frac{2r^6}{6}\Big)\Big|_0^1 \\
&= 2\pi\Big(-\frac{1}{2} + \frac{1}{3} + \frac{1}{3}\Big) = 2\pi\cdot\frac{1}{6} = \frac{\pi}{3}.
\end{aligned}$$

b) Stožac i paraboloid zatvaraju tijelo Ω koje ćemo opisati u cilindričnim koordinatama. Prema teoremu o divergenciji vrijedi:

$$\iint_{\vec{\Sigma}} \vec{a}\,\mathrm{d}\vec{S} = \iiint_\Omega (z + z + 1)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \mathrm{d}\rho \int_{\rho^2}^{\rho} (2z+1)\rho\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 (\rho z^2 + \rho z)\Big|_{\rho^2}^{\rho} \mathrm{d}\rho$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 (\rho^3 + \rho^2 - \rho^5 - \rho^3)\mathrm{d}\rho = 2\pi\Big(\frac{\rho^3}{3} - \frac{\rho^6}{6}\Big)\Big|_0^1 = \frac{\pi}{3}.$$

$\square$

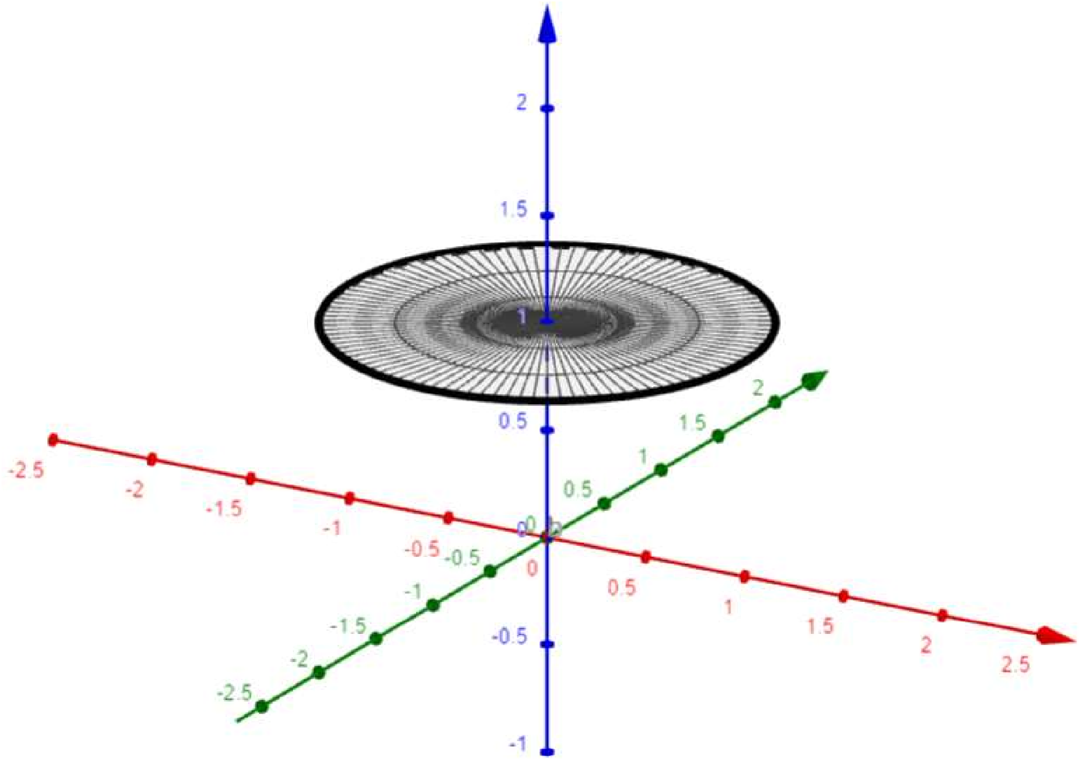
Stokesov teorem

Teorem. *(Stokes) Neka je* $\vec{\Gamma} \subseteq \mathbb{R}^3$ *zatvorena krivulja koja omeđuje plohu* $\vec{\Sigma}$ *(*$\partial\vec{\Sigma} = \vec{\Gamma}$*) orijentirana pozitivno ako se gleda iz vrha normale na plohu* $\vec{\Sigma}$ *i neka je* $\vec{a}$ *dovoljno glatko vektorsko polje definirano na području koje sadrži plohu* $\vec{\Sigma}$*. Tada vrijedi:*

$$\int_{\vec{\Gamma}} \vec{a}\, d\vec{r} = \iint_{\vec{\Sigma}} \operatorname{rot} \vec{a}\, d\vec{S}.$$

Zadatak 6.23. Primjenom Stokesovog teorema izračunajte $\int_{\vec{\Gamma}} xy^2 \mathrm{d}x + \mathrm{d}y + z\,\mathrm{d}z$ ako je $\vec{\Gamma}$ kružnica $x^2 + y^2 = 1$ i $z = 1$. Skicirajte krivulju i plohu.

Rješenje: Odredimo rotaciju vektorskog polja $\vec{a}$:



Slika 6.38: Ploha $\vec{\Sigma}$

$$\operatorname{rot} \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 1 & z \end{vmatrix} = -2xy\,\vec{k}.$$

Budući da jednadžba plohe $\vec{\Sigma}$ ima oblik $z = f(x, y) = 1$, normala na plohu je $\vec{n} = \vec{k}$, tako da vrijedi $\operatorname{rot} \vec{a} \cdot \vec{n} = -2xy$. Sada prema Stokesovom teoremu vrijedi

$$\int_{\vec{\Gamma}} \vec{a}\, \mathrm{d}\vec{r} = \iint_{\vec{\Sigma}} \operatorname{rot} \vec{a}\, \mathrm{d}\vec{S} = \iint_{D} (-2xy) \mathrm{d}x\, \mathrm{d}y$$

$$= -2 \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 r^2 \cos\varphi \sin\varphi\, r\, \mathrm{d}r = -2 \int_0^{2\pi} \cos\varphi \sin\varphi\, \frac{r^4}{4}\Big|_0^1 \mathrm{d}\varphi$$

$$= -\frac{1}{2} \int_0^{2\pi} \sin\varphi \cos\varphi \mathrm{d}\varphi = -\frac{1}{2} \cdot \frac{\sin^2\varphi}{2}\Big|_0^{2\pi} = 0.$$

Integral također možemo izračunati direktno pomoću formule za krivuljni integral 2. vrste. Parametriziramo krivulju $\vec{\Gamma}$:

$$\begin{aligned} x(t) &= \cos t \Rightarrow x'(t) = -\sin t \\ y(t) &= \sin t \Rightarrow y'(t) = \cos t \\ z(t) &= 1 \Rightarrow z'(t) = 0 \\ t &\in [0, 2\pi]. \end{aligned}$$

Sada računamo integral:

$$\int_{\vec{\Gamma}} \vec{a}\, \mathrm{d}\vec{r} = \int_0^{2\pi} (\cos t \sin^2 t \cdot (-\sin t) + \cos t + 1 \cdot 0) \mathrm{d}t$$

$$= \int_0^{2\pi} (-\sin^3 t \cos t + \cos t) \mathrm{d}t = \Big(-\frac{\sin^4 t}{4} + \sin t \Big)\Big|_0^{2\pi} = 0.$$

□

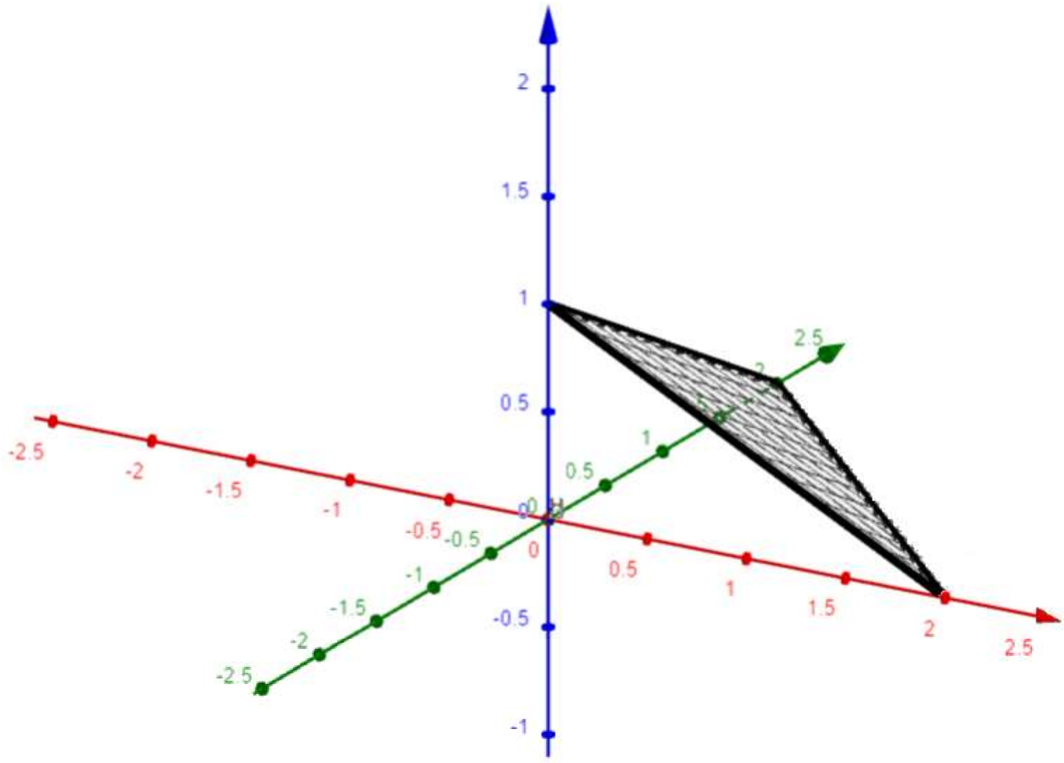
Zadatak 6.24. Primjenom Stokesovog teorema izračunajte $\int_{\vec{\Gamma}} \vec{a}\, \mathrm{d}\vec{r}$, ako je $\vec{a} = 2xz\vec{i} - x\vec{j} + z\vec{k}$, a $\vec{\Gamma}$ krivulja nastala presjekom ravnine $x + y + 2z = 2$ s koordinatnim ravninama. Skicirajte krivulju i plohu.

Rješenje: Jednadžba plohe $\vec{\Sigma}$ (dio ravnine u 1. oktantu) ima oblik:

$$z = f(x, y) = 1 - \frac{x}{2} - \frac{y}{2} \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = -\frac{1}{2},$$

pa je normala na plohu koja zatvara šiljsti kut s vektorom $\vec{k}$ jednaka $\vec{n} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}$. Odredimo rotaciju vektorskog polja $\vec{a}$:

$$\operatorname{rot} \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & -x & z \end{vmatrix} = 0 \cdot \vec{i} - (0 - 2x)\vec{j} + (-1 - 0)\vec{k} = 2x\vec{j} - \vec{k}.$$



Slika 6.39: Ploha $\vec{\Sigma}$

Sada je $\operatorname{rot} a \cdot \vec{n} = x - 1$, pa prema Stokesovom teoremu vrijedi (prijekcija D plohe na xy-ravninu je pravokutan trokut s katetama na koordinatnim osima i hipotenuzom na pravcu $y = 2 - x$):

$$\int_{\vec{\Gamma}} \vec{a}\, \mathrm{d}\vec{r} = \iint_{\vec{\Sigma}} \operatorname{rot} \vec{a}\, \mathrm{d}\vec{S} = \iint_D (x-1) \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^2 \mathrm{d}x \int_0^{2-x} (x-1) \mathrm{d}y = \int_0^2 (x-1)(x-2) \mathrm{d}x$$

$$= \int_0^2 (2x - 2 - x^2 + x) \mathrm{d}x = \left(-\frac{x^3}{3} + \frac{3x^2}{2} - 2x \right) \Big|_0^2$$

$$= -\frac{8}{3} + 6 - 4 = -\frac{2}{3}.$$

□