MATEMATIKA 2, 8. 2. 2021.

1. Riješite diferencijalne jednadžbe:

   (a) **(12 bodova)** $y'' - 5y' + 6y = xe^x$,

   (b) **(8 bodova)** $xy' - y = 1$.

2. (a) **(7 bodova)** Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \arcsin(x^2 + y^2) + \ln(y - x).$$

   (b) **(13 bodova)** Odredite ekstreme funkcije

$$f(x,y) = 4 - 4x - 3y$$

   uz uvjet $x^2 + y^2 = 16$.

3. **(20 bodova)** Izračunajte masu tijela omeđenog plohama $x^2 + (y-2)^2 = 4$, $z = 0$ i $z = 2$, ako mu je gustoća dana s $\rho(x,y,z) = \frac{y^2}{\sqrt{x^2+y^2}}$. Skicirajte tijelo.

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4. (a) **(15 bodova)** Odredite konstante $A, B, C \in \mathbb{R}$ tako da vektorsko polje

$$\vec{v} = (Axz + y)\vec{i} + (Bx + Cz)\vec{j} + (x^2 + y)\vec{k}$$

   bude potencijalno, te mu odredite potencijal.

   (b) **(10 bodova)** Izračunajte $\int_\Gamma \frac{9}{64} x\,ds$, ako je Γ luk polukubne parabole $y^2 = \frac{1}{16}x^3$ od točke $A(0,0)$ do točke $B(4,2)$.

5. **(15 bodova)** Izračunajte površinu dijela polusfere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, koju isijeca ploha $x^2 + y^2 = 4x$. Skicirajte plohu.

Prvi dio čine prva tri zadatka. **Drugi dio** čine 4. i 5. zadatak.

Za polaganje ispita treba skupiti 50 bodova (od toga barem 24 boda iz prvog dijela i barem 16 bodova iz drugog dijela).

MAT2. 8.2.2021. (12. 26)

1. a) (12b) $y'' - 5y' + 6y = xe^x$

$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_{1,2} = \frac{5 \pm \sqrt{25-24}}{2}$ $\nearrow \lambda_1 = 2$, $\searrow \lambda_2 = 3$

$y_H = C_1 e^{2x} + C_2 e^{3x}$

$f(x) = xe^x$, $\alpha = 1 \neq \lambda_1, \lambda_2$

$y_p = (Ax+B)e^x$

$y_p' = Ae^x + (Ax+B)e^x$

$y_p'' = Ae^x + Ae^x + (Ax+B)e^x$

$2Ae^x + (Ax+B)e^x - 5Ae^x - 5(Ax+B)e^x + 6(Ax+B)e^x = xe^x$

$-3Ae^x + 2(Ax+B)e^x = xe^x \quad / : e^x$

$2Ax + (-3A+2B) = x \quad \Rightarrow$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$-3A + 2B = 0 \Rightarrow -\frac{3}{2} + 2B = 0 \Rightarrow B = \frac{3}{4}$

$y_p = (\frac{1}{2}x + \frac{3}{4})e^x$

$$\boxed{y = y_H + y_p = C_1 e^{2x} + C_2 e^{3x} + (\frac{x}{2} + \frac{3}{4})e^x}$$

b) (8b) $xy' - y = 1 \quad / : x \Rightarrow y' \underbrace{- \frac{1}{x}}_{f(x)} y = \underbrace{\frac{1}{x}}_{g(x)}$

$y = e^{-\int f(x)dx} \left[\int e^{\int f(x)dx} g(x)dx + C \right]$

$\int f(x)dx = -\int \frac{dx}{x} = -\ln x$

$$\int e^{\int f(x)dx} g(x)\,dx = \int e^{-\ln x}\cdot \frac{1}{x}\,dx = \int e^{\ln\frac{1}{x}}\cdot\frac{1}{x}\,dx$$

$$= \int \frac{dx}{x^2} = -\frac{1}{x}$$

$$y = \underbrace{e^{\ln x}}_{=x}\left[-\frac{1}{x}+C\right] \Rightarrow \boxed{y = Cx - 1}$$

(12.35)

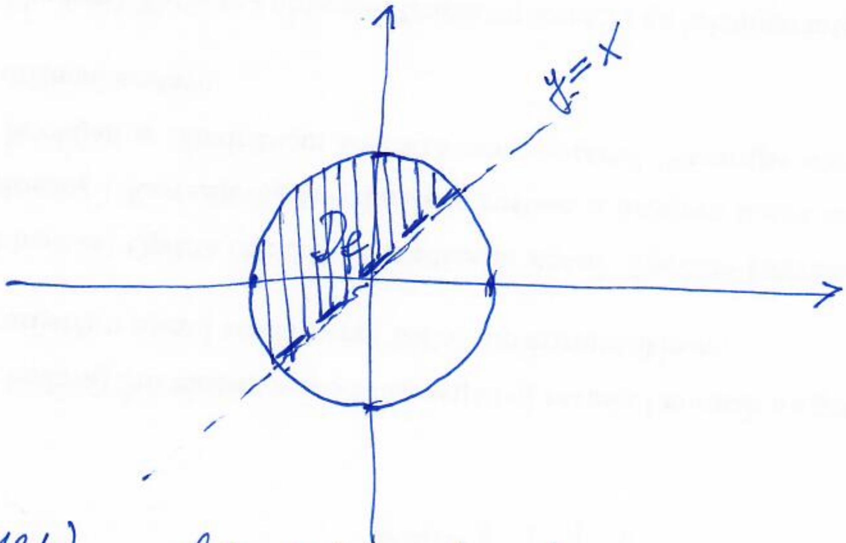
2.

a) (7b) Domena?

$$f(x,y) = \arcsin(x^2+y^2) + \ln(y-x)$$

$$\underbrace{-1 \le x^2+y^2 \le 1}_{\text{uvjet zadovoljen!}} \quad i \quad y-x>0 \Rightarrow y>x$$

$$D_f = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1,\ y>x\}$$



b) (13b) $f(x,y)=4-4x-3y$, $x^2+y^2=16$

$$F(x,y,\lambda) = 4-4x-3y+\lambda(x^2+y^2-16)$$

$$\frac{\partial F}{\partial x} = -4+2\lambda x = 0 \Rightarrow x=\frac{2}{\lambda}$$

$$\frac{\partial F}{\partial y} = -3+2\lambda y = 0 \Rightarrow y = \frac{3}{2\lambda}$$

$$x^2+y^2=16 \Rightarrow \frac{4}{\lambda^2}+\frac{9}{4\lambda^2}=16 \Rightarrow \lambda^2=\frac{25}{64} \Rightarrow \lambda_{1,2}=\pm\frac{5}{8}$$

$x_1 = \frac{2}{\frac{5}{8}} = \frac{16}{5}$ $\quad y_1 = \frac{3}{\cancel{2}\cdot\frac{5}{\cancel{8}_4}} = \frac{12}{5}$ $\quad \boxed{T_1\left(\frac{16}{5}, \frac{12}{5}\right)}$

$\boxed{T_2\left(-\frac{16}{5}, -\frac{12}{5}\right)}$

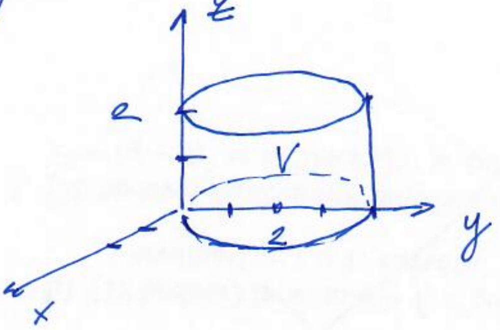
$f(T_1) = 4 - \frac{64}{5} - \frac{36}{5} < f(T_2) = 4 + \frac{64}{5} + \frac{36}{5}$

$\Rightarrow$ u T_1 f postiže minimum, a u T_2 f postiže maksimum

-2 min

12,49 12,47

3. (20 b)



$V \ldots \; 0 \le \varphi \le \pi$

$0 \le \rho \le 4\sin\varphi$

$0 \le z \le 2$

$$m = \iiint_V \frac{y^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^\pi d\varphi \int_0^{4\sin\varphi} \frac{\rho^2\sin^2\varphi}{\cancel{\rho}}\cancel{\rho}\,d\rho \int_0^2 dz$$

$$= 2\int_0^\pi \sin^2\varphi \left.\frac{\rho^3}{3}\right|_0^{4\sin\varphi} d\varphi = \frac{2}{3}\int_0^\pi 64\sin^5\varphi\,d\varphi$$

$$= \frac{128}{3}\int_0^\pi \sin\varphi(1-\cos^2\varphi)^2\,d\varphi = \left| \begin{array}{l} t=\cos\varphi \\ dt = -\sin\varphi\,d\varphi \\ \varphi=0 \Rightarrow t=1 \\ \varphi=\pi \Rightarrow t=-1 \end{array} \right|$$

$$= \frac{128}{3}\int_{-1}^1 (1-t^2)^2\,dt = \frac{256}{3}\int_0^1 (1-2t^2+t^4)\,dt$$

$$= \frac{256}{3}\left(t - \frac{2t^3}{3} + \frac{t^5}{5}\right)\Big|_0^1 = \frac{256}{3}\left(1-\frac{2}{3}+\frac{1}{5}\right) =$$

$= \frac{256}{3} \frac{15-10+3}{15} = \frac{256}{3} \cdot \frac{8}{15} = \boxed{\frac{2048}{45}}$

13.00

1. dio 10 min + 12 min + 11 min = 33 min

9,36

4. (15b) $\vec{v} = (Axz + y)\vec{i} + (Bx + Cz)\vec{j} + (x^2 + y)\vec{k}$

$\text{rot } \vec{v} = \vec{0} \Rightarrow \vec{v}$ potencijalno

$$\vec{0} = \text{rot } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axz+y & Bx+Cz & x^2+y \end{vmatrix}$$

$= (1-C)\vec{i} - \vec{j}(2x - Ax) + \vec{k}(B-1)$

$\Rightarrow C = 1, \; A = 2, \; B = 1$

$\vec{v} = (2xz + y)\vec{i} + (x + z)\vec{j} + (x^2 + y)\vec{k}$ je potencijalno

$\boxed{\varphi(x,y,z)} = \int_0^x (2tz + y)\,dt + \int_0^y z\,dt = \left(\cancel{2}\frac{t^2}{\cancel{2}} z + yt\right)\Big|_0^x$

$+ zt\Big|_0^y = \boxed{x^2 z + xy + yz + C}$

5. ~~(15b)~~ b) (10b)

$\Gamma \ldots \; x(t) = t$

$y(t) = \frac{1}{4} t^{\frac{3}{2}}$

$t \in [0, 4]$

$x'(t) = 1$

$y'(t) = \frac{1}{4} \cdot \frac{3}{2} t^{\frac{1}{2}}$

$= \frac{3}{8}\sqrt{t}$

$$\int_{\Gamma} \frac{9}{64}x\,ds = \int_0^4 \frac{9}{64}t\sqrt{1+\frac{9}{64}t}\,dt$$

$$= \left|\begin{matrix} u = 1+\frac{9}{64}t \\ du = \frac{9}{64}dt \\ t=0 \Rightarrow u=1 \\ t=4 \Rightarrow u=\frac{25}{16} \end{matrix}\right| = \int_1^{\frac{25}{16}} (u-1)\sqrt{u}\cdot\frac{64}{9}\,du$$

$$= \frac{64}{9}\int_1^{\frac{25}{16}}\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du = \frac{64}{9}\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}}-\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^{\frac{25}{16}}$$

$$= \frac{64}{9}\left(\frac{\cancel{2}}{\cancel{5}}\left(\frac{25}{16}\right)^2\cdot\frac{\cancel{5}}{\cancel{4}_2}-\frac{2}{3}\left(\frac{25}{16}\right)\cdot\frac{5}{\cancel{4}_2}-\frac{2}{5}+\frac{2}{3}\right)$$

$$= \frac{64}{9}\left(\frac{25}{32}\underbrace{\left(\frac{25}{16}-\frac{5}{3}\right)}_{-\frac{2}{48}}\underbrace{-\frac{2}{5}+\frac{2}{3}}_{\frac{4}{15}}\right) = \dots ?$$

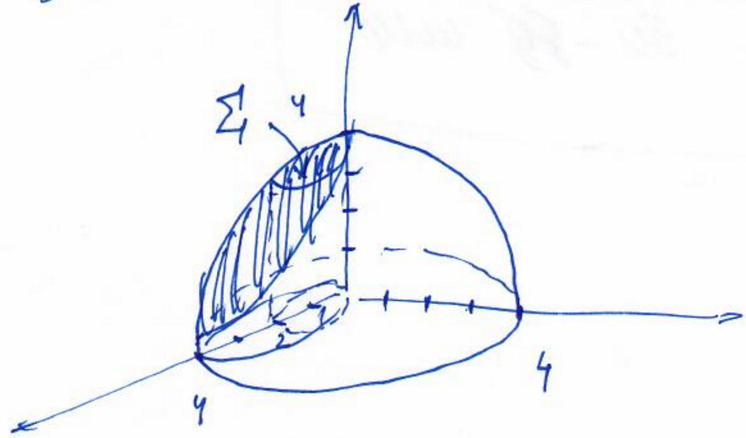
75-

$$\cancel{= \frac{64}{9}\left(-\frac{25}{32}+1\right)\left(\frac{2}{3}-\frac{2}{5}\right) = \frac{64}{9}\left(\frac{7}{32_{16}}\cdot\frac{2}{3}-\frac{2}{5}\right)}$$

9,58 =

9,58

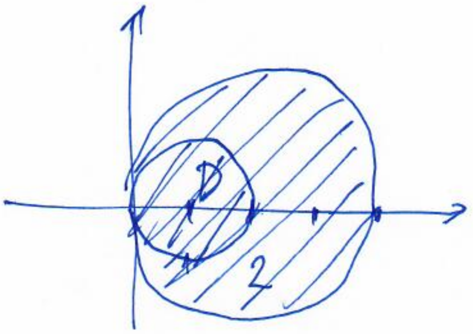
5(20b)



$\Sigma_1 \ldots\ z = f(x,y) = \sqrt{16-x^2-y^2}$

$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{16-x^2-y^2}}$

$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{16-x^2-y^2}}$



$D \ldots\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

$0 \leq r \leq 4\cos\varphi$

$$P(\Sigma_1) = \iint_{\Sigma_1} dS = \iint_D \sqrt{1 + \frac{x^2}{16-x^2-y^2} + \frac{y^2}{16-x^2-y^2}}\,dx\,dy$$

$$= 4\iint_D \frac{dx\,dy}{\sqrt{16-x^2-y^2}} = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{4\cos\varphi} \frac{r\,dr}{\sqrt{16-r^2}}$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\sqrt{16-r^2}\right)\Big|_0^{4\cos\varphi} d\varphi = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(4 - \underbrace{\sqrt{16-16\cos^2\varphi}}_{4|\sin\varphi|}\right) d\varphi$$

10/12

$$= \left| \quad -\frac{\pi}{2} \quad \frac{\pi}{2} \quad \right| = 16\left(\pi - 2\int_0^{\frac{\pi}{2}} \sin\varphi\,d\varphi\right)$$

$$= 16\left(\pi - 2(-\cos\varphi)\Big|_0^{\frac{\pi}{2}}\right) = \boxed{16(\pi-2)}$$

2.dio 12 + 13 min = 25 min